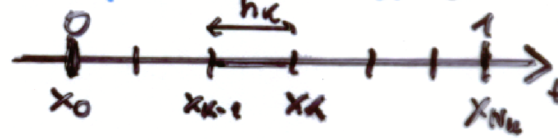
• **Example 2.15:** Our model problem (5) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f, \ldots$

VLM with linear elements:



Then $K_h = K_h^T > 0$ (SPD)

(28)
$$\boxed{\frac{3}{h^2} = \frac{3}{\max_k h_k^2} \le \lambda_{max}(M_h^{-1}K_h) \le \frac{12}{\max_k h_k^2} = \frac{12}{h^2}}$$

for a uniform grid: $h = h_k \ \forall k$

Proof: mms, but see also Subsection 1.4.2. □

$\lambda_{max}(M_h^{-1}K_h) \le \, ?$:

$$(K_h \underline{v}_h, \underline{v}_h) = \sum_{k=1}^{N_h} \Big(K_h^{(k)} \underbrace{\begin{pmatrix} v_{k-1} \\ v_k \end{pmatrix}}_{\underline{v}^{(k)}}, \begin{pmatrix} v_{k-1} \\ v_k \end{pmatrix}\Big) = \quad K_h^{(k)} = \frac{1}{h_k}\hat{K}$$

$$= \sum_{k=1}^{N_h} \frac{1}{h_k}(\hat{K}\underline{v}^{(k)}, \underline{v}^{(k)}) \le 12 \sum_{k=1}^{N_h} \frac{1}{h_k}(\hat{M}\underline{v}^{(k)}, \underline{v}^{(k)})$$

$$\hat{K}\underline{\varphi} = \lambda \hat{M} \underline{\varphi}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{\varphi} = \frac{\lambda}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \underline{\varphi} \quad \Downarrow \quad \lambda_{max}(\hat{M}^{-1}\hat{K}) = 12$$

$$\le \frac{12}{\min_k h_k^2} \sum_{k=1}^{N_h} \frac{1}{h_k}(\hat{M}\underline{v}^{(k)}, \underline{v}^{(k)}) = \frac{12}{\min_k h_k^2} \sum_{k=1}^{N_h} (M_h^{(k)}\underline{v}^{(k)}, \underline{v}^{(k)})$$

$$= \frac{12}{\min_k h_k^2}(M_h \underline{v}_h, \underline{v}_h), \text{ i.e. } \lambda_{max}(M_h^{-1}K_h) \le \frac{12}{\min_k h_k^2}.$$

$\lambda_{max}(M_h^{-1}K_h) \ge \, ?$ We choose now $\underline{v}_h = \underline{e}_i = (0,\ldots,0,1,0,\ldots,0)^T$

$$\lambda_{max}(M_h^{-1}K_h) = \max_{\underline{v}_h} \frac{(K_h \underline{v}_h, \underline{v}_h)}{(M_h \underline{v}_h, \underline{v}_h)} \ge \frac{(K_h \underline{e}_i, \underline{e}_i)}{(M_h \underline{e}_i, \underline{e}_i)}$$

$$= \frac{1/h_i + 1/h_{i+1}}{1/3\,(h_i + h_{i+1})} = \frac{3}{h_i h_{i+1}} \ge \frac{3}{\max_k h_k^2}.$$

q.e.d

This implies, e.g., for an equidistant subdivision

$$L = O(h^{-2}) \xrightarrow[h \to 0]{} \infty \quad \Downarrow \quad e^{LT} = e^{ch^{-2}T} \xrightarrow[h\to 0]{} \infty.$$

Therefore, the stability bound $c_s = e^{LT}$ is completely useless!