Isogeometric Analysis for Nonlinear Dynamics of Timoshenko Beams

Stanislav Stoykov[1], Clemens Hofreither[1], and Svetozar Margenov[1]

Institute of Information and Communication Technologies,
Bulgarian Academy of Sciences,
Acad. G. Bonchev St., Bl. 25A, 1113 Sofia, Bulgaria

Abstract. The dynamics of beams that undergo large displacements is analyzed in frequency domain and comparisons between models derived by isogeometric analysis and p-FEM are presented. The equation of motion is derived by the principle of virtual work, assuming Timoshenko's theory for bending and geometrical type of nonlinearity.
As a result, a nonlinear system of second order ordinary differential equations is obtained. Periodic responses are of interest and the harmonic balance method is applied. The nonlinear algebraic system is solved by an arc-length continuation method in frequency domain.
It is shown that IGA gives better approximations than the p-FEM when models with the same number of degrees of freedom are used.

Keywords: p-FEM, Bifurcation diagrams, Isogeometric analysis, B-Splines, Continuation method, Nonlinear frequency-response function.

1 Introduction

Isogeometric analysis (IGA) is a relatively novel discretization technique which was introduced by Hughes et al. [5]. The key idea of representing both the computational geometry as well as the trial functions using spaces of appropriate spline functions afford the method several interesting advantages. It has been demonstrated that IGA has superior spectral approximation properties compared to standard finite element method (FEM) discretizations [6]. The method has been applied successfully to the structural analysis of various mechanical models [3, 1, 4, 8]. Recently, nonlinear dynamics of beams in frequency domain has been studied by isogeometric approach [15].
Higher natural frequencies become important in nonlinear dynamics of elastic structures [7, 13]. Thus, it is expected that space discretization by isogeometric analysis will give better approximations in the computations of nonlinear normal modes and nonlinear frequency-response function than discretization by the finite element method. In the current work, the equation of motion of beams is used for comparison of the nonlinear frequency-response function of models obtained by isogeometric analysis and p-FEM. The equation of motion is derived by the principle of virtual work, Timoshenko's beam theory is assumed and geometrical nonlinearity is taken into account. External harmonic excitation is

considered and the response is assumed to be periodic function of time, thus the vector of generalized coordinates is expressed in Fourier series. The frequency-response function is obtained by the continuation method. It is shown that IGA gives better approximations than a p-FEM, when models with the same number of degrees of freedom are used.

2 Isogeometric analysis

We present very briefly the construction of an isogeometric method in 1D. We begin by setting up a B-spline basis

$$\{B_j : [-1, 1] \to \mathbb{R}_0^+\}_j$$

of degree p over an open knot vector which spans the parameter interval $[-1, 1]$. *Open* means that the first and last knots are repeated $p + 1$ times. The interior knots are repeated at most p times. For the definition of B-splines, see $[12, 9, 2]$.

To each basis function B_j, we associate a control point (coefficient) $C_j \in \mathbb{R}$ in such a way that we obtain an invertible geometry mapping $F = \sum_j C_j B_j :$ $[-1, 1] \to \Omega = [-\ell/2, \ell/2]$, where Ω is an open interval representing the beam of length ℓ. The isogeometric basis functions on Ω are given by $\psi_j(x) := B_j(F^{-1}(x))$ and their span $\mathcal{V}_h := \text{span}\{\psi_j\}$ is the isogeometric trial space on Ω.

Note that in the 1D case it is always possible to choose the control points in such a way that the parametrization F is linear. However, it is also possible to use a nonlinear parametrization, which can be advantageous in certain situations.

For problems in higher space dimensions, NURBS, i.e., rational versions of the B-spline basis functions, are commonly used to represent the geometry. For the one-dimensional geometry of beams, however, we stick to the use of B-splines.

The linear longitudinal equation of motion of beams is used as an example of space discretization using isogeometric analysis:

$$(EAu'(x,t))' + F_u(x,t) = \mu\ddot{u}(x,t) \qquad \forall x \in \Omega, t \in (0, T), \tag{1}$$

where E denotes the elastic modulus, A is the area of the cross section, μ the mass per unit length, F_u the longitudinal external load, and $u(x,t)$ the longitudinal displacement at point x and time t. The variational form of (1) is

$$\int_\Omega \mu\ddot{u}v\,dx + \int_\Omega EAu'v'\,dx = \int_\Omega F_u v\,dx \quad \forall v \in \mathcal{V} \tag{2}$$

with an appropriate test space $\mathcal{V}$. It is assumed that the ends of the beam are fixed, i.e., $u(x) = 0$ at $x = \pm\ell/2$.

The isogeometric discretization of the linear longitudinal equation of motion is obtained by restricting the variational equation (2) to the isogeometric space $\mathcal{V}_{h,0} \subset \mathcal{V}_h$, where the first and last basis function have been removed from $\mathcal{V}_h$ to enforce the essential boundary condition $u(x) = 0$ at $x = \pm\ell/2$. That is, we seek time-dependent coefficients $(q_j^u(t))_j$ such that

$$u_h(x,t) = \sum_j q_j^u(t)\psi_j^u(x) \in \mathcal{V}_{h,0}$$

satisfies (2) for all $v_h \in \mathcal{V}_{h,0}$. Here we write $\psi_j^u := \psi_j$ for the basis functions to make clear that they correspond to the displacement u. Later on, we will use different bases for the transverse displacement and rotation components.

With the stiffness matrix, mass matrix, and load vector, respectively,

$$K_{ij} = \int_\Omega EA\psi_j'\psi_i' \, dx, \quad M_{ij} = \int_\Omega \mu\psi_j\psi_i \, dx, \quad f_j = \int_\Omega F_u\psi_j \, dx,$$

we obtain the system of linear ordinary differential equations

$$\mathbf{M\ddot{q}}^u + \mathbf{Kq}^u = \mathbf{f}$$

for the vector-valued function $q^u(t)$ of displacement coefficients.

3   An isogeometric formulation of Timoshenko beams

The equation of motion is derived by the principle of virtual work considering Timoshenko's theory for bending. It is assumed that the cross section remains plain after deformation, but the transverse shear strain ε_{xz} is not neglected. We denote the longitudinal and transverse displacements by $u(x, z, t)$ and $w(x, z, t)$, respectively, they depend on space coordinates x and z and on time t. The displacement field on the middle line is expressed as

$$u(x, z, t) = u_0(x, t) + z\phi_y(x, t), \tag{3a}$$
$$w(x, z, t) = w_0(x, t), \tag{3b}$$

where $u_0(x, t)$ and $w_0(x, t)$ denote the longitudinal and transverse displacements on the middle line and $\phi_y(x, t)$ is the rotation of the cross section about y axis. Nonlinearity is taken into account by assuming nonlinear strain-displacement relations. The axial and shear strains are derived from Green's strain tensor and the longitudinal terms of second order are neglected:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}. \tag{4}$$

Elastic isotropic materials are of interest, hence the stresses are related to the strains by Hooke's law,

$$\boldsymbol{\sigma} = \left\{\begin{array}{c} \sigma_x \\ \tau_{xz} \end{array}\right\} = \begin{bmatrix} E & 0 \\ 0 & \lambda G \end{bmatrix} \left\{\begin{array}{c} \varepsilon_x \\ \gamma_{xz} \end{array}\right\} = \boldsymbol{D\varepsilon},$$

where E is the Young's modulus of the material, G is the shear modulus and λ is the shear correction factor. In the numerical experiments, $\lambda = \frac{5+5\nu}{6+5\nu}$ is used.

The middle line displacements are discretized as

$$u_0(x,t) = \sum_{i=1}^{p_u} \psi_i^u(x) q_i^u(t), \tag{5a}$$

$$w_0(x,t) = \sum_{i=1}^{p_w} \psi_i^w(x) q_i^w(t), \tag{5b}$$

$$\phi_y(x,t) = \sum_{i=1}^{p_{\phi_y}} \psi_i^{\phi_y}(x) q_i^{\phi_y}(t), \tag{5c}$$

where $\mathbf{q}^u(t)$, $\mathbf{q}^w(t)$ and $\mathbf{q}^{\phi_y}(t)$ are coefficient vectors depending on time t. Above, p_u, p_w and p_{ϕ_y} are the number of basis functions used for the discretization of each of the displacement components and $\boldsymbol{\psi}_i^u(x)$, $\boldsymbol{\psi}_i^w(x)$ and $\boldsymbol{\psi}_i^{\phi_y}(x)$ are the basis functions for space discretization. For the isogeometric case, these functions are B-spline basis functions, and for the p-FEM, these sets are hierarchical shape functions [11]. The basis functions must satisfy the geometric boundary conditions. The sets of B-splines and hierarchical shape functions used in the current work for the transverse displacement w_0 are shown in Fig. 1.

The equation of motion is derived by the principle of virtual work,

$$\delta W_V + \delta W_{in} + \delta W_E = 0, \tag{6}$$

where δW_V is the virtual work of internal forces, δW_{in} is the virtual work of inertia forces and δW_E is the virtual work of external forces. They are defined as

$$\delta W_V = -\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV, \quad \delta W_{in} = -\int_V \rho \delta\mathbf{d}^T \ddot{\mathbf{d}}\, dV, \quad \delta W_E = \int_V \delta\mathbf{d}^T \mathbf{P}\, dV, \tag{7}$$

where $\delta\boldsymbol{\varepsilon}$ is the vector of virtual strains, $\boldsymbol{\sigma}$ is the vector of stresses, ρ is the density, $\delta\mathbf{d}$ represents the vector of virtual displacements, $\ddot{\mathbf{d}}$ is the acceleration of a point of the beam and $\mathbf{P}$ is the vector of external forces. Inserting equations (7) into (6), we obtain the equation of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}(\mathbf{q}(t))\mathbf{q}(t) = \mathbf{f}(t), \tag{8}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{C}$ is the damping matrix, $\mathbf{K}$ is the stiffness matrix which depends on the coefficient vector $\mathbf{q}$, and $\mathbf{f}$ is the load vector. A mass proportional and frequency dependent damping is used. The longitudinal inertia is neglected and the degrees of freedom of the equation of motion are reduced, without neglecting the longitudinal displacement.

Since periodic responses are of interest, the coefficient vector is expressed in Fourier series and inserted into the equation of motion (8). The harmonic balance method (HBM) is employed and a nonlinear system of algebraic equations is obtained. The new unknowns are the coefficients of the harmonic functions, from the Fourier series, and the frequency of vibration. The nonlinear algebraic system is solved by an arc-length continuation method in frequency domain. Details about the application of the HBM and the continuation methods can be found, for example in [14, 10].
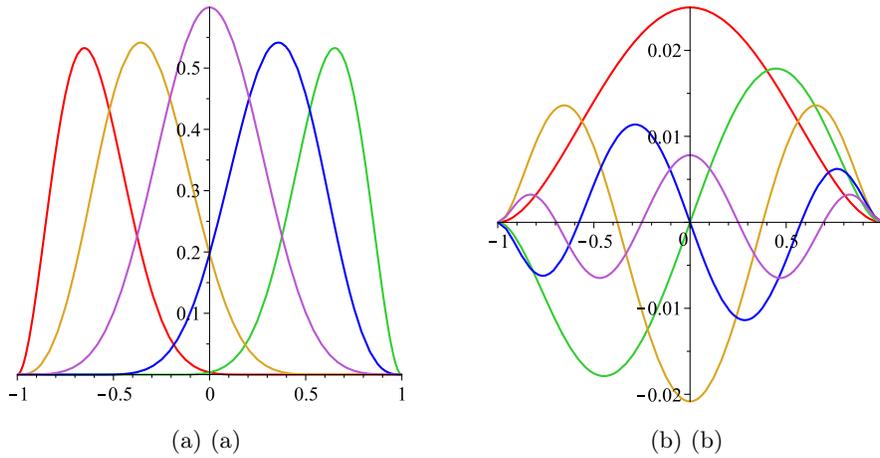
(a) (a)　　　　　　　　　　　　　(b) (b)

Fig. 1: (a) B-spline basis functions of degree 4; (b) hierarchical basis of shape functions.

4 Results

In this section, numerical results of Timoshenko beam, obtained by isogeometric analysis and by p-FEM, are compared. First, the natural frequencies are presented, and then, the nonlinear frequency-response functions are compared. The beam is assumed to be clamped-clamped with dimensions $l = 0.58m$, $b = 0.02m$, $h = 0.002m$, and isotropic material with properties (aluminum) $E = 70GPa$, $\nu = 0.34$, $\rho = 2778kg/m^3$.

Comparison of natural frequencies. It has been shown that the isogeometric analysis gives better approximation to the higher natural frequencies than the the FEM [6]. In this section, the natural frequencies of both approaches are compared, keeping the same number of degrees of freedom. The bending natural frequencies are presented in Table 1, where five functions are used for the transverse displacement w_0 from (5b), and the corresponding derivatives are used for the rotation of the cross section ϕ_y from (5c). In addition, the results are also compared with the conveged frequencies from [8].

B-splines of varying degree are considered in the isogeometric analysis. The p-FEM is used with hierarchical shape functions [13] and the beam is modeled as one finite element. It can be seen that the fourth degree B-splines yield much better approximations to the natural frequencies than those obtained by the p-FEM.

The longitudinal natural frequencies are presented in Table 2. Six functions are used for the longitudinal displacement given in (5a). In this case, quadratic B-splines are more suitable than higher order B-splines or p-FEM.

Table 1: Comparison of bending natural frequencies (rad/s) obtained by IGA and p-FEM, 10 DOF.

Mode	Converged values	ref. [8]	IGA p = 3	IGA p = 4	IGA p = 5	p-FEM
1	192.74	192.74	192.79	192.74	192.74	192.74
2	531.24	531.29	532.71	531.47	531.58	531.42
3	1041.27	1041.49	1055.76	1046.73	1041.68	1043.24
4	1720.99	1721.52	1795.28	1865.64	1947.05	1910.51
5	2570.24	2571.47	2668.90	2601.51	2758.14	3041.97

Table 2: Comparison of longitudinal natural frequencies (rad/s) obtained by IGA and p-FEM, 6 DOF.

Mode	Converged values	IGA p = 2	IGA p = 3	IGA p = 4	p-FEM
1	2.7190E+04	2.7191E+04	2.7190E+04	2.7190E+04	2.7190E+04
2	5.4379E+04	5.4436E+04	5.4385E+04	5.4381E+04	5.4379E+04
3	8.1569E+04	8.2106E+04	8.1714E+04	8.1818E+04	8.1728E+04
4	1.0876E+05	1.1156E+05	1.0955E+05	1.0877E+05	1.0947E+05
5	1.3595E+05	1.4415E+05	1.5600E+05	1.6433E+05	1.6214E+05
6	1.6314E+05	1.6421E+05	1.7369E+05	1.8766E+05	2.0673E+05

Comparison of nonlinear frequency-response functions. Taking into account the results from Tables 1 and 2, and considering that in nonlinear dynamics of elastic structures the higher modes of vibration become important, an isogeometric Timoshenko beam model is generated using five B-splines of degree four for the transverse displacement w_0 and six quadratic B-splines for the longitudinal displacement u_0. The total number of degrees of freedom is 16, and after neglecting the longitudinal inertia the model has 10 DOFs. A nonlinear model with the same number of DOFs is generated by the p-FEM, by using hierarchical set of shape functions. External harmonic force is applied on the middle of the beam in transverse direction, $F_z(t) = 0.035 \cos(\omega t) N$. Harmonics up to fifth order are used in the Fourier series, hence the algebraic nonlinear equation has 110 DOFs. The bifurcation diagrams of these models are presented and compared in Fig. 2. In addition, a model based on the isogeometric analysis but using B-splines of degree four for the longitudinal and transverse displacement is generated and included in the comparisons. The reference solution is obtained by p-FEM with 15 shape functions for each displacement component. It was verified that the results of this model are converged.

The nonlinear frequency-response function of the first harmonic is approximated very well by both methods, there is no visible difference. The difference appears in the third harmonic (Fig. 2(b)), where it is obvious that the isogeometric analysis yields a better approximation. The better approximation of the
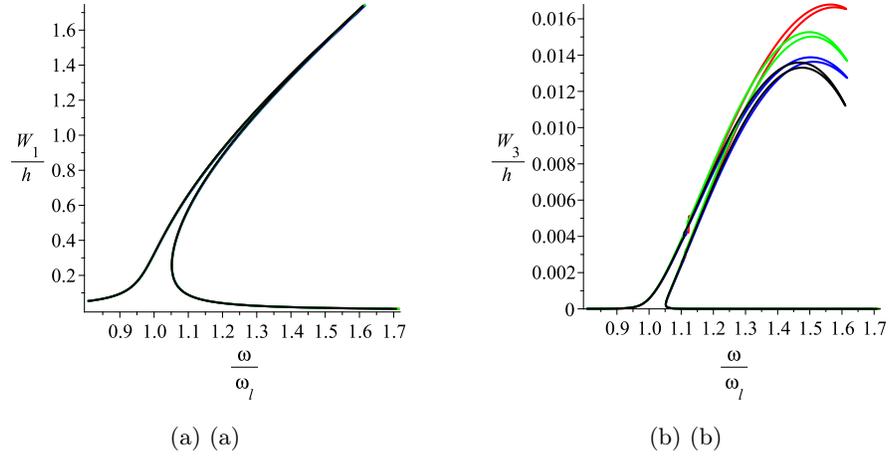
(a) (a) (b) (b)

Fig. 2: Comparison of nonlinear frequency-response functions obtained with 10 DOF. (a) amplitude of the first harmonic, (b) amplitude of the third harmonic; black: reference results, red: p-FEM, green: IGA, $p = 4$ for longitudinal and transverse displacements, blue: IGA, $p = 2$ for longitudinal and $p = 4$ for transverse displacements.
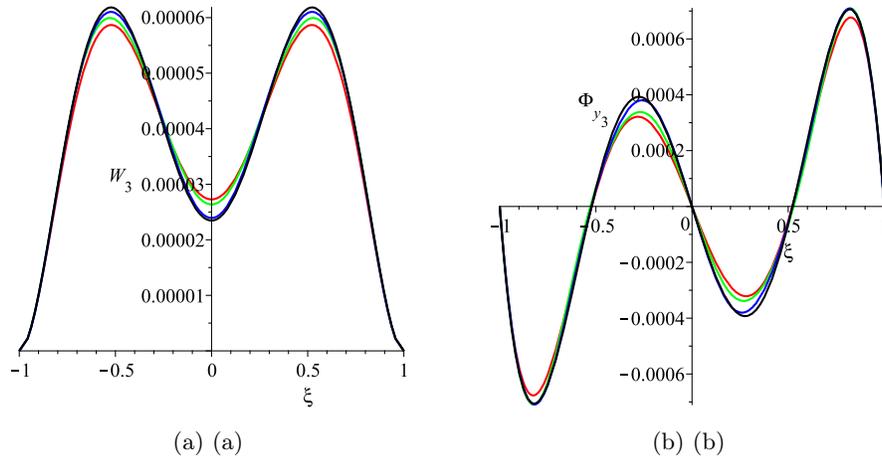

(a) (a) (b) (b)

Fig. 3: Comparison of shapes of vibration for point $\omega/\omega_l = 1.45$ from the bifurcation diagram. (a) transverse displacement w_0, (b) rotation of the cross section ϕ_y; black: reference results, red: p-FEM, green: IGA, $p = 4$ for longitudinal and transverse displacements, blue: IGA, $p = 2$ for longitudinal and $p = 4$ for transverse displacements.

nonlinear frequency-response function is consequence of the better approximation of the higher linear frequencies of vibration.

The shapes of vibration of the transverse displacement w_0 and the rotation of the cross section ϕ_y, for point $\omega/\omega_l = 1.45$ from the bifurcation diagram, are shown in Fig. 3.

Acknowledgments

References

1. D. J. Benson, Y. Bazilevs, M. C. Hsu, and T. J. R. Hughes. Isogeometric shell analysis: The Reissner-Mindlin shell. *Computer Methods in Applied Mechanics and Engineering*, 199(5-8):276–289, 2010.
2. J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*. John Wiley and Sons, Chichester, England, 2009.
3. J. A. Cottrell, A. Reali, Y. Bazilevs, and T. J. R. Hughes. Isogeometric analysis of structural vibrations. *Computer Methods in Applied Mechanics and Engineering*, 195(41–43):5257–5296, 2006. John H. Argyris Memorial Issue, Part II.
4. L. Beirão da Veiga, A. Buffa, C. Lovadina, M. Martinelli, and G. Sangalli. An isogeometric method for the Reissner-Mindlin plate bending problem. *Computer Methods in Applied Mechanics and Engineering*, 209-212(0):45–53, 2012.
5. T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering*, 194(39-41):4135–4195, October 2005.
6. T. J. R. Hughes, A. Reali, and G. Sangalli. Duality and unified analysis of discrete approximations in structural dynamics and wave propagation: Comparison of p-method finite elements with k-method NURBS. *Computer Methods in Applied Mechanics and Engineering*, 197(49–50):4104–4124, 2008.
7. G. Kerschen, M. Peeters, J.C. Golinval, and A.F. Vakakis. Nonlinear normal modes, part I: A useful framework for the structural dynamicist. *Mechanical Systems and Signal Processing*, 23(1):170–194, 2009. cited By (since 1996)118.
8. S. J. Lee and K. S. Park. Vibrations of Timoshenko beams with isogeometric approach. *Applied Mathematical Modelling*, 37(22):9174–9190, 2013.
9. L. Piegl and W. Tiller. *The NURBS Book*. Monographs in visual communications. Springer-Verlag GmbH, 1997.
10. J. Remmers C. Verhoosel R. de Borst, M. Crisfield. *Non-linear Finite Element Analysis of Solids and Structures*. Computational Mechanics. John Wiley and Sons Ltd., 2012.
11. P. Ribeiro and M. Petyt. Non-linear vibration of beams with internal resonance by the hierarchical finite-element method. *Journal of Sound and Vibration*, 224(4):591–624, 1999. cited By (since 1996)53.

12. L. Schumaker. *Spline Functions: Basic Theory*. Cambridge Mathematical Library. Cambridge University Press, 2007.

13. S. Stoykov and P. Ribeiro. Nonlinear free vibrations of beams in space due to internal resonance. *Journal of Sound and Vibration*, 330:4574–4595, 2011.

14. W. Szemplinska-Stupnicka. *The Behavior of Nonlinear Vibrating Systems*, volume 12 of *Computational Mechanics*. Kluwer Academic Publishers, 1990.

15. O. Weeger, U. Wever, and B. Simeon. Isogeometric analysis of nonlinear Euler-Bernoulli beam vibrations. *Nonlinear Dynamics*, 72(4):813–835, 2013.