T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T X Monday, 22 January 2007 (Time: 13:45 - 15:15, Room: P 004)

1.10 Runge-Kutta methods for IVPs of ODEs

45 If to evaluate the integral

$$\int_t^{t+\tau} f(s, u(s))ds$$

we apply the trapezoidal rule (TR):

$$\int_t^{t+\tau} f(s, u(s))ds \overset{\mathrm{TR}}{\approx} \frac{\tau}{2}\,[f(t, u(t)) \;+\; f(t+\tau, u(t+\tau))]\;,$$

and we approximate $u(t+\tau) \approx u(t) + \tau f(t, u(t))$ according to the forward Euler method, we obtain the so-called Heun method:

$$\begin{aligned} g_1 &= u \\ g_2 &= u + \tau f(t, u)\,. \end{aligned}$$

We can associate to the Heun method the following table:

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

which shows that the Heun method is a 2-step Runge-Kutta method, and it holds

$$u(t+\tau) = u(t) + \frac{\tau}{2}[f(t, u) + f(t+\tau, u + \tau f(t, u))]\,.$$

Considering a Taylor expansion of the local error $u(t+\tau) - u_\tau(t+\tau)$, show that the Heun method has order of consistency equal to 2.

46 Consider the initial-value problem

$$\left\{ \begin{array}{rcll} u'(t) &=& B(u + A), & 0 < t < 1\,, \\ u(0) &=& 0\,, & \end{array} \right. \tag{1.26}$$

with $B = 3$ and $A = e^B - 1$.

a) Compute analitically the exact solution u of (1.26).

b) Consider the discretization steps $\tau_i = 2^{-i}$ for $i = 0, \ldots, 4$, and compute the numerical approximation u_τ of u using the following explicit Runge-Kutta methods:

- the (explicit) Forward Euler method (FE)
- the Heun method (H)
- the classical fourth-order Runke-Kutta method (RK4):

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array} \tag{1.27}$$

Plot the exact and the numerical solution obtained for each τ_i.

c) Fill in the following table reporting the approximation $u_\tau(1)$ of $u(1)$:

Method \ τ		1	1/2	1/4	1/8	1/16	$u(1)$
FE	(*)						
	(E)					–	
H	(*)						
	(E)					–	
RK4	(*)						
	(E)					–	

Notice that

(*) denotes the 'original' method

(E) corresponds to a post-processing of the computed values using a global extrapolation strategy considering the formula:

$$\hat{u}_\tau(t) = u_{\tau/2}(t) + \frac{u_{\tau/2}(t) - u_\tau(t)}{2p-1},$$

p being the order of the considered method.

47 Consider the following initial-value problem

$$\begin{cases} u'(t) = -50(u(t) - \cos(t)), & 0 < t < 1.5, \\ u(0) = 0. \end{cases} \tag{1.28}$$

a) Compute analitically the exact solution u of (1.28).

b) Consider the discretization steps $\tau = \frac{1}{20}, \frac{3}{80}, \frac{1}{30}, \frac{1}{40}$, and compute the numerical approximation u_τ of u using the following explicit Runge-Kutta methods:

- the (explicit) Forward Euler method (FE)
- the Heun method (H)
- the (implicit) Backward Euler method (BE).

Compare graphically the exact and the computed numerical solutions. Moreover, fill in the following table with the approximation $u_\tau(1.5)$ of the exact solution u at $t = 1.5$:

Method \ τ	1/20	3/80	1/30	1/40	$u(1.5)$
FE					
H					
BE					

c) Start the explicit Forward Euler method from $t_0 = 1/2$ using the exact solution $u(t_0) = u(1/2)$ and the discretization step $\tau = 1/20$. Compare graphically the exact and the numerical solution obtained in this case. Which value of τ must be considered in order to guarantee that the method is stable?

d) Compute the solution of (1.28) considering the (implicit) Backward Euler method and the discretization step $\tau = 1/2$. Compare graphically the computed solution and the exact one.

$\boxed{48}$ The orbit of a satellite in the plane of the Earth-Moon system can be modeled by the following second-order ODEs system:

$$\begin{aligned}
&y_1'' = y_1 + 2y_2' - (1-\mu)\,\frac{y_1+\mu}{D_1} - \mu\,\frac{y_1-(1-\mu)}{D_2}, \quad 0 < t < T\,,\\
&y_2'' = y_2 - 2y_1' - (1-\mu)\,\frac{y_2}{D_1} - \mu\,\frac{y_2}{D_2}, \quad 0 < t < T\,,\\
&y_1(0) = 0.994\,,\\
&y_2(0) = 0\,,\\
&y_1'(0) = 0\,,\\
&y_2'(0) = -2.001\,585\,106\,379\,082\,522\,405\,378\,622\,24\,,
\end{aligned} \tag{1.29}$$

where

$$\begin{aligned}
D_1 &= \left[(y_1+\mu)^2 + y_2^2\right]^{3/2}\,,\\
D_2 &= \left[(y_1-(1-\mu))^2 + y_2^2\right]^{3/2}\,,\\
\mu &= 0.012277471\,.
\end{aligned}$$

For the given data, the system has a solution whose period is

$$t_{per} = t = \underline{17.065\,216\,560\,157\,96}2\,558\,891\,720\,6249\,.$$

a) Rewrite (1.29) in an equivalent form as a system of first-order ODEs.

b) Solve (1.29) numerically considering

- the classical fourth-order Runge-Kutta method (1.27), with $\tau = T/m$ for $m = 6000$ and another suitable choice of m
- another suitably chosen method.

c) Plot the computed trajectory $(y_{1\tau}(t), y_{2\tau}(t))$, considering a linear intepolation of the computed positions.