T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T VI Thursday, 30 November 2006 (Time: 8:30 - 10:00, Room: HS 13)

1.6 FEM for BVPs of Second-order ODEs

30 Consider the problem

$$\text{find } u \in V_0 \cap H^1(0,1): \quad a(u,v) = \langle F, v\rangle \qquad \forall v \in V_0,$$

with $V_0 = \{v \in H^1(0,1) |\ v(0) = 0\}$, and let u_h be a conforming finite element approximation of u based on Courant elements. Show that we have convergence at least in $H^1(0,1)$:

$$\lim_{h\to 0} \|u - u_h\|_1 \le \frac{\mu_2}{\mu_1} \lim_{h\to 0} \inf_{v_h \in V_h} \|u - v_h\|_1 \to 0\,,$$

where V_h is a conforming internal approximation of $H^1(0,1)$, and μ_1, μ_2 are the coercivity and continuity constants of the bilinear form $a(u,v)$, respectively.

Hint: Recall that $H^2(0,1)$ is dense in $H^1(0,1)$.

31 Consider the model problem

$$\text{find } u \in H^1_0(0,1): \quad a(u,v) = \langle F, v\rangle \qquad \forall v \in H^1_0(0,1)$$

where $a(u,v) = \int_0^1 u'v'dx$ and $\langle F, v\rangle = \int_0^1 fv dx$, and consider its finite element approximation based on linear Courant elements. Let μ_1 be the coercivity constant of the bilinear form $a(\cdot,\cdot)$. Then, the following a-posteriori error estimate holds:

$$\|u - u_h\|_1 \le \frac{C}{\mu_1}\eta(u_h), \quad C > 0, \tag{1.19}$$

where

$$\eta^2(u_h) = \sum_{k=1}^{N_h} \eta_k^2, \qquad \eta_k^2 = h_k^2 \|f\|^2_{0,T_k}\,.$$

Write a function `ErrorEstimator(↓mesh,↓(*f)(x),↑error)` which implements the a-posteriori error estimator (1.19).

32 Use the functions implemented in Tutorials III-IV to discretize the following one-dimensional problem: find u such that

$$\begin{aligned} -u''(x) &= 8 \qquad x \in (0,1),\\ u(0) &= -1,\\ u'(1) &= -4. \end{aligned}$$

Solve the discretized problem

$$K_h \underline{u}_h = \underline{f}_h$$

by the preconditioned Richardson method with Jacobi preconditioner $C_h = D_h = \text{diag}(K_h)$, and compute the error estimate (1.19).

33 Let K be a symmetric and positive definite matrix, and let $u^{(n+1)}$ be an approximation of the exact solution u of the linear system

$$Ku = f,$$

obtained with the iterative method

$$u^{(n+1)} = u^{(n)} + \alpha^{(n)} p^{(n)} \tag{1.20}$$

where $p^{(n)}$ is a given search direction and $\alpha^{(n)} \in \mathbb{R}$. Determine the parameter $\alpha^{(n)}$ such that (1.20) satifies the following condition:

$$\|f - Ku^{(n+1)}\|_{\ell_2} = \min_{v \in u^{(n)} + \text{span}(p^{(n)})} \|f - Kv\|_{\ell_2}.$$

Hint: the function $q(\alpha) = \|f - K[u^{(n)} + \alpha p^{(n)}]\|_{\ell_2}^2$ is a quadratic function in α.