2.4. Full Discretization of Parabolic IBVP

Space-Time Discretization:

(6) **LVF of parabolic IBVP:** $V_0 \subset H^1(\Omega)$, $H = L_2(\Omega)$

Find $u \in H^1((0,T), V; H)$:

$$\frac{d}{dt}(u(t), v)_H + a(u(t), v) = \langle F(t), v \rangle \quad \forall v \in V \ \forall t \in (0,T)$$

IC: $u(0) = u_0$ in H, i.e. $(u(0), v)_H = (u_0, v)_H \quad \forall v \in V \subset H$

VML $\Downarrow$ Space discretization by the VML+FEM (lin. El.)

- $V_h = V_{0h} = \operatorname{span}\{\varphi_i : i = \overline{1, N_h}\} \subset V = V_0$ – FE-subspace
- $u_h(x,t) = \sum_{k=1}^{N_h} u_k(t)\varphi_k(x) \longleftrightarrow \underline{u}_h(t) \in [H^1(0,T)]^{N_h}$

$(6)_h$ Find $\underline{u}_h(t) = [u_k(t)]_{k=\overline{1,N_h}} \in [H^1(0,T)]^{N_h}$:

$$M_h \underline{u}_h'(t) + K_h(t)\underline{u}_h(t) = \underline{f}_h(t) \quad \forall t \in (0,T)$$

IC: $M_h \underline{u}_h(0) = \underline{g}_h := [(u_0, \varphi_i)_H]_{i=\overline{1,N_h}}$

with $M_h = [(\varphi_k, \varphi_i)_H]$, $K_h = [a(\varphi_k, \varphi_i)]$, $\underline{f}_h = [\langle F(t), \varphi_i \rangle]$

$\Downarrow$ Time discretization by RKF
e.g. θ-method (A-stable for $\theta \in [\frac{1}{2}, 1)$)

(52) Find $\underline{u}_h^{j+1} \in \mathbb{R}^{N_h}$ $(\underline{u}_h^{j+1} \approx \underline{u}_h(t_{j+1}))$:

$$[\tilde{M}_h + \tau\theta K_h]\underline{u}_h^{j+1} = [M_h + \tau(1-\theta)K_h]\underline{u}_h^j + \tau[(1-\theta)\underline{f}_h(t_j) + \theta \underline{f}_h(t_{j+1})]$$

$j = 0, 1, 2, \ldots, M-1$

IC: $\underline{u}_h^0 \in \mathbb{R}^{N_h}$: $M_h \underline{u}_h^0 = \underline{g}_h$

$\Downarrow$

$$u_h^j(x) = \sum_{k=1}^{N_h} u_k^j \varphi_k(x) \approx u_h(x, t_j) = \sum_{k=1}^{N_h} u_k(t_j)\varphi_k(x) \approx u(x, t_j)$$

Error: $\|u_h^j - u(t_j)\|_H = \begin{cases} O(\tau + h^2) & \text{for } \theta \in (\frac{1}{2}, 1] \\ O(\tau^2 + h^2) & \text{for } \theta = 1/2 \end{cases}$

see Nu III for the proof $\Big[$ $\|\cdots - \cdots\|_H = O(\tau + h^2)$ for $\theta \in [0, 1/2)$ if $\tau \leq 2/(\lambda_{max}(M_h^{-1}K_h)(1-\theta))$