Lemma 2.51 = uniqueness = !

Ass.: Assume that there is a constant $\mu_1 \geq 0$:

$$\mu_1 \|v\|_V^2 \leq a(v,v) \quad \forall v \in V.$$

St.: Then there exists at most one solution of the initial value problem $(6)_{LVF}$, i.e.

$$(6)_{LVF} \quad \begin{cases} \langle u'(t), v\rangle + a(u(t),v) = \langle F(t), v\rangle \quad \forall v \in V \ \dot{\forall} t \in [0,T], \\ u(0) = u_0 \quad \text{in } H \end{cases}$$

Proof: Assume that $u_1(t)$ and $u_2(t)$ are solutions of the IVP $(6)_{LVF}$. Then

$$u(t) = u_2(t) - u(t)$$

is a solution of the IVP

$$\begin{cases} \langle u'(t), v\rangle + a(u(t),v) = 0 \quad \forall v \in V \\ u(0) = 0 \end{cases}$$

Now we have

$$\frac{d}{dt}\left[\frac{1}{2}\|u(t)\|_H^2\right] \overset{!}{=} \langle u'(t), u(t)\rangle = -a(u(t),u(t))$$

$$\leq -\mu_1 \|u(t)\|_V^2 \leq -\mu_1 c^{-2} \|u(t)\|_H^2$$

$$\frac{d}{dt}\|u(t)\|_H + \mu_1 c^{-2}\|u(t)\|_H \leq 0$$

$\int_0^t$

$$\frac{d}{dt}\left(e^{\mu_1 t/c^2}\|u(t)\|_H\right) \leq 0$$

$$\|u(t)\|_H \leq e^{-\mu_1 t/c^2}\|u(0)\|_H = 0,$$

i.e. $u(t) = 0 \ \dot{\forall} t \in [0,T]$.

This is also true for $\mu_1 = 0$!

q.e.d.