• Theorem 2.10:

Ass.: Assume that the bilinear form $a: V \times V \to \mathbb{R}$ is V-elliptic and V-bounded, i.e. $\exists \mu_2 \geq \mu_1 > 0$:

$$a(v,v) \geq \mu_1 \|v\|_V^2 \quad \forall v \in V, \text{ and}$$

$$|a(w,v)| \leq \mu_2 \|v\|_V \|w\|_V \quad \forall w, v \in V.$$

St.: Then we have:

$$(13) \quad \|\underset{(6)_h}{u_h(t)} - \underset{(6)}{u(t)}\|_H \leq e^{-\mu_1 t/c^2} \|u_{0h} - R_h u_0\|_H + \|(I - R_h)u(t)\|_H + \int_0^t \|[(I - R_h)u(s)]'\|_H \, e^{-\mu_1 (t-s)/c^2} ds,$$

where the constant c: $\|v\|_H \leq c \|v\|_V \; \forall v \in V$.

Proof:

▸ First, we divide the discretization error into two parts:

$$u_h(t) - u(t) = \underbrace{u_h(t) - R_h u(t)}_{= \theta_h(t) \in V_h} + \underbrace{R_h u(t) - u(t)}_{= \rho_h(t) \in V \subset H} = \theta_h(t) + \rho_h(t)$$

can be estimated by the approximation error using Cea's Lemma $\begin{cases} V \\ H \end{cases}$

▸ For the first part $\theta_h(t)$, it follows:

$$\begin{array}{rl} - & \langle u'(t), v_h \rangle + a(u(t), v_h) = \langle F(t), v_h \rangle \quad \forall v_h \in V_h \subset V \\ & \qquad\qquad\qquad \overset{(11)}{\|} \\ & \qquad\qquad\quad a(R_h u(t), v_h) \\ + & \langle u_h'(t), v_h \rangle + a(u_h(t), v_h) = \langle F(t), v_h \rangle \quad \forall v_h \in V_h \\ \hline \end{array}$$

$$\langle \underbrace{u_h'(t) - u'(t)}_{= \theta_h'(t) + \rho_h'(t)}, v_h \rangle + a(\underbrace{u_h(t) - R_h u(t)}_{= \theta_h(t)}, v_h) = 0 \quad \forall v_h \in V_h$$

$$\Rightarrow (14) \quad \left[\langle \theta_h'(t), v_h \rangle + a(\theta_h(t), v_h) = -\langle \rho_h'(t), v_h \rangle \quad \forall v_h \in V_h \right.$$