Friedrichs-type Inequalities:

Let us consider the following setting:
\[ \Gamma_1 \subset \Gamma = \partial \Omega : \text{meas}_{d-1}(\Gamma_1) = \text{meas}_{d-1}(\Gamma_1) := \int_{\Gamma_1} ds > 0, \]
\[ V_0 = \{ v \in W^1_p(\Omega) : \gamma_{\partial \Omega} v = v|_{\Gamma_1} = 0 \} \subset W^1_p(\Omega), \]
\[ V_0 = W^1_p(\Omega) \text{ if } \Gamma_1 = \Gamma, \quad \Gamma_1 \cap \partial \Omega = \emptyset, \]
\[ 1 \leq p < \infty \]

Lemma 2.15:
Ass.: \[ 1 \leq p < \infty; \Gamma_1 \subset \Gamma : \text{meas}_{d-1}(\Gamma_1) = |\Gamma_1| > 0. \]
St.: Then there exists a positive constant \( \bar{c} = \text{const} > 0 \) such that
\[ (M) \quad \frac{1}{\Omega} \int_{\Omega} \| u \|_p^p dx \leq \bar{c} \frac{1}{\Omega} \int_{\Omega} \| \nabla u \|_p^p dx \quad \forall u \in V_0. \]

In the case \( \Gamma_1 = \Gamma \), inequality \((M)\) is also called Friedrichs' inequality: \( c_F = \bar{c} \).

Proof: Using Sobolev's norm equivalence Theorem 2.13, we first show that
\[ \| u \|_{W^1_p(\Omega)} = (\int_{\Gamma_1} (u_{\Gamma_1}^p + |u|_{W^1_p(\Omega)}^p)\, ds)^{1/p} \leq \| u \|_{W^1_p(\Omega)} \quad \text{in } W^1_p(\Omega) \]
with \( f_1(u) := \left( \frac{1}{\Gamma_1} \int_{\Gamma_1} |u|^p ds \right)^{1/p} \).

Indeed, \( f_1(\cdot) \) fulfills the assumptions of Th.2.13:
1) \( f_1(\cdot) : W^1_p(\Omega) \to \mathbb{R}_+ := [0, \infty) \) is a semi-norm: (mm)
2) \[ 0 \leq f_1(u) = \left( \frac{1}{\Gamma_1} \int_{\Gamma_1} |u|^p ds \right)^{1/p} \leq \left( \frac{1}{\Gamma_1} \int_{\Gamma_1} |u|^p ds \right)^{1/p} \]
\[ = \| u \|_{L_p(\Gamma)} \leq c \| u \|_{W^1_p(\Omega)}. \]
\[(9)_{\bar{c}}\]