Adaptive High-Order Finite Element Method in Elastoplasticity

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1 Introduction

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Example - Beam under load of $g(t)$
Deformation and Stress at Time $t$
Elastic (blue) and plastic (red) zones
Elastoplasticity - Weak Problem

\[ V = \left[ H^1(\Omega) \right]^2 \]

\[ V_0 = \{ u \in V : u = 0 \text{ on } \Gamma_D \} \quad \text{and} \quad V_D = \{ u \in V : u = u_D \text{ on } \Gamma_D \} \]

**Problem \((k-)\text{th time step})**

Find \( u_k \in V_D \) such, that for all \( w \in V_0 \) there holds

\[
\int_{\Omega} C \left( \varepsilon(u_k) - p_k(\varepsilon(u_k)) \right) : \varepsilon(w) \, dx - \int_{\Gamma_N} g w \, dS(x) = 0.
\]

\[
= F_k(u_k)[w]
\]

\[
\varepsilon(v) = \left( \nabla v + (\nabla v)^T \right)/2
\]
Approximation by a Newton-like Method

Newton-like Iterates
For a given displacement $v_j \in V_D$ find $v_{j+1} \in V_D$ such that
\[
\mathcal{D} F_k(v_j)[v_{j+1} - v_j, w] = -F_k(v_j)[w] \quad (\forall w \in V_0).
\]

After that, postprocessing of $p_k = p_k(\varepsilon(u_k))$. 
Regularity of the Solution $u_k$

Let the volume force and the coefficients of the PDE be sufficiently smooth. Then

- $u_k \in H^2(\omega)$, for any compact $\omega \subset \Omega$,
- $u_k \in C^\infty(\omega)$, for any compact $\omega \subset \{x \in \Omega : p_k(x) = 0\}$.

**hp-FEM**

- Where the solution $u$ has **high regularity**, approximate with **high order** polynomials on a **coarse** grid.
- Where the solution $u$ has **low regularity**, approximate with **low order** polynomials on a **fine** grid.

**Polynomial Degree Vector**

$$ p := (p_K)_{K \in \mathcal{T}} $$

**Discrete Spaces**

$$ S^p(\Omega, \mathcal{T}) = \{ u \in H^1(\Omega) : u \circ F_K \in \text{span}\{\Psi^{p_K}(\hat{K})\}, \ K \in \mathcal{T}\}, $$

$$ S^p_0(\Omega, \mathcal{T}) = S^p(\Omega, \mathcal{T}) \cap H^1_0(\Omega). $$
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Strategy 1 - Zone Concentrated FEM

(based on BC-FEM by Khoromskij, Melenk – 2001)

BC-FEM is defined by the use of

- *Geometric* Meshes,
- *Linear* Polynomial Degree Vector $p$. 
Geometric Mesh
Linear Polynomial Degree Vector

**Definition**

A polynomial degree vector \( \mathbf{p} := (p_K)_K \) is said to be *linear* with slope \( \alpha > 0 \) if

\[
1 + \alpha c_1 \log \frac{h_K}{h} \leq p_k \leq 1 + \alpha c_2 \log \frac{h_K}{h}.
\]

**Convergence**

If the slope \( \alpha \) is chosen large enough:

\[
\|u - u_{FE}\|_{H^1(\Omega)} \leq C h^\delta,
\]

where \( u \in H^{1+\delta}(\Omega) \), \( \delta \in (0,1) \).

Degrees of Freedom: \( O(h^{-1}) \)
Strategy 1 - Zone Concentrated FEM
Strategies 2-4: Basic $hp$-Adaptive Algorithm

- **Input:** $\mathcal{T}, p := (p_K)_{K \in \mathcal{T}}, u_{FE} \in S_0^p(\Omega, \mathcal{T})$.
- **Output:** $\mathcal{T}_{\text{ref}}, p := (p_K)_{K \in \mathcal{T}_{\text{ref}}}$.

1. Calculate $\eta_K^2$ for all $K \in \mathcal{T}$.
2. Determine $\mathcal{T}_{h-\text{ref}}$.
3. Determine $\mathcal{T}_{p-\text{ref}}$.
4. Determine preliminary $\mathcal{T}'_{\text{ref}}$.
5. Elimination of hanging nodes $\rightarrow \mathcal{T}_{\text{ref}}$.
6. Increase the polynomial degree $p_K = p_K + 1$ for all elements $K \in \mathcal{T}_{p-\text{ref}} \cap \mathcal{T}_{\text{ref}}$. In particular: elements to which an $h$-refinement is applied inherit the polynomial degree from their father.
Strategies 2-4: Basic $hp$-Adaptive Algorithm

How to determine $T_{h\text{-ref}}$ and $T_{p\text{-ref}}$?
Remark: ZZ-Error-Estimator for Elasticity

\[ \eta^2_K = \int_K (\sigma_{\text{FE}} - \sigma_{\text{FE}}^*) : C^{-1}(\sigma_{\text{FE}} - \sigma_{\text{FE}}^*) \, dx \]

\[ \eta^2 = \frac{\sum_{K \in \mathcal{T}} \eta^2_K}{\sum_{K \in \mathcal{T}} \int_K \sigma_{\text{FE}}^*: C^{-1} \sigma_{\text{FE}}^* \, dx} \]

\[ \sigma_{\text{FE}} = C \varepsilon(u_{\text{FE}}) \text{ element-wise} \]

\[ \sigma_{\text{FE}}^* \ldots \text{ Clement-Interpolation of } \sigma_{\text{FE}} \]

- The ZZ-error-estimator of elasticity \((\eta^2)\) is reliable but not efficient for elastoplastic problems.
- There is no reliable and efficient error estimator known in elastoplasticity.
Strategy 2 (Eibner, Melenk – 2007)

Determine whether the solution is locally smooth (analytic) by expanding the finite element solution $u_{FE}$ in $L_2$-orthogonal polynomials.
Strategy 2 (Eibner, Melenk – 2007)

1. Choose coefficients $\sigma, \delta > 0$.
2. Compute the mean error $\bar{\eta}^2 = \frac{1}{\#T} \sum_{K \in T} \eta_K^2$.
3. For all $K \in T$ with $\eta_K^2 \geq \sigma \bar{\eta}^2$ compute
   
   $u_{ij;K} = ||\psi_{ij}||_{L^2(\hat{K})}^2 \langle u_{FE}|_K \circ F_K , \psi_{ij} \rangle_{L^2(\hat{K})}$

   for $0 \leq i + j \leq p$ and estimate the decay coefficient $b_K$ by a least squares fit of
   
   $\ln|u_{ij;K}| \approx C_K - b_K (i + j)$.

4. Determine
   
   - $T_{p-ref} = \{K \in T \mid \eta_K^2 \geq \sigma \bar{\eta}^2 \land b_k \geq \delta\}$,
   - $T_{h-ref} = \{K \in T \mid \eta_K^2 \geq \sigma \bar{\eta}^2 \land b_k < \delta\}$.
Strategy 2 - Adaptive Refinement

**Mesh**

**Slope**

**New mesh**

**Refinement**
Strategy 2 - Adaptive Refinement

Mesh

Slope

New Mesh

Refinement
Strategy 2 - Adaptive Refinement

Mesh

New mesh

Slope

Refinement
Strategy 2 - Adaptive Refinement

mesh

slope

new mesh

refinement
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slope

refinement
Strategy 2 - Adaptive Refinement
Strategy 2 (Eibner, Melenk – 2007)

Drawback: Initial polynomial degree has to be high (e.g., 4, 5)
Strategy 3

Same as Strategy 2, but:

- mark element $K$ (plus neighbourhood) for $h$-refinement, if $p_k \neq 0$ on $K$.
- the initial polynomial degree may be kept low in plastic zones.
Strategy 3
Strategy 4 (Demkowicz, Oden, Rachowicz & co. – 1986)

1. Choose coefficients $0 < \delta_1 < \delta_2 < 1$.
2. Compute the maximum error $\eta_{\text{max}} = \max_{K \in \mathcal{T}} \{ \eta_K \}$.
3. Determine
   - $\mathcal{T}_{p-\text{ref}} = \{ K \in \mathcal{T} \mid \delta_1 \eta_{\text{max}} \leq \eta_K \leq \delta_2 \eta_{\text{max}} \}$,
   - $\mathcal{T}_{h-\text{ref}} = \{ K \in \mathcal{T} \mid \eta_K > \delta_2 \eta_{\text{max}} \}$. 
Strategy 4 (Demkowicz, Oden, Rachowicz & co. – 1986)
Convergence

![Convergence Graph]

The graph illustrates the convergence of various adaptive high-order finite element strategies in elastoplasticity. The axes represent the degrees of freedom and the convergence rate, with different strategies depicted by distinct markers. The X-axis shows the degrees of freedom, ranging from $10^{-6}$ to $10^1$, while the Y-axis represents the convergence rate, ranging from $10^{-1}$ to $1$. The graph compares strategies such as Uniform Refinement, ZC-FEM, Strategy 1, Strategy 2, Strategy 3, Strategy 4, and (η only). The performance of these strategies is visually analyzed to assess their efficiency and accuracy in the context of elastoplasticity.
Convergence

![Graph showing convergence vs degrees of freedom for different refinement strategies.](image)
Convergence

![Graph showing convergence with different refinement strategies.](image)
Conclusions

- **Strategy 4 (Error Estimator):** Not reliable for elastoplastic problems.

- **Strategy 3 (Analyticity + PZones):** Converges very fast from the beginning. Minus: A little sensitive w.r.t. calibration of parameters.

- **Strategy 2 (Analyticity):** Converges late, but asymptotically best. Minus: Very sensitive w.r.t. calibration.

- **Strategy 1 (ZC-FEM):** Converges rather slow; Plus: No calibration, always works.
Thank You!
Appendix
Attempt to Differentiate $J_k$ for the Second Time

Explicit Structure

\[
A_k = C(\varepsilon(v) - p_{k-1}),
\]
\[
\text{dev } A_k = A_k - \frac{\text{tr}(A_k)}{\text{tr}(I)} I,
\]
\[
f_1(\varepsilon(v)) = \|\text{dev } A_k\|_F - \sigma_y (1 + H\alpha_{k-1}),
\]
\[
f_2(\varepsilon(v)) = \frac{1}{2\mu + \sigma_y^2 H^2} \frac{\text{dev } A_k}{\|\text{dev } A_k\|_F},
\]
\[
p_k(\varepsilon(v)) = \max\{0, f_1(\varepsilon(v))\} f_2(\varepsilon(v)) + p_{k-1},
\]

where
\[
H > 0, \quad \sigma_y > 0, \quad \mu > 0, \quad \alpha_{k-1} = \alpha_{k-2} + \sigma_y H\|p_{k-1} - p_{k-2}\|_F.
\]
Hierarchic Shape Functions

- Define $\Phi^p = (\Phi_V, \Phi_E^{\min(p, p_{\text{neighbour}})}, \Phi_B^p)$ on $\hat{Q}$.
- Obtain $\Psi^p = \Phi^p \circ D^{-1}$ on $\hat{K}$.

Figure: Duffy transformation $\hat{K} \leftrightarrow \hat{Q}$. 
Theorem (Melenk – 2002)

Define on the reference triangle $\hat{K}$ the $L_2(\hat{K})$-orthogonal basis $\psi_{pq}$, $p, q \in \mathbb{N}_0$ by

$$\psi_{pq} = \tilde{\psi}_{pq} \circ D^{-1}$$

with

$$\tilde{\psi}_{pq} = P_p^{(0,0)}(\eta_1) \left( \frac{1 - \eta_2}{2} \right)^p P_q^{(2p+1,0)}(\eta_2),$$

where $P_p^{(\alpha,\beta)}$ is the $p$th Jacobi polynomial with respect to the weight $\eta \mapsto (1 - \eta)^\alpha (1 + \eta)^\beta$ and $D$ the Duffy transformation. Let $u \in L^2(\hat{K})$ be written as $u = \sum_{p,q\in\mathbb{N}_0} u_{pq} \psi_{pq}$. Then $u$ is analytic on $\hat{K}$ if and only if there exist constants $C, b > 0$ s.t.

$$|u_{pq}| \leq Ce^{-b(p+q)}$$

for all $p, q$. 