- **Der Raum $H^1(\Omega) \equiv W_2^1(\Omega)$:** (Sobolev-Räume)

$$H^1(\Omega) := \{u \in L_2(\Omega) : \exists \tfrac{\partial u}{\partial x_i} \in L_2(\Omega), i=\overline{1,d}\}$$

$$\|u\|^2_{H^1(\Omega)} \equiv \|u\|^2_{1,\Omega} := \|u\|^2_{0,\Omega} + |u|^2_{1,\Omega} = \int_\Omega (|u|^2 + |\nabla u|^2)\,dx,$$

$$(u,v)_{H^1(\Omega)} \equiv (u,v)_{1,\Omega} := \int_\Omega \{uv + \nabla^T u \nabla v\}\,dx,$$

$$\nabla u = \left(\tfrac{\partial u}{\partial x_1},\dots,\tfrac{\partial u}{\partial x_d}\right)^T, \quad |\nabla u|^2 = \left(\tfrac{\partial u}{\partial x_1}\right)^2 + \dots + \left(\tfrac{\partial u}{\partial x_d}\right)^2,$$

Cauchy-Ungl.: $|(u,v)_1| \le \|u\|_1 \|v\|_1$ [illegible]

- **Die Sobolev-Räume $H^k(\Omega) \equiv W_2^k(\Omega)$:**

$$H^k(\Omega) := \{u \in L_2(\Omega) : \exists \partial^\alpha u \in L_2(\Omega)\ \forall |\alpha| \le k\}$$

$$\|u\|_{H^k(\Omega)} = \|u\|_{k,\Omega} \equiv \|u\|_k := (u,u)_k^{0.5}$$

$$(u,v)_k \equiv (u,v)_{k,\Omega} \equiv (u,v)_{H^k(\Omega)} := \sum_{|\alpha| \le k} \int_\Omega \partial^\alpha u\, \partial^\alpha v\, dx$$

- **Formeln der partiellen Integration:** $|\vec{n}| = 1$
  - Grundformel: $\vec{n} = \vec{n}(x) = (n_1(x),\dots,n_d(x))^T$

(2) $$\int_\Omega \frac{\partial w}{\partial x_i}\,dx = \int_{\partial\Omega} w(x)\, n_i(x)\, ds_x$$

- **Folgerungen:**

a) Formel der partiellen Integration: $w = u \cdot v$

(2)a) $$\boxed{\int_\Omega \frac{\partial u}{\partial x_i} \cdot v\,dx = -\int_\Omega u \frac{\partial v}{\partial x_i}\,dx + \int_{\partial\Omega} u \cdot v \cdot n_i\,ds_x}$$

b) Gauß: $w = w_i$, Σ $\left(\operatorname{div} \vec{w} = \sum_{i=1}^d \frac{\partial w_i}{\partial x_i}\right)$

(2)b) $$\int_\Omega \operatorname{div} \vec{w}\,dx = \int_{\partial\Omega} \vec{w}^T \vec{n}\,ds_x$$

c) Green: $u \mapsto \frac{\partial u}{\partial x_i}$, Σ in a)

(2)c) $$\int_\Omega \Delta u \cdot v\,dx = -\sum_{i=1}^d \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\,dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v\,ds_x \quad 1.$$

$$= \int_\Omega u \cdot \Delta v\,dx - \int_{\partial\Omega} u \frac{\partial v}{\partial n}\,ds_x + \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,ds_x \quad 2.$$