T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T II Monday, 23 October 2006 (Time: 10:15 - 11:45, Room: T 041)

1.2 BVPs for Second-order PDEs

07 Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$. Find the variational formulation $(V = ?, V_0 = ?, V_g = ?, a(.,.) = ?, \langle F, . \rangle = ?)$ for the following boundary value problem:

$$\begin{aligned} -\text{div}(a(x)\nabla u(x)) &= f(x), \quad x \in \Omega, && (1.8)\\ u(x) &= g_D(x), \quad x \in \Gamma_D, && (1.9)\\ \frac{\partial u}{\partial n}(x) &= g_N(x), \quad x \in \Gamma_N, && (1.10)\\ \frac{\partial u}{\partial n}(x) &= \alpha(x)(g_R(x) - u(x)), \quad x \in \Gamma_R. && (1.11) \end{aligned}$$

08 Show that the variational problem associated to (1.8)-(1.11) has a unique solution provided that the following conditions imposed on the data are fulfilled:

1. $a \in L_\infty(\Omega) : 0 < a_1 = const \leq a(x) \leq a_2 = const$ for almost all $x \in \Omega$,
2. $f \in L_2(\Omega)$,
3. $g_D = \gamma_D g := g_{|\Gamma_D}$ with a given function g from $H^1(\Omega)$,
4. $g_N \in L_2(\Gamma_N)$, $g_R \in L_2(\Gamma_R)$,
5. $\alpha \in L_\infty(\Gamma_R)$: $0 \leq a(x) \leq \alpha_2 = const$ for almost all $x \in \Gamma_R$.

09 Due to the Corollary 1.8, the solution of the variational problem of the BVP (1.8)-(1.11) can be approximated by the fixed point iteration (18)=(19) given in the lectures. Give the classical formulation of this fixed point iteration for the variational problem derived in Exercise **07** !

10 Show that the variational problem: find $u \in V = V_g = V_0 = H^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \left(\int_\Omega u(x)\, dx\right)\left(\int_\Omega v(x)\, dx\right) = \int_\Omega f(x)v(x)\, dx, \quad \forall v \in V, \quad (1.12)$$

has a unique solution for given $f \in L_2(\Omega)$! Furthermore, prove that if the right-hand side fulfills the solvability condition

$$\langle F, c \rangle := \int_\Omega f(x)c\, dx = 0, \quad \forall c \in \mathbb{R}, \quad (1.13)$$

the solution u of (1.12) solves also the Neumann problem:

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad \partial u / \partial n = 0 \text{ on } \Gamma, \tag{1.14}$$

and it satisfies the orthogonality condition

$$\int_\Omega u(x)\, dx = 0. \tag{1.15}$$

11 Let $\Omega = (0,1) \times (0,1)$ and $\Gamma_D = [0,1] \times \{0\}$. Show the Friedrichs inequality in a constructive way, i.e. determine a constant $c_F > 0$ such that

$$\|v\|_0 \le c_F\, |v|_1, \quad \forall v \in V_0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}. \tag{1.16}$$

12 Let $\widetilde{\Gamma} \subset \Gamma = \partial\Omega$ with $\text{meas}(\widetilde{\Gamma}) = \int_{\widetilde{\Gamma}} ds > 0$. Show the equivalence of the norm

$$\|v\|_1^* := \left(\int_{\widetilde{\Gamma}} (v(x))^2 ds + |v|_1^2 \right)^{1/2} \tag{1.17}$$

with the standard norm $\|v\|_1$ in $H^1(\Omega)$!
Hints: Use Sobolev's norm equivalence theorem !