Newton-Like Solver for Elastoplastic Problems with Hardening and its Local Superlinear Convergence

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Slanting Functions

Definition (X. Chen, Z. Nashed, L. Qi — 2000)

A function $F : X \to Y$ is said to be *slantly differentiable at* x if

1. $\exists\, F^o : X \to L(X,Y)$:
$$\lim_{h\to 0} \frac{\|F(x+h) - F(x) - F^o(x+h)\,h\|}{\|h\|} = 0\,.$$
2. F^o is uniformly bounded in an open neighbourhood of x.

$F^o(x)$ is said to be a *slanting function for* F *at* x.

Detail

Slanting Function for the max–Function

Example

$$F : \mathbb{R} \to \mathbb{R}, \; y \mapsto \max\{0, y\}$$

$$F^o(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 & \text{if } y > 0 \\ \gamma \in \mathbb{R} & \text{if } y = 0 \end{cases}$$

since

$$\lim_{h \to 0} \frac{|F(y+h) - F(y) - F^o(y+h)\,h|}{|h|} = 0$$

Slanting Function for the max–Function in Lebesgue Spaces

Theorem (M. Hintermüller, K. Ito, K. Kunisch — 2002)

Let

$$F : L_q(\Omega) \to L_p(\Omega)\,,\; y \mapsto \max\{0, y\}\,,$$

$$F^o(y) = \begin{cases} 0 & \textit{on } \{x \mid y(x) < 0 \quad a.e.\}\,, \\ 1 & \textit{on } \{x \mid y(x) > 0 \quad a.e.\}\,, \\ \gamma \in \mathbb{R} & \textit{else}\,. \end{cases}$$

Then F^o serves as a slanting function for F if $p < q$.
This is, in general, not true if $p \geq q$.

Detail

Newton-Like Method Utilizing Slanting Functions

Theorem (X. Chen, Z. Nashed, L. Qi — 2000)

Let $U \subseteq X$ *be an open subset,* $F : U \to Y$ *be a slantly differentiable function with a slanting function* $F^o : U \to L(X, Y)$*, and* $x^* \in U$ *be a solution to the nonlinear problem* $F(x) = 0$*.*
If $F^o(x)$ *is non-singular for all* $x \in U$ *and* $\{\|F^o(x)^{-1}\| : x \in U\}$ *is bounded, then the Newton-like iteration*

$$x^{j+1} = x^j - F^o(x^j)^{-1} F(x^j)$$

converges super-linearly to x^**, provided that* $\|x^0 - x^*\|$ *is sufficiently small.*

Elastoplasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma(x,t) = f(x,t) \quad \text{in } \Omega \times [0,T]$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Additive Splitting	$\varepsilon = e + p$
Hook's Law	$\sigma = \mathbb{C}\, e$
Plastic Flow	$\sigma : (q - \dot{p}) + D(\dot{p}) \leq D(q) \quad \forall q$
Boundary Conditions	$u = u_D \quad \text{on } \Gamma_D, \quad \sigma\, n = g \quad \text{on } \Gamma_N$
Initial Condition	$p(x,0) = p_0(x)$

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Minimization Problem

Discretizing in Time

$$[0,T] \rightsquigarrow \{0 = t_0\,,\ t_1\,,\ \ldots\,,\ t_{n-1}\,,\ t_n = T\}$$

$$u_k(x) := u(x,t_k),\ \text{etc.}$$

$$\dot{p}(x,t_k) \approx \frac{p_k - p_{k-1}}{t_k - t_{k-1}}$$

Primal Variables

$$u_k \in V := H^1(\Omega)^3 \qquad\qquad p_k \in Q := L_2(\Omega)^{3\times 3}_{\mathrm{sym}}$$

Minimization Problem

Problem (k–th time step, C. Carstensen — 1997)

Find $(u_k, p_k) \in V_D \times Q$ *such, that*

$$J_k(u_k, p_k) = \inf \{ J_k(v, q) \mid (v, q) \in V_D \times Q \}$$

where

$$J_k(v, q) = \frac{1}{2} \| \varepsilon(v) - q \|_{\mathbb{C}}^2 + \psi_k(q) - l_k(v)\,.$$

$$\langle q_1 \;,\; q_2 \rangle_{\mathbb{C}} = \int_\Omega \mathbb{C}\, q_1 : q_2 \,\mathrm{d}x \qquad \psi_k \ldots \text{ convex, not smooth}$$

$$\|q\|_{\mathbb{C}} = \langle q \;,\; q \rangle_{\mathbb{C}}^{1/2} \qquad l_k \ldots \text{ linear}$$

Minimization Problem

Problem (k–th time step)

Find $u_k \in V_D$ such, that

$$J_k(u_k) = \inf_{v \in V_D} J_k(v)\,,$$

where

$$J_k(v) = \frac{1}{2}\|\varepsilon(v) - p_k(\varepsilon(v))\|_{\mathbb{C}}^2 + \psi_k(p_k(\varepsilon(v))) - l_k(v)\,.$$

Properties: (shown by a theorem of J. J. Moreau — 1965)

J_k is strictly convex and differentiable with

$$\mathrm{D}\,J_k(v\,;\,w) = \langle \varepsilon(v) - p_k(\varepsilon(v))\,,\,\varepsilon(w)\rangle_{\mathbb{C}} - l_k(w)\,.$$

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Attempt to Differentiate J_k for the Second Time

Plastic Strain Minimizer (Alberty, Carstensen, Zarrabi – 1999)

$$p_k(\varepsilon(v)) = \max\{0, f_1(\varepsilon(v))\}\; f_2(\varepsilon(v)) + p_{k-1}\,,$$

$$f_1 : Q \to L_2(\Omega) \text{ smooth}\,, \qquad f_2 : Q \to Q \text{ smooth}\,.$$

▸ Detail

Derivatives of J_k

$$\begin{aligned} \mathrm{D}\, J_k(v\,;\,w_2) &= \langle \varepsilon(v) - p_k(\varepsilon(v))\,,\ \varepsilon(w_2)\rangle_{\mathbb{C}} - l_k(w_2) \\ \mathrm{D}^2\, J_k(v\,;\,w_1, w_2) &= \langle \varepsilon(w_1) - \mathrm{D}\, p_k(\varepsilon(v)\,;\,\varepsilon(w_1))\,,\ \varepsilon(w_2)\rangle_{\mathbb{C}} \end{aligned}$$

Slanting Functions in Elastoplasticity

Slanting Functions for D J_k

$$\mathrm{D}\, J_k(v\,;\,w_2) = \langle \varepsilon(v) - p_k(\varepsilon(v))\,,\, \varepsilon(w_2)\rangle_{\mathbb{C}} - l_k(w_2)$$

$$(\mathrm{D}\, J_k)^o\,(v\,;\,w_1, w_2) = \langle \varepsilon(w_1) - p_k^o(\varepsilon(v)\,;\,\varepsilon(w_1))\,,\, \varepsilon(w_2)\rangle_{\mathbb{C}}$$

Plastic Strain Minimizer

$$p_k(\varepsilon(v)) = \max\{0, f_1(\varepsilon(v))\}\; f_2(\varepsilon(v)) + p_{k-1}\,,$$

Slanting Functions for p_k

$\max : L_2(\Omega) \to L_2(\Omega)$	NOT OK ,
$\max : L_{2+\delta}(\Omega) \to L_2(\Omega) \quad (\delta > 0)$	OK ,
$\max : \mathbb{R} \to \mathbb{R}$ (discretized case)	OK .

Newton-Like Method applied to Elastoplasticity

Newton-Like Iterates

$$v_{j+1} = v_j - \left[(\mathrm{D}\, J_k)^o \, (v_j) \right]^{-1} \mathrm{D}\, J_k(v_j)$$

Theorem (here, hardening is required)

$\left[(\mathrm{D}\, J_k)^o \, (v) \right]^{-1}$ *exists and is uniformly bounded.*

Detail

Final Result

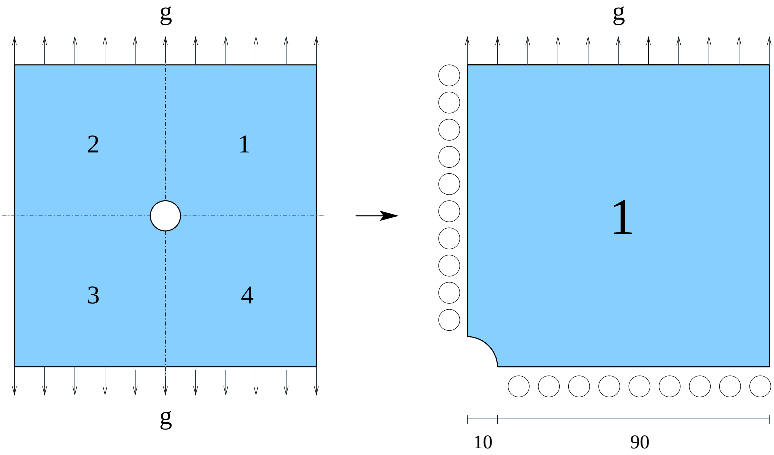
Theorem

Let J_k be defined as above. Then the sequence (v_j) defined by

$$v_{j+1} = v_j - \left[(\mathrm{D}\, J_k)^o\,(v_j)\right]^{-1} \mathrm{D}\, J_k(v_j)$$

converges super-linearly to the solution u_k of the elastoplastic problem, provided that $\|v_0 - u_k\|$ is small enough and that $f_1(\varepsilon(v_j)) \in L_{2+\delta}(\Omega)$ with $\delta > 0$.

Problem Description



Graphical Solution

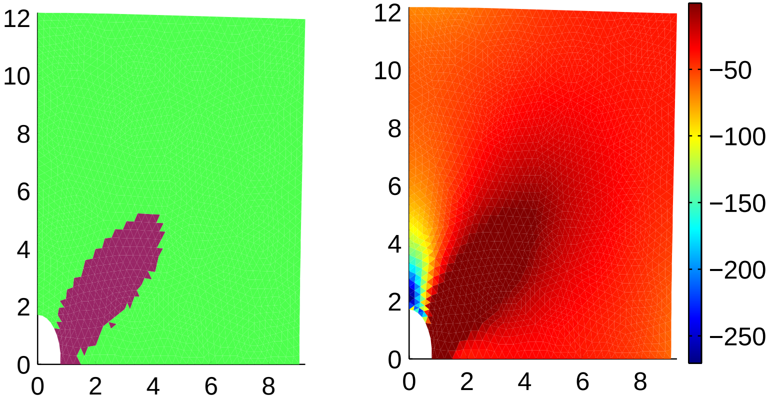


Figure: Left: Plastic zones colored pink Right: Stress field

Convergence Table – One Time Step

Level	0	...	3	4	5
DOF	245	...	14560	57920	231040
$\lvert u_j - u_{j-1} \rvert$:					
step 1	2.1826e-02	...	4.5238e-02	4.6300e-02	4.6603e-02
step 2	2.2225e-03	...	8.0839e-03	8.3886e-03	8.5454e-03
step 3	1.0478e-04	...	3.4440e-04	4.0032e-04	4.1602e-04
step 4	1.4404e-08	...	1.5206e-05	1.2050e-05	1.3944e-05
step 5	7.2634e-16	...	2.4947e-07	7.2972e-07	3.2631e-07
step 6		...	3.5062e-13	5.3972e-12	1.6473e-12
step 7		...		7.2441e-15	1.4518e-14
VPZ (%):					
step 0-1	4.889	...	7.042	7.116	7.129
step 1-2	1.333	...	5.444	5.549	5.546
step 2-3	0.8889	...	1.056	1.125	1.098
step 3-4	0	...	0.1597	0.1233	0.1215
step 4-5	0	...	0.01389	0.01042	0.008247
step 5-6		...	0	0	0
step 6-7		...		0	0
Time (sec.)	2	...	64	286	1195

The Main Contribution in Short

- Proof of local super-linear convergence concerning the one time step problem in the spatially discrete case.
- Determination of sufficient regularity assumptions in the spatially continuous case.

Future Investigations

- Sufficient conditions to guarantee $f_1(\varepsilon(v_j)) \in L_{2+\delta}(\Omega)$.

Slanting Functions

Definition (X. Chen, Z. Nashed, L. Qi — 2000)

Let X and Y be vector spaces, $U \subseteq X$ be an open subset and $x \in U$. A function $F : U \to Y$ is said to be *slantly differentiable at* x if

1. there exist mappings $F^o : U \to L(X, Y)$ and $r : X \to Y$ with $\lim_{h \to 0} \frac{||r(h)||}{||h||} = 0$ such, that
$$F(x + h) = F(x) + F^o(x + h)\, h + r(h)$$
holds for all $h \in X$ which satisfy $(x + h) \in U$,
2. F^o is uniformly bounded in an open neighbourhood of x, i. e.
$$\exists\, \delta > 0\,,\ C > 0\ :\quad \sup_{||h|| < \delta} ||F^o(x + h)|| < C\,.$$

We say, that $F^o(x)$ is a *slanting function for* F *at* x.

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Slanting Function for the max-Function in Lebesgue Spaces

Theorem (M. Hintermüller, K. Ito, K. Kunisch)

Let p and q in $\mathbb{R}$ be fixed arbitrarily such that $1 \leq p \leq q \leq +\infty$ is satisfied, and let F be a mapping of $L_q(\Omega)$ into $L_p(\Omega)$ defined as $F(y) := \max\{0, y\}$. Then there holds, that for γ fixed arbitrarily in $\mathbb{R}$, the function

$$F^o(y)(x) := \begin{cases} 0 & \text{for } y < 0 \text{ a.e.}, \\ 1 & \text{for } y > 0 \text{ a.e.}, \\ \gamma & \text{else}. \end{cases}$$

serves as a slanting function for F if $p < q$, but F^o does in general not serve as a slanting function for F if $p = q$.

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Attempt to Differentiate J_k for the Second Time

Explicit Structure

$$\begin{aligned} A_k &= \mathbb{C}\left(\varepsilon(v) - p_{k-1}\right), \\ f_1(\varepsilon(v)) &= \|\text{dev } A_k\|_F - \sigma_y\left(1 + H\alpha_{k-1}\right), \\ f_2(\varepsilon(v)) &= \frac{\text{dev } A_k}{\|\text{dev } A_k\|_F}, \\ p_k(\varepsilon(v)) &= \frac{1}{2\mu + \sigma_y^2 H^2} \max\{0, f_1(\varepsilon(v))\}\, f_2(\varepsilon(v)), \end{aligned}$$

where

$$H > 0\,, \quad \sigma_y > 0\,, \quad \mu > 0\,, \quad \alpha_{k-1} = \alpha_{k-2} + \sigma_y H \|p_{k-1} - p_{k-2}\|_F\,.$$

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Newton-Like Method applied to Elastoplasticity

Newton-Like Iterates

$$v_{j+1} = v_j - \left[\left(\mathrm{D}\, J_k \right)^o (v_j) \right]^{-1} \mathrm{D}\, J_k(v_j)$$

Notice: $\mathrm{D}\, J_k(v_j) \in {V_0}^*$ and $\left(\mathrm{D}\, J_k \right)^o (v_j) \in L(V_0, {V_0}^*)$.
Thus, we derive:

Newton-Like Iterates

Find v_{j+1} in V_D such, that

$$\left(\mathrm{D}\, J_k \right)^o (v_j\, ;\, v_{j+1} - v_j\, ,\, w) = -\, \mathrm{D}\, J_k(v_j\, ;\, w) \quad \forall w \in V_0\, .$$

We have to check:

Theorem (Bijectivity of $(\mathrm{D}\, J_k)^o\,(v)$)

Let $v \in V_D$ be fixed arbitrarilly. For all $F \in V_0^*$, there exists a unique element $w_1 \in V_0$ satisfying

$$(\mathrm{D}\, J_k)^o\,(v\,;\, w_1\,,\, w_2) = \langle F\,,\, w_2\rangle \quad \forall w_2 \in V_0\,.$$

Proof.

Show, that there exist κ_1 and κ_2 (independent from v) s. t.

$$(\mathrm{D}\, J_k)^o\,(v\,;\, w\,,\, w) \geq \kappa_1\,\|w\|^2 \qquad w \in V_0\,,$$

$$(\mathrm{D}\, J_k)^o\,(v\,;\, w_1\,,\, w_2) \leq \kappa_2\,\|w_1\|\,\|w_2\| \qquad w_1,\ w_2 \in V_0\,.$$

Succeed then by using the Lax-Milgram theorem. □

More Requirements

Theorem (Boundedness of the Inverse)

$$\exists\, C \in \mathbb{R}\ \forall\, v \in V_D \quad : \quad \| [(\mathrm{D}\, J_k)^o\,(v)]^{-1} \| \leq C\,.$$

Proof.

Done by using the ellipticity property. ▸ Detail □

◂ Back

Proof.

$$
\begin{aligned}
\left\| \left[(\mathrm{D}\, J_k)^o\, (v) \right]^{-1} \right\| &= \sup_{w^* \in V_0^*} \frac{\left\| \left[(\mathrm{D}\, J_k)^o\, (v) \right]^{-1}\; w^* \right\|_{V_0}}{\| w^* \|_{V_0^*}} \\
&= \sup_{w \in V_0} \frac{\| w \|_{V_0}}{\| (\mathrm{D}\, J_k)^o\, (v\, ;\, w, \cdot) \|_{V_0^*}} \\
&= \sup_{w \in V_0} \inf_{\overline{w} \in V_0} \frac{\| w \|_{V_0} \| \overline{w} \|_{V_0}}{\left| (\mathrm{D}\, J_k)^o\, (v\, ;\, w, \overline{w}) \right|} \\
&\leq \sup_{w \in V_0} \frac{\| w \|_{V_0}^2}{\left| (\mathrm{D}\, J_k)^o\, (v\, ;\, w, w) \right|} \leq \frac{1}{\kappa_1}\,.
\end{aligned}
$$

□

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