Beweis für den expliziten Euler ($p=1$):

- Für den lokalen Fehler gilt (↑):

$$\max_{j=0,m} |d_\tau(t_j)| \le \underbrace{\frac{1}{2} \max_{t\in[0,T]} |u''(t)|}_{=:\, c_K(u)} \tau^2$$

mit $d_\tau(t_j) = u(t_j) - (u(t_{j-1}) + \tau f(t_{j-1}, u(t_{j-1})))$

- Dann gilt für den globalen Fehler $e_j = u(t_j) - u_j$

$$|u(t_j)-u_j| = |u(t_j) - (u_{j-1} + \tau f(t_{j-1}, u_{j-1}))|$$

$$= |u(t_j) - (u(t_{j-1}) + \tau f(t_{j-1}, u(t_{j-1}))) + (u(t_{j-1}) + \tau f(t_{j-1}, u(t_{j-1}))) - (u_{j-1} + \tau f(t_{j-1}, u_{j-1}))|$$

$$\le |u(t_j) - (u(t_{j-1}) + \tau f(t_{j-1}, u(t_{j-1})))| + |u(t_{j-1}) - u_{j-1}| + \tau |f(t_{j-1}, u(t_{j-1})) - f(t_{j-1}, u_{j-1})|$$

$$\le |d_\tau(t_j)| + |u(t_{j-1}) - u_{j-1}| + \tau L |u(t_{j-1}) - u_{j-1}|$$

$$\le c_K(u)\tau^2 + (1+L\tau) |u(t_{j-1}) - u_{j-1}|$$

$$\le c_K\tau^2 + (1+L\tau)\left(c_K\tau^2 + (1+L\tau)|u(t_{j-2}) - u_{j-2}|\right)$$

$$\le \ldots \le c_K\tau^2 \sum_{i=0}^{j-1} (1+L\tau)^i + (1+L\tau)^j |u(t_0) - u_0|$$

$1+L\tau \le e^{L\tau} = 1 + L\tau + \frac{1}{2}(L\tau)^2 + \ldots$

$$\le c_K\tau^2 \sum_{i=0}^{j-1} (e^{L\tau})^i + e^{L\tau j} |u(t_0) - u_0|$$

$$\underset{\uparrow}{\le} c_K\tau^2 \frac{e^{L\tau m} - 1}{e^{L\tau} - 1} + e^{LT} |u(t_0) - u_0| \le c\,\tau\, \frac{e^{LT}-1}{L} + e^{LT} |\ldots|$$

$j \le m$, $m\tau = T$

$e^{L\tau} - 1 \ge 1 + L\tau - 1 = L\tau$