T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T XI Monday, 29 January 2007 (Time: 13:45 - 15:15, Room: P 004)

1.11 Solution methods for IVPs of ODEs

49 Consider the initial-value problem

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(0) = u_0 \,. \end{cases}$$

Under the assumption that

$$\|f(t, w) - f(t, v)\| \leq L\|w - v\| \qquad \forall t, v, w,$$

show that, for each t_j and u_j, there exists a unique solution u_{j+1} of the equation

$$u_{j+1} = u_j + \tau f(t_j + \tau, u_{j+1}),$$

if $\tau < 1/L$.

Hint: apply Banach's Fixed Point Theorem.

50 Consider the initial-value problem

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(0) = u_0 \,. \end{cases}$$

Assuming that

$$\|f(t, w) - f(t, v)\| \leq L\|w - v\| \qquad \forall t, v, w,$$

and

$$(f(t, w) - f(t, v), w - v) \leq 0 \qquad \forall t, v, w.$$

show that, for each $\tau > 0$, t_j and u_j, there exists a unique solution u_{j+1} of the equation

$$u_{j+1} = u_j + \tau f(t_j + \tau, u_{j+1}).$$

Hint: apply Banach's Fixed Point Theorem to the following equivalent equation

$$u_{j+1} = G(u_{j+1}) = (1 - \rho)u_{j+1} + \rho(u_j + \tau f(t_j + \tau, u_{j+1}))$$

for some parameter $\rho \in (0, 1)$. Estimate

$$\|G(w) - G(v)\|^2 = (G(w) - G(v), G(w) - G(v)),$$

and choose $\rho \in (0, 1)$ such that G is a contraction.

$\boxed{51}$ Let

$$\psi_\tau(t+\tau) = \frac{1}{\tau}\left(u(t+\tau) - u(t)\right) - f(t+\tau, u(t+\tau))$$

denote the consistency error of the implicit Euler method, where $u(t)$ is the solution of the differential equation

$$u'(t) = f(t, u(t)).$$

Show that the following estimate holds:

$$\|\psi_\tau(t+\tau)\| \leq \int_t^{t+\tau} \|u''(s)\| ds \ .$$

$\boxed{52}$ Consider the same notation and the same assumptions as in exercises $\boxed{50}$ and $\boxed{51}$. Let $u(t)$ be the exact solution of the initial-value problem and consider the implicit Euler method:

$$u_{j+1} = u_j + \tau f(t_{j+1}, u_{j+1})\,.$$

Show that

$$u(t_{j+1}) = u(t_j) + \tau f(t_{j+1}, u(t_{j+1})) + \tau\psi_\tau(t_{j+1})\,,$$

and

$$e_{j+1} = e_j + \tau(f(t_{j+1}, u(t_{j+1})) - f(t_{j+1}, u_{j+1})) + \tau\psi_\tau(t_{j+1})\,, \tag{1.30}$$

where $e_k = u(t_k) - u_k$ denotes the global error.

$\boxed{53}$ Consider the same notations as in exercise $\boxed{52}$. Show that the following estimate holds:

$$\|e_{j+1}\| \leq \|e_j\| + \tau\|\psi_\tau(t_{j+1})\|\,.$$

Hint: multiply (1.30) by $e_{j+1} = u(t_{j+1}) - u_{j+1}$ and use Cauchy's inequality at the right-hand side.

$\boxed{54}$ Using the same assumptions and notations as in exercise $\boxed{52}$, show that

$$\|u(t_j) - u_j\| \leq \tau \int_0^{t_j} \|u''(s)\|\, ds.$$