T U T O R I A L

"Numerical Methods for Solving Partial Differential Equations"

to the Lectures on NuPDE

T I Monday, 16 October 2006 (Time: 10:15 - 11:45, Room: T 041)

1 Elliptic Differential Equations

1.1 BVPs for Second-order ODEs

01 Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(.,.) = ?, \langle F, . \rangle = ?$) for the following boundary value problems:

(a) $$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u(0) &= g_0, \\ u(1) &= g_1. \end{aligned}$$

(b) $$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u'(0) &= g_0, \\ -u'(1) &= g_1. \end{aligned}$$

(c) $$\begin{aligned} -u''(x) &= f(x), \quad x \in (0,1), \\ u'(0) &= \alpha_0 u(0) - \beta_0 g_0, \\ -u'(1) &= \alpha_1 u(1) - \beta_1 g_1. \end{aligned}$$

02 Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(.,.) = ?, \langle F, . \rangle = ?$) for the following boundary value problem:

$$-\bar{a}(x)u''(x) + \bar{b}(x)u'(x) + \bar{c}(x)u(x) = f(x) + \delta(x-y) \quad x \in (0,1), \tag{1.1}$$
$$u'(0) = \alpha_0(u(0) - g_0), \tag{1.2}$$
$$-u'(1) = \alpha_1(u(1) - g_1), \tag{1.3}$$

where $\bar{a}(.), \bar{b}(.), \bar{c}(.), f(x)$ are given sufficiently smooth functions and $\delta(.)$ denotes the delta-function with given $y \in (0,1)$, and $\alpha_0, \alpha_1, g_0, g_1 \in \mathbb{R}$ are also given.
*Hint: Rewrite first the differential operator of the differential equation (**??**) in the so-called divergence form $-(a(x)u'(x))' + b(x)u'(x) + c(x)u(x)$!*

03 Show for the variational formulation of the Neumann boundary value problem (b) of Exercise **01** :

(a) If u is a solution, then, for any constant $c \in \mathbb{R}$, $u + c$ is also a solution !

(b) If the boundary value problem has a solution u, then

$$\langle F, c \rangle = 0, \quad \forall c \in \mathbb{R}. \tag{1.4}$$

04 Show that the variational problem: find $u \in V = V_g = V_0 = H^1(0,1)$ such that

$$\int_0^1 u'(x)v'(x)\,dx + (\int_0^1 u(x)\,dx)(\int_0^1 v(x)\,dx) = \int_0^1 f(x)v(x)\,dx =: \langle F, v \rangle, \quad (1.5)$$

for all $v \in V$, has a unique solution for given $f(.) \in L_2(0,1)$! Furthermore, prove that if the right-hand side fulfills the solvability condition (**??**) of the Neumann problem, the solution u of (**??**) also solves the Neumann problem (b) of Exercise **01** with $g_0 = g_1 = 0$, and satisfies the orthogonality condition

$$\int_0^1 u(x)\,dx = 0. \quad (1.6)$$

Hints: Use the Lax-Milgram Theorem and Poincare's inequality !

05 Show that the variational problem: find $u \in V = V_g = V_0 = H^1(0,1)$ such that

$$\int_0^1 (a(x)u'(x)v'(x) \; + b(x)u'(x)v(x) + c(x)u(x)v(x))dx = \int_0^1 f(x)v(x)\,dx, \quad (1.7)$$

for all $v \in V$, has a unique solution provided that the following assumptions are fulfilled:

1. $a(.) \in L_\infty(0,1)$ and $a(x) \geq a_0 = const > 0$ for almost all $x \in (0,1)$,
2. $b(x) \equiv b = const \in \mathbb{R}$,
3. $c(.) \in L_\infty(0,1)$ and $c(x) \geq 0$ for almost all $x \in (0,1)$,
4. $f(.) \in L_2(0,1)$.

Hint: Use the identity $u'u = \frac{1}{2}(u^2)'$ *!*

06 Due to the Corollary 1.8, the solution of the variational problem (**??**) can be approximated by the fixed point iteration (18)=(19) given in the lectures. Give the classical formulation of this fixed point iteration for the variational problem (**??**) !

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2 Elliptic Differential Equations

2.1 BVPs for Second-order ODEs

01 Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(.,.) = ?, \langle F, . \rangle = ?$) for the following boundary value problems:

(a) $$-u''(x) = f(x), \quad x \in (0,1),$$
$$u(0) = g_0,$$
$$u(1) = g_1.$$

(b) $$-u''(x) = f(x), \quad x \in (0,1),$$
$$u'(0) = g_0,$$
$$-u'(1) = g_1.$$

(c) $$-u''(x) = f(x), \quad x \in (0,1),$$
$$u'(0) = \alpha_0 u(0) - \beta_0 g_0,$$
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02 Find the variational formulations ($V = ?, V_0 = ?, V_g = ?, a(.,.) = ?, \langle F, . \rangle = ?$) for the following boundary value problem:

$$-\bar{a}(x)u''(x) + \bar{b}(x)u'(x) + \bar{c}(x)u(x) = f(x) + \delta(x-y) \quad x \in (0,1), \tag{2.1}$$
$$u'(0) = \alpha_0(u(0) - g_0), \tag{2.2}$$
$$-u'(1) = \alpha_1(u(1) - g_1), \tag{2.3}$$

where $\bar{a}(.), \bar{b}(.), \bar{c}(.), f(x)$ are given sufficiently smooth functions and $\delta(.)$ denotes the delta-function with given $y \in (0,1)$, and $\alpha_0, \alpha_1, g_0, g_1 \in \mathbb{R}$ are also given.
*Hint: Rewrite first the differential operator of the differential equation (**??**) in the so-called divergence form $-(a(x)u'(x))' + b(x)u'(x) + c(x)u(x)$!*

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04 Show that the variational problem: find $u \in V = V_g = V_0 = H^1(0,1)$ such that

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for all $v \in V$, has a unique solution for given $f(.) \in L_2(0,1)$! Furthermore, prove that if the right-hand side fulfills the solvability condition (**??**) of the Neumann problem, the solution u of (**??**) also solves the Neumann problem (b) of Exercise 01 with $g_0 = g_1 = 0$, and satisfies the orthogonality condition

$$\int_0^1 u(x)\,dx = 0. \quad (2.6)$$

Hints: Use the Lax-Milgram Theorem and Poincare's inequality !

05 Show that the variational problem: find $u \in V = V_g = V_0 = H^1(0,1)$ such that

$$\int_0^1 (a(x)u'(x)v'(x) \; + b(x)u'(x)v(x) + c(x)u(x)v(x))dx = \int_0^1 f(x)v(x)\,dx, \quad (2.7)$$

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