11 Let $\Omega = (0,1)\times(0,1)$ and $\Gamma_D = [0,1]\times\{0\}$. Show the Friedrichs inequality in a constructive way, i.e. determine a constant $c_F > 0$ such that

$$\|v\|_0 \le c_F\,|v|_1, \quad \forall v \in V_0 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}. \tag{1.16}$$

Solution We consider a function $v \in C^\infty(\Omega)$ such that $v = 0$ on Γ_D.

Then, we have

$$|v(x,y)| = \left|\int_0^y \frac{d}{dy}v(x,t)dt\right| \le |y|^{\frac{1}{2}}\left|\int_0^y \left(\frac{\partial}{\partial y}v(x,t)\right)^2 dt\right|^{\frac{1}{2}}.$$

Then,

$$\begin{aligned}\|v\|_0 = \int_\Omega v^2(x,y)d\Omega &\le \int_\Omega |y|\int_0^1\left(\frac{\partial}{\partial y}v(x,t)\right)^2 dt\ d\Omega\\ &\le \int_0^1\int_0^1 |y|dy\int_0^1 |\nabla v(x,t)|^2 dtdx\\ &\le \frac{1}{2}\int_\Omega |\nabla v(x,t)|^2 dxdt = \frac{1}{2}\|\nabla v\|_0^2\,.\end{aligned}$$

Thus,

$$\|v\|_0 \le \frac{1}{\sqrt{2}}|v|_1 \qquad \text{and } c_P = \frac{1}{\sqrt{2}}.$$

12 Let $\widetilde{\Gamma} \subset \Gamma = \partial\Omega$ with $\text{meas}(\widetilde{\Gamma}) = \int_{\widetilde{\Gamma}} ds > 0$. Show the equivalence of the norm

$$\|v\|_1^* := \left(\int_{\widetilde{\Gamma}}(v(x))^2 ds + |v|_1^2\right)^{1/2} \tag{1.17}$$

with the standard norm $\|v\|_1$ in $H^1(\Omega)$!

Hints: Use Sobolev's norm equivalence theorem !

Solution Let us consider

$$f(v) := \left(\int_{\tilde{\Gamma}} |v|^2 ds\right)^{\frac{1}{2}} \qquad \forall v \in H^1(\Omega)\ (= W_2^1(\Omega)\ ).$$

$f(\cdot) : H^1(\Omega) \to [0,+\infty)$ is a seminorm and it satisfies:

a) for all $v \in H^1(\Omega)$

$$0 \le f(v) \le \left(\int_\Gamma |v|^2 ds\right)^{\frac{1}{2}} = \|v\|_{L^2(\Gamma)} \le C\|v\|_1 \quad \text{with } C > 0.$$

b) let $v \in \mathbb{P}_0$, i.e. $v = c$ (constant). Then,

$$f(c) = \left(\int_{\tilde{\Gamma}} |c|^2\right)^{\frac{1}{2}} = |c|(meas(\tilde{\Gamma}))^{\frac{1}{2}} = 0 \qquad \Leftrightarrow \qquad c = 0\,.$$

Thus, $f(\cdot)$ satisfies the hypotheses of the Sobolev's norm equivalence theorem, so that there exist two positive constants $\underline{c}, \overline{c} > 0$ such that

$$\begin{aligned}\underline{c}\left(f^2(v) + |v|_1^2\right)^{\frac{1}{2}} \le \|v\|_1 \le \overline{c}\left(f^2(v) + |v|_1^2\right)^{\frac{1}{2}} &\qquad \forall v \in H^1(\Omega)\\ \Leftrightarrow \qquad \underline{c}\|v\|_1^* \le \|v\|_1 \le \overline{c}\|v\|_1^* &\qquad \forall v \in H^1(\Omega)\,.\end{aligned}$$