Variational Inequalities in Elastoplasticity

Seminar – Numerical Analysis of Variational Inequalities

Peter Gruber

10/24/2006

Mathematical Modelling

Weak Formulation

Notation

- With $\cdot$ we denote the scalar produkt of two vector valued quantifiers
$$u \cdot v = u^T v \, .$$
- With $:$ we denote the scalar product of two matrix valued quantifiers
$$A : B = \sum_{i,j} a_{ij} b_{ij} \, .$$
- With $\| \cdot \|_F$ we denote Frobenius' norm of respectively a matrix or vector valued quantifier, i. e.,
$$\|M\|_F = (M : M)^{1/2} \quad \text{and} \quad \|v\|_F = (v \cdot v)^{1/2} \, .$$

Linear Elasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma = f \quad \text{in } \Omega$
B. Angular Momentum	$\sigma = \sigma^T$
Hook's Law	$\sigma = \mathbb{C}\,\varepsilon$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Boundary Conditions	$u = u_D \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma\, n = g \quad \text{on } \Gamma_N$

What is Different in Elastoplasticity?

An elastoplastic problem is time dependent, i. e.,

$$u = u(x,t),\ \sigma = \sigma(x,t),\ f = f(x,t),\ \text{etc.} \quad \text{for } (x,t) \in \Omega \times [0,\mathrm{T}]\,.$$

The balance of momentum would then read

$$\rho\,\ddot{u} - \operatorname{div}\sigma = f \quad \text{in } \Omega \times [0,\mathrm{T}]\,.$$

Accelerations $\ddot{u}$ are considered to be very small, such that they can be neglected and we have to solve

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega \times [0,\mathrm{T}]\,.$$

A problem of such kind is said to be *quasistatic*.

What Else is Different in Elastoplasticity?

The strain ε is additively split into parts

$$\varepsilon = \mathrm{e} + p\,.$$

We denote e the *elastic strain*, and p the *plastic strain*. Only the elastic strain is associated with the stress tensor by Hook's law

$$\sigma = \mathbb{C}\mathrm{e} = \mathbb{C}\left(\varepsilon - p\right)\,.$$

We further prescribe the initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad \sigma(x,0) = \sigma_0(x)\,.$$

Elastoplasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma = f \quad \text{in } \Omega \times [0, \mathrm{T}]$
B. Angular Momentum	$\sigma = \sigma^T$
Hook's Law	$\sigma = \mathbb{C}\,(\varepsilon - p)$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Boundary Conditions	$u = u_D$ on Γ_D and $\sigma\, n = g$ on Γ_N
Initial Conditions	$u(x,0) = u_0(x)$ and $\sigma(x,0) = \sigma_0(x)$

Missing Prescriptions

- unknowns: displacement u, plastic strain p.
- We need more prescriptions to determine the plastic strain p.
- Physics: $\sigma = \mathbb{C}(\varepsilon - p)$ must satisfy an admissibility condition.
- Physics: Time development of p given by the plastic flow law.

Admissibility Condition for the Stress σ

We introduce the so called *yield function* ϕ, which is a convex mapping into $\mathbb{R}$. A stress tensor σ is *admissible* if there holds

$$\phi(\sigma) \leq 0 .$$

An example of such yield function would be

$$\phi(\sigma) = \|\operatorname{dev} \sigma\|_F - \sigma_y ,$$

where $\operatorname{dev} \sigma = \sigma - \frac{\sigma : I}{I : I} I$ and the constant $\sigma_y > 0$ are said to be the *deviator* and the *yield stress* respectively. The zero-level set of ϕ,

$$K = \{\sigma \,:\, \phi(\sigma) \leq 0\},$$

is called the *set of admissible stresses*.

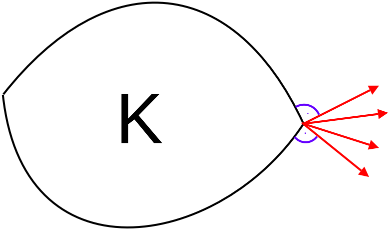
Plastic Flow Law for the Plastic Strain p

The plastic flow law states, that

$$\dot{p} \in \mathcal{N}_K(\sigma),$$

where $\mathcal{N}_K(\sigma)$ denotes the normal cone of K in σ:

$$\mathcal{N}_K(\sigma) = \{n : \langle n, \theta - \sigma \rangle \leq 0 \quad \forall \theta \in K\}.$$



Elastoplasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma = f \quad \text{in } \Omega$
B. Angular Momentum	$\sigma = \sigma^T$
Hook's Law	$\sigma = \mathbb{C}(\varepsilon - p)$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Boundary Conditions	$u = u_D \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma n = g \quad \text{on } \Gamma_N$
Initial Condition	$u(x,0) = 0\,, \quad \sigma(x,0) = 0$

Admissibility	$\phi(\sigma) \leq 0$
Plastic Flow	$\langle \dot{p}\,,\ \theta - \sigma \rangle \leq 0 \quad \forall \theta \in K$

Equivalent Subgradient Condition

- We define the *dissipation functional* D_ϕ by

$$D_\phi(\sigma) = \begin{cases} 0 & \text{if } \phi(\sigma) \leq 0\,, \\ +\infty & \text{else}\,, \end{cases}$$

- and the *subgradient* of a function f as the set

$$\partial f(x) = \{g \,:\, f(y) \geq f(x) + \langle g\,, y - x \rangle \quad \forall y\}\,.$$

Then, one can show:

$$\underbrace{[\dot{p} \in \mathcal{N}_K(\sigma)] \wedge [\phi(\sigma) \leq 0]}_{\text{dual formulation}} \Leftrightarrow \underbrace{\dot{p} \in \partial D_\phi(\sigma)}_{\text{primal formulation}}\,.$$

Elastoplasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma = f \quad \text{in } \Omega$
B. Angular Momentum	$\sigma = \sigma^T$
Hook's Law	$\sigma = \mathbb{C}(\varepsilon - p)$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Boundary Conditions	$u = u_D \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma n = g \quad \text{on } \Gamma_N$
Initial Condition	$u(x,0) = 0\,, \quad \sigma(x,0) = 0$

Primal	$\langle \dot{p}\,,\ \theta - \sigma\rangle + D_\phi(\sigma) \le D_\phi(\theta) \quad \forall\theta$
Dual	$[\sigma \in K] \wedge [\langle \dot{p}\,,\ \theta - \sigma\rangle \le 0 \quad \forall\theta \in K]$

Conjugate Function

Definition (Conjugate Function)

Let X be a Banach space, X^* its topological dual, and $f \in X^*$. Then, the function $f^* : H^* \to \mathbb{R}$ is said to be *conjugate* to f if for all $x^* \in X^*$ there holds

$$f^*(x^*) = \sup_{x \in X} \left(\langle x^* \, , \, x \rangle - f(x) \right) .$$

Theorem

Let some assumptions be satisfied. Then there holds

$$x^* \in \partial f(x) \ \Leftrightarrow \ x \in \partial f^*(x^*) .$$

Equivalent Primal Formulation

Corollary

Thanks to the previous theorem, there holds

$$
\begin{aligned}
& \langle \dot{p}\,,\ \theta - \sigma \rangle + D_\phi(\sigma) \ \leq \ D_\phi(\theta) \quad \forall \theta \\
\Leftrightarrow \quad & \langle \sigma\,,\ q - \dot{p} \rangle + D_\phi^*(\dot{p}) \ \leq \ D_\phi^*(q) \quad \forall q
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& \langle \dot{p}\,,\ \theta - \sigma \rangle + D_\phi(\sigma) \ \leq \ D_\phi(\theta) \qquad \forall \theta \\
\Leftrightarrow \quad & \dot{p} \ \in \ \partial D_\phi(\sigma) \\
\Leftrightarrow \quad & \sigma \ \in \ \partial D_\phi^*(\dot{p}) \\
\Leftrightarrow \quad & \langle \sigma\,,\ q - \dot{p} \rangle + D_\phi^*(\dot{p}) \ \leq \ D_\phi^*(q) \qquad \forall q
\end{aligned}
$$

□

Elastoplasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma = f \quad \text{in } \Omega$
B. Angular Momentum	$\sigma = \sigma^T$
Hook's Law	$\sigma = \mathbb{C}(\varepsilon - p)$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Boundary Conditions	$u = u_D \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma n = g \quad \text{on } \Gamma_N$
Initial Condition	$u(x,0) = 0\,, \quad \sigma(x,0) = 0$

Primal	$\sigma : (q - \dot{p}) + D_\phi^*(\dot{p}) \leq D_\phi^*(q) \quad \forall q$
Dual	$[\sigma \in K] \wedge [\langle \dot{p}\,,\ \theta - \sigma \rangle \leq 0 \quad \forall \theta \in K]$

Mathematical Modelling

Weak Formulation

Function Spaces

- $V = \left[H^1(\Omega)\right]^3$
- $V_D = \{v \in V \ : \ v = u_D \text{ on } \Gamma_D\}$
- $V_0 = \{v \in V \ : \ v = 0 \text{ on } \Gamma_D\}$
- $Q = [L_2(\Omega)]^{3\times 3}_{\text{sym}}$
- $u \in V_D; \quad \dot{u} \in V_0; \quad p, \sigma \in Q$

Primal Variational Formulation

The unknowns are (u, p). Integration of the primal condition yields

$$\int_\Omega D_\phi^*(\dot{p})\,\mathrm{d}x + \int_\Omega \mathbb{C}\,(\varepsilon(u) - p) : (q - \dot{p})\,\mathrm{d}x \leq \int_\Omega D_\phi^*(q)\,\mathrm{d}x \qquad \forall q \in Q\,.$$

We multiply the balance law with $v - \dot{u}$, where $v \in V_D$, combine it with $\sigma = \mathbb{C}(\varepsilon(u) - p)$ and integrate by parts (cf. variational equation of linear elasticity):

$$\int_\Omega \mathbb{C}(\varepsilon(u) - p) : (\varepsilon(v) - \varepsilon(\dot{u}))\,\mathrm{d}x = \int_\Omega f \cdot (v - \dot{u})\,\mathrm{d}x + \int_{\Gamma_N} g \cdot (v - \dot{u})\,\mathrm{d}s \qquad \forall v \in V_D\,.$$

Abstract Primal Variational Formulation

We define

- $w = (u, p) \in V_D \times Q$
- $z = (v, q) \in V_0 \times Q$
- $a_1(w, z) = \int_\Omega \mathbb{C}\,(\varepsilon(u) - p) : q \; \mathrm{d}x$
- $a_2(w, z) = \int_\Omega \mathbb{C}\,(\varepsilon(u) - p) : \varepsilon(v) \; \mathrm{d}x$
- $a(w, z) = a_1(w, z) + a_2(w, z)$
- $j(z) = \int_\Omega D_\phi^*(q) \; \mathrm{d}x$
- $l(z) = \int_\Omega f \cdot v \; \mathrm{d}x + \int_{\Gamma_N} g \cdot v \; \mathrm{d}s$

Abstract Primal Variational Formulation

Then we have to solve:

Problem

Find $w \in V_D \times Q$ with $\dot{w} \in V_0 \times Q$ such, that w satisfies initial conditions and for all $z \in V_0 \times Q$ there hold

$$\begin{aligned} a_1(w, \dot{w} - z) + j(\dot{w}) - j(z) &\leq 0 \\ a_2(w, \dot{w} - z) + l(\dot{w} - z) &= 0\,, \end{aligned}$$

or equivalently (elementary calculation),

$$a(w, \dot{w} - z) + j(\dot{w}) - j(z) + l(\dot{w} - z) \leq 0\,.$$

Problem to show unique solvability: time derivatives.

Time Discretization

- $[0,\mathrm{T}] \quad \rightsquigarrow \quad \{0 = t_0, t_1, t_2, \ldots, t_{n-1}, t_n = T\}$
- $t_i - t_{i-1} = k \qquad \forall i \in 1, \ldots, n$
- define $w_i(x) = w(x, t_i)$ and $z_i(x) = z(x, t_i)$ a.e.
- approximate $\dot{w}_i \approx \frac{w_i - w_{i-1}}{k}$
- redefine a_1, a_2, j and l properly

Problem

Find $w_i \in V_D \times Q$ such, that for all $z \in V_D \times Q$ there hold

$$a(w_i, w_i - z) + j(w_i) - j(z) + l(w_i - z) \leq 0 \quad \forall z \in V_D \times Q\,.$$

Notice, the quantifiers a_1, a_2, j and l depend on the solution of the last time step, w_{i-1}. At the initial time step, w_0 is set according to the initial conditions.

Dual Variational Formulation

We now derive a dual variational formulation (mixed form). In the classical formulation of the dual problem, we prescribed $\sigma \in K$ and

$$\langle \dot{p}\,,\ \theta - \sigma \rangle \leq 0 \quad \forall \theta \in K\,.$$

The latter condition can be reformulated due to $\dot{p} = \dot{\varepsilon} - \mathbb{C}^{-1}\,\dot{\sigma}$,

$$\langle \dot{\varepsilon} - \mathbb{C}^{-1}\,\dot{\sigma}\,,\ \theta - \sigma \rangle \leq 0 \quad \forall \theta \in K\,. \tag{1}$$

So we have to find (u, σ), such that $\sigma \in K$, and the balance of momentum, $-\operatorname{div}\sigma = f$, and expression (1) are satisfied.

Dual Variational Formulation

We multiply the balance of momentum with a test function $v \in V_0$, integrate it by parts, and also integrate (1) over Ω. Then, by the definitions

- $\mathscr{K} = \{\sigma \in Q : \sigma(x) \in K \; a.e.\}$
- $a(\sigma, \theta) = \int_\Omega \mathbb{C}^{-1} \sigma : \theta \,\mathrm{d}x$
- $b(v, \theta) = -\int_\Omega \varepsilon(v) : \theta \,\mathrm{d}x$
- $l(v) = -\int_\Omega f \cdot v \,\mathrm{d}x - \int_{\Gamma_N} g \cdot v \,\mathrm{d}s$

we obtain the following problem:

Dual Variational Formulation

Problem

Find $(u, \sigma) \in V_D \times Q$ with $\dot{u} \in V_0$ and $\sigma \in \mathscr{K}$, such that the initial conditions are satisfied and there holds

$$\begin{aligned} b(v, \sigma) &= l(v) \quad \forall v \in V_0 \\ a(\dot{\sigma}, \sigma - \theta) + b(\dot{u}, \sigma - \theta) &\leq 0 \qquad \forall \theta \in \mathscr{K} \,. \end{aligned}$$

Again, by time discretization as done before, and proper redefinitions of a, b and l we obtain

Problem

Find $(u_i, \sigma_i) \in V_D \times Q$ with $\sigma_i \in \mathscr{K}$, such that there holds

$$\begin{aligned} b(v, \sigma_i) &= l(v) \quad \forall v \in V_0 \\ a(\sigma_i, \sigma_i - \theta) + b(u_i, \sigma_i - \theta) &\leq 0 \qquad \forall \theta \in \mathscr{K} \,. \end{aligned}$$

Thank you for your attention!