Variational Inequalities in Elastoplasticity

Seminar – Numerical Analysis of Variational Inequalities

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Notation

- With $\cdot$ we denote the scalar produkt of two vector valued quantifiers

$$u \cdot v = u^T v \, .$$

- With $:$ we denote the scalar product of two matrix valued quantifiers

$$A : B = \sum_{i,j} a_{ij} b_{ij} \, .$$

- With $\|\cdot\|_F$ we denote Frobenius' norm of respectively a matrix or vector valued quantifier, i. e.,

$$\|M\|_F = (M : M)^{1/2} \quad \text{and} \quad \|v\|_F = (v \cdot v)^{1/2} \, .$$

Linear Elasticity – Classical Formulation

Balance of Momentum	$-\operatorname{div}\sigma = f \quad \text{in } \Omega$
B. Angular Momentum	$\sigma = \sigma^T$
Hook's Law	$\sigma = \mathbb{C}\,\varepsilon$
Linearized Strain	$\varepsilon(u) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$
Boundary Conditions	$u = u_D \quad \text{on } \Gamma_D \quad \text{and} \quad \sigma\, n = g \quad \text{on } \Gamma_N$

What is Different in Elastoplasticity?

An elastoplastic problem is time dependent, i. e.,

$$u = u(x,t),\ \sigma = \sigma(x,t),\ f = f(x,t),\ \text{etc.} \quad \text{for } (x,t) \in \Omega \times [0,\mathrm{T}]\,.$$

The balance of momentum would then read

$$\rho\,\ddot{u} - \operatorname{div}\sigma = f \quad \text{in } \Omega \times [0,\mathrm{T}]\,.$$

Accelerations $\ddot{u}$ are considered to be very small, such that they can be neglected and we have to solve

$$-\operatorname{div}\sigma = f \quad \text{in } \Omega \times [0,\mathrm{T}]\,.$$

A problem of such kind is said to be *quasistatic*.

What Else is Different in Elastoplasticity?

The strain ε is additively split into parts

$$\varepsilon = \mathrm{e} + p\,.$$

We denote e the *elastic strain*, and p the *plastic strain*. Only the elastic strain is associated with the stress tensor by Hook's law

$$\sigma = \mathbb{C}\,\mathrm{e} = \mathbb{C}\,(\varepsilon - p)\,.$$

We further prescribe the initial conditions

$$u(x,0) = u_0(x) \quad \text{and} \quad \sigma(x,0) = \sigma_0(x)\,.$$

Elastoplasticity – Classical Formulation

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Initial Conditions	$u(x,0) = u_0(x)$ and $\sigma(x,0) = \sigma_0(x)$

Missing Conditions

- unknowns: displacement u, plastic strain p.
- We need more prescriptions to determine the plastic strain p.
- Physics: $\sigma = \mathbb{C}(\varepsilon - p)$ must satisfy an admissibility condition.
- Physics: Time development of p given by the plastic flow law.

Admissibility Condition for the Stress σ

We introduce a convex mapping ϕ into $\mathbb{R}$, which steers by the relation

$$\phi(\sigma) \leq 0\,,$$

if a stress σ is allowed or not. The zero-level set of ϕ,

$$K = \{(\Sigma) \,:\, \phi(\Sigma) \leq 0\},$$

is called the *set of admissible states*.

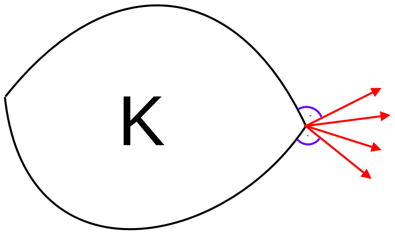
Plastic Flow Law for the Plastic Strain p

The plastic flow law states, that

$$\dot{\Pi} = (\dot{p}, \dot{\xi}) \in \mathscr{N}_K(\Sigma),$$

where $\mathscr{N}_K(\Sigma)$ denotes the normal cone of K in (Σ):

$$\mathscr{N}_K(\Sigma) = \{n \,:\, \langle n \,,\, \Theta - \Sigma\rangle \leq 0 \quad \forall \Theta \in K\}.$$



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Initial Condition	$u(x,0) = 0\,, \quad \sigma(x,0) = 0$

Constitution	$\chi = -\mathbb{H}\,\xi$
Admissibility	$\phi(\sigma,\chi) \leq 0$
Plastic Flow	$\langle (\dot{p},\dot{\xi})\,,\,(\theta,\rho) - (\sigma,\chi)\rangle \leq 0 \quad \forall \theta,\rho$

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Admissibility	$\phi(\sigma,\chi) \leq 0$
Plastic Flow	$\langle(\dot{p}, -\mathbb{H}^{-1}\dot{\chi})\,,\,(\theta,\rho) - (\sigma,\chi)\rangle \leq 0 \quad \forall \theta, \rho$

Equivalent Subgradient Condition

- We define the *dissipation functional* D_ϕ by

$$D_\phi(\Sigma) = \begin{cases} 0 & \text{if } \phi(\Sigma) \leq 0\,, \\ +\infty & \text{else}\,, \end{cases}$$

- and the *subgradient* of a function f as the set

$$\partial f(x) = \{g \,:\, f(y) \geq f(x) + \langle g\,, y - x\rangle \quad \forall y\}\,.$$

Then, one can show:

$$\underbrace{\left[\dot{\Pi} \in \mathcal{N}_K(\Sigma)\right] \wedge \left[\phi(\Sigma) \leq 0\right]}_{\text{dual formulation}} \Leftrightarrow \underbrace{\dot{\Pi} \in \partial D_\phi(\Sigma)}_{\text{primal formulation}}\,.$$

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Initial Condition	$u(x,0) = 0\,, \quad \sigma(x,0) = 0$

Primal $\langle(\dot{p}, -\mathbb{H}^{-1}\dot{\chi})\,,\,(\theta,\rho) - (\sigma,\chi)\rangle + D_\phi(\sigma,\chi) \le D_\phi(\theta,\rho) \quad \forall \theta,\, \rho$

Dual $[(\sigma,\chi) \in K] \wedge \left[\langle(\dot{p}, -\mathbb{H}^{-1}\dot{\chi})\,,\,(\theta,\rho) - (\sigma,\chi)\rangle \le 0 \quad \forall (\theta,\rho) \in K\right]$

Conjugate Function

Definition (Conjugate Function)

Let X be a Banach space, X^* its topological dual, and $f \in X^*$. Then, the function $f^* : H^* \to \mathbb{R}$ is said to be *conjugate* to f if for all $x^* \in X^*$ there holds

$$f^*(x^*) = \sup_{x \in X} \left(\langle x^* \, , \, x \rangle - f(x) \right) .$$

Theorem

Let some assumptions be satisfied. Then there holds

$$x^* \in \partial f(x) \;\Leftrightarrow\; x \in \partial f^*(x^*) .$$

Equivalent Primal Formulation

$$
\begin{aligned}
\langle (\dot{p}, -\mathbb{H}^{-1}\dot{\chi}) \,,\, (\theta,\rho) - (\sigma,\chi) \rangle + D_\phi(\sigma,\chi) \leq &\quad D_\phi(\theta,\rho) \quad \forall \theta,\rho \\
\Leftrightarrow (\dot{p}, -\mathbb{H}^{-1}\dot{\chi}) \in &\quad \partial D_\phi(\sigma,\chi) \\
\Leftrightarrow (\sigma,\chi) \in &\quad \partial D_\phi^*(\dot{p}, -\mathbb{H}^{-1}\dot{\chi}) \\
\Leftrightarrow \langle (\sigma,\chi) \,,\, (q,\rho) - (\dot{p}, -\mathbb{H}^{-1}\dot{\chi}) \rangle + D_\phi^*(\dot{p}, -\mathbb{H}^{-1}\dot{\chi}) \leq &\quad D_\phi^*(q,\rho) \quad \forall q,\rho \\
\Leftrightarrow \langle (\sigma, -\mathbb{H}\xi) \,,\, (q,\rho) - (\dot{p},\dot{\xi}) \rangle + D_\phi^*(\dot{p},\dot{\xi}) \leq &\quad D_\phi^*(q,\rho) \quad \forall q,\rho \\
\Leftrightarrow \sigma : (q - \dot{p}) - \mathbb{H}\xi \cdot (\rho - \dot{\xi}) + D_\phi^*(\dot{p},\dot{\xi}) \leq &\quad D_\phi^*(q,\rho) \quad \forall q,\rho
\end{aligned}
$$

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Initial Condition	$u(x,0) = 0\,, \quad \sigma(x,0) = 0$

Primal $\quad \sigma : (q - \dot{p}) - \mathbb{H}\xi \cdot (\rho - \dot{\xi}) + D^*_\phi(\dot{p}, \dot{\xi}) \leq D^*_\phi(q, \rho) \quad \forall q, \rho$

Dual $\quad [(\sigma,\chi) \in K] \wedge [\langle(\dot{p}, -\mathbb{H}\dot{\chi})\,,\,(\theta,\rho) - (\sigma,\chi)\rangle \leq 0 \quad \forall(\theta,\rho) \in K]$

Function Spaces

- $V = \left[H^1(\Omega)\right]^3$
- $V_D = \{v \in V \ : \ v = u_D \text{ on } \Gamma_D\}$
- $V_0 = \{v \in V \ : \ v = 0 \text{ on } \Gamma_D\}$
- $Q = [L_2(\Omega)]^{3\times 3}_{\text{sym}}$
- $u \in V_D; \quad \dot{u} \in V_0; \quad p, \sigma \in Q; \quad \chi, \xi \in [L_2(\Omega)]^m$

Primal Variational Formulation

The unknowns are (u, p, ξ). Integration of the primal condition yields

$$\int_\Omega D_\phi^*(\dot{p}, \dot{\xi})\,\mathrm{d}x + \int_\Omega \Big[\mathbb{C}\,(\varepsilon(u) - p) : (q - \dot{p}) - \mathbb{H}\xi \cdot \Big(\rho - \dot{\xi}\Big)\Big]\,\mathrm{d}x$$

$$\leq \int_\Omega D_\phi^*(q, \rho)\,\mathrm{d}x \qquad \forall (q, \rho) \in Q \times L_2(\Omega)\,.$$

Next we multiply the balance law with $v - \dot{u}$, where $v \in V_D$, combine it with $\sigma = \mathbb{C}(\varepsilon(u) - p)$ and integrate by parts:

$$\int_\Omega \mathbb{C}(\varepsilon(u) - p) : (\varepsilon(v) - \varepsilon(\dot{u}))\,\mathrm{d}x = \int_\Omega f \cdot (v - \dot{u})\,\mathrm{d}x + \int_{\Gamma_N} g \cdot (v - \dot{u})\,\mathrm{d}s$$

$$\forall v \in V_D\,.$$

Abstract Primal Variational Formulation

We define

- $w = (u, p, \xi) \in V_D \times Q \times L_2(\Omega)$
- $z = (v, q, \rho) \in V_0 \times Q \times L_2(\Omega)$
- $a_1(w,z) = \int_\Omega [\mathbb{C}(\varepsilon(u) - p) : q - \mathbb{H}\xi \cdot \rho] \, \mathrm{d}x$
- $a_2(w,z) = \int_\Omega \mathbb{C}(\varepsilon(u) - p) : \varepsilon(v) \, \mathrm{d}x$
- $a(w,z) = a_1(w,z) + a_2(w,z)$
- $j(z) = \int_\Omega D^*_\phi(q, \rho) \, \mathrm{d}x$
- $l(z) = \int_\Omega f \cdot v \, \mathrm{d}x + \int_{\Gamma_N} g \cdot v \, \mathrm{d}s$

Abstract Primal Variational Formulation

Then we have to solve:

Problem

Find $w \in V_D \times Q \times L_2(\Omega)$ *with* $w \in V_0 \times Q \times L_2(\Omega)$ *such, that for all* $z \in V_0 \times Q \times L_2(\Omega)$ *there hold*

$$\begin{aligned} a_1(w, \dot{w} - z) + j(\dot{w}) - j(z) &\leq 0 \quad \forall z\,, \\ a_2(w, \dot{w} - z) + l(\dot{w} - z) &= 0 \quad \forall z\,, \end{aligned}$$

or equivalently (elementary calculation)

$$a(w, \dot{w} - z) + j(\dot{w}) - j(z) + l(\dot{w} - z) \leq 0 \quad \forall z\,.$$

Problem to show unique solvability: time derivatives.

Time Discretization

- $[0,\mathrm{T}] \quad \rightsquigarrow \quad \{0=t_0,t_1,t_2,\dots,t_{n-1},t_n=T\}$
- $t_i - t_{i-1} = k \qquad \forall i \in 1,\dots,n$
- define $w_i(x) = w(x,t_i)$ and $z_i(x) = z(x,t_i)$ a.e.
- approximate $\dot{w}_i \approx \frac{w_i - w_{i-1}}{k}$
- define $z_i = z + w_{i-1} \in V_D$!
- redefine a_1, a_2, j and l properly

Problem

Find $w_i \in V_D \times Q \times L_2(\Omega)$ such, that for all $z \in V_D \times Q \times L_2(\Omega)$ there hold

$$a(w_i, w_i - z) + j(w_i) - j(z) + l(w_i - z) \leq 0 \quad \forall z \in V_D \times Q \times L_2(\Omega)\,.$$

Dual Variational Formulation

In the classical formulation of the dual problem, we prescribed $(\sigma, \chi) \in K$ and

$$\langle (\dot{p}, -\mathbb{H}\dot{\chi}) \, , \, (\theta, \rho) - (\sigma, \chi) \rangle \leq 0 \quad \forall (\theta, \rho) \in K \, .$$

The latter condition can be reformulated due to $\dot{p} = \dot{\varepsilon} - \mathbb{C}^{-1} \dot{\sigma}$,

$$\langle (\dot{\varepsilon} - \mathbb{C}^{-1} \dot{\sigma}, -\mathbb{H}\dot{\chi}) \, , \, (\theta, \rho) - (\sigma, \chi) \rangle \leq 0 \quad \forall (\theta, \rho) \in K \, .$$

Thus, the unknown variables are (u, σ, χ).

Dual Variational Formulation

We define

- $\mathscr{K} = \{(\theta,\rho) \in Q \times [L_2(\Omega)]^m : (\theta(x),\rho(x)) \in K \; a.e.\}$
- $a(\sigma,\theta) = \int_\Omega \sigma : \mathbb{C}^{-1}\theta \, \mathrm{d}x$
- $b(v,\theta) = -\int_\Omega \varepsilon(v) : \theta \, \mathrm{d}x$
- $c(\chi,\rho) = \int_\Omega \chi \cdot \mathbb{H}^{-1}\rho \, \mathrm{d}x$
- $l(v) = -\int_\Omega f \cdot v \, \mathrm{d}x - \int_{\Gamma_N} g \cdot v \, \mathrm{d}s$

and obtain

Problem

Find $(u,\sigma,\chi) \in V_D \times Q \times [L_2(\Omega)]^m$ *such, that* $(\sigma,\chi) \in \mathscr{K}$ *and*

$$\begin{aligned} b(v,\sigma) &= l(v) \quad \forall v, \in V_0 \\ a(\dot\sigma,\sigma-\theta) + c(\dot\chi,\chi-\rho) + b(\dot u,\sigma-\theta) &\leq 0 \quad \forall(\theta,\rho) \in \mathscr{K} \, . \end{aligned}$$

Thank you for your attention!