TUTORIAL

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T V Monday, 20 November 2006 (Time: 13:45 - 15:15, Room: P 004)

1.5 FEM for BVPs of Second-order ODEs

25 Let

$$\hat{K} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \hat{M} = \begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that

$$\frac{1}{6}\hat{D} \le \hat{M} \quad \text{and} \quad \hat{K} \le 2\hat{D}.$$

26 Consider the one-dimensional boundary value problem

$$\begin{aligned} -u''(x) &= f(x) \quad x \in (0,1), \\ u(0) &= g_0, \\ u'(1) &= g_1. \end{aligned}$$

Let K_h be the stiffness matrix obtained by the finite element method using the Courant elements on a subdivision $0 = x_0 < x_1 < \ldots < x_{N_h} = 1$.

Show that

$$\frac{\min_k h_k^2}{6c_F^2} D_h \le K_h \le 2D_h,$$

where $D_h = \text{diag}(K_h)$, c_F is the constant arising from Friedrichs' inequality, and $h_k = x_k - x_{k-1}$.

Hint: Use

$$(D_h \underline{v}_h, \underline{v}_h)_{\ell_2} = D_h^{(1)} v_1^2 + \sum_{k=2}^{N_h} \left(D_h^{(k)} \begin{pmatrix} v_{k-1} \\ v_k \end{pmatrix}, \begin{pmatrix} v_{k-1} \\ v_k \end{pmatrix} \right)_{\ell_2}$$

with

$$D_h^{(1)} = K_h^{(1)} = \frac{1}{h_1} \quad \text{and} \quad D_h^{(h)} = \text{diag}(K_h^{(k)}) = \frac{1}{h_k}\text{diag}(\hat{K}) = \frac{1}{h_k}\hat{D}.$$

27 Consider the variational problem: find $u \in V_g = V_0 = H_0^1(0,1)$:

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \quad \forall v \in V_0.$$

Solve this variational problem using the Galerkin method considering the finite element space

$$V_{0h} = V_{0n} = \text{span}\{x(1-x), x^2(1-x), \ldots, x^{n-1}(1-x)\}.$$

Let $f(x) = \cos(k\pi x)$, $k = l + 1$, and l is the last digit from your study code (or your age). Compute the stiffness matrix K_h analytically and solve the linear system $K_h \underline{u}_h = \underline{f}_h$ numerically using the Gauss method. Consider $n = 2, 4, 8, 10, 50, 100$.

$\boxed{28}$ Let M_h be the mass matrix obtained by the finite element method using the Courant elements on a uniform mesh $0 = x_0 < x_1 < \ldots < x_{N_h} = 1$, $h_k = x_k - x_{k-1} = h$, $\forall k$. Show that the condition number $\kappa(M_h) = \text{cond}_2(M_h) = O(1)$.

$\boxed{29^*}$ Let K_h be the stiffness matrix obtained by the finite element method using the Courant elements on a uniform mesh $0 = x_0 < x_1 < \ldots < x_{N_h} = 1$, $h_k = x_k - x_{k-1} = h$, $\forall k$. Show that the estimate on the condition number $\kappa(K_h)$ given in the course is sharp, i.e.

$$\kappa(K_h) = O(h^{-2}).$$

Hint: considering a special choice of $\underline{v}_h \in \mathbb{R}^{N_h}$, show that

$$\lambda_{min}(K_h) = \min_{\underline{v}_h} \frac{(K_h \underline{v}_h, \underline{v}_h)}{(\underline{v}_h, \underline{v}_h)} \leq Ch \quad \text{and} \quad \lambda_{max}(K_h) = \max_{\underline{v}_h} \frac{(K_h \underline{v}_h, \underline{v}_h)}{(\underline{v}_h, \underline{v}_h)} \geq Ch^{-1}.$$