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On a new mixed formulation of Kirchhoff plates on curvilinear polygonal domains

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Abstract. For Kirchhoff plate bending problems on domains whose boundaries are curvilinear polygons a discretization method based on the consecutive solution of three second-order problems is presented.

In Rafetseder and Zulehner (preprint, arXiv:1703.07962) a new mixed variational formulation of this problem is introduced using a nonstandard Sobolev space (and an associated regular decomposition) for the bending moments. In case of a polygonal domain the coupling condition for the two components in the decomposition can be interpreted as standard boundary conditions, which allows for an equivalent reformulation as a system of three (consecutively to solve) second-order elliptic problems.

The extension of this approach to curvilinear polygonal domains poses severe difficulties. Therefore, we propose in this paper an alternative approach based on Lagrange multipliers.

1 The Kirchhoff plate bending problem

We consider the Kirchhoff plate bending problem, where the undeformed mid-surface is described by a domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary Γ. The plate is considered to be clamped on a part $\Gamma_c \subset \Gamma$, simply supported on $\Gamma_s \subset \Gamma$, and free on $\Gamma_f \subset \Gamma$ with $\Gamma = \Gamma_c \cup \Gamma_s \cup \Gamma_f$. Furthermore, $n = (n_1, n_2)^T$ and $t = (-n_2, n_1)^T$ represent the unit outer normal vector and the unit counterclockwise tangent vector to Γ, respectively.

Then the problem reads: For given load f, find a deflection w such that

$$\operatorname{div} \operatorname{Div} \left(\mathcal{C} \nabla^2 w\right) = f \quad \text{in } \Omega, \tag{1}$$

where div denotes the standard divergence of a vector-valued function, Div the row-wise divergence of a matrix-valued function, ∇^2 the Hessian, and $\mathcal{C}$ a fourth-order material tensor. The boundary conditions are given by

$$\begin{aligned}
&w = 0, && \partial_n w = 0 && \text{on } \Gamma_c,\\
&w = 0, && \boldsymbol{M}n \cdot n = 0 && \text{on } \Gamma_s,\\
&\boldsymbol{M}n \cdot n = 0, && \partial_t(\boldsymbol{M}n \cdot t) + \operatorname{Div} \boldsymbol{M} \cdot n = 0 && \text{on } \Gamma_f,
\end{aligned}$$

and the corner conditions

$$[\![\boldsymbol{M}_{nt}]\!]_x = (\boldsymbol{M}n_1 \cdot t_1)(x) - (\boldsymbol{M}n_2 \cdot t_2)(x) = 0 \quad \text{for all } x \in \mathcal{V}_{\Gamma,f},$$

where $\boldsymbol{M}$ denotes the bending moment tensor, given by $\boldsymbol{M} = -\mathcal{C}\nabla^2 w$, and $\mathcal{V}_{\Gamma,f}$ denotes the set of corner points whose two adjacent edges (with corresponding normal and tangent vectors n_1, t_1 and n_2, t_2) belong to Γ_f.

As an example, the material tensor $\mathcal{C}$ for isotropic materials is given by

$$\mathcal{C}\boldsymbol{N} = D\big((1-\nu)\boldsymbol{N} + \nu\ \mathrm{tr}(\boldsymbol{N})\boldsymbol{I}\big), \tag{2}$$

for matrices $\boldsymbol{N}$, where ν is the Poisson ration, $D > 0$ depends on ν, Young's modulus, and the thickness of the plate, $\boldsymbol{I}$ is the identity matrix, and tr is the trace operator for matrices.

A standard (primal) variational formulation of (1) is given as follows: Find $w \in W$ such that

$$\int_\Omega \mathcal{C}\nabla^2 w : \nabla^2 v\ dx = \langle F, v\rangle \quad \text{for all } v \in W, \tag{3}$$

with the Frobenius inner product $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j} \boldsymbol{A}_{ij}\, \boldsymbol{B}_{ij}$ for matrices $\boldsymbol{A}, \boldsymbol{B}$, the right-hand side $\langle F, v\rangle = \int_\Omega f\ v\ dx$, and the function space

$$W = \{v \in H^2(\Omega) :\ v = 0,\ \partial_n v = 0 \text{ on } \Gamma_c, \quad v = 0 \text{ on } \Gamma_s\}. \tag{4}$$

Here and throughout the paper $L^2(\Omega)$ and $H^m(\Omega)$ denote the standard Lebesgue and Sobolev spaces of functions on Ω with corresponding norms $\|.\|_0$ and $\|.\|_m$ for positive integers m. Moreover, $H^1_{0,\Gamma'}(\Omega)$ denotes the set of function in $H^1(\Omega)$ which vanish on a part Γ' of Γ. The L^2-inner product on Ω and Γ' are always denoted by $(.,.)$ and $(.,.)_{\Gamma'}$, respectively, no matter whether it is used for scalar, vector-valued, or matrix-valued functions. We use H^* to denote the dual of a Hilbert space H and $\langle .,.\rangle$ for the duality product on $H^* \times H$.

2 New mixed formulation

In our previous work [4] a new mixed variational formulation for the Kirchhoff plate bending problem with the bending moment tensor $\boldsymbol{M}$ as additional unknown is derived. The new mixed formulation satisfies Brezzi's conditions and is equivalent to the original problem without additional convexity assumption on Ω. These important properties come at the expense of an appropriate nonstandard Sobolev space for $\boldsymbol{M}$. In order to make this space computationally accessible, we show in [4, Theorem 4.2] a regular decomposition of it, which provides the following representation of the solution $\boldsymbol{M}$

$$\boldsymbol{M} = p\boldsymbol{I} + \mathrm{symCurl}\,\phi,$$

with $p \in Q = H^1_{0,\Gamma_c\cup\Gamma_s}(\Omega)$ and $\phi \in (H^1(\Omega))^2$ satisfying the coupling condition

$$\langle \partial_t \phi, \nabla v\rangle_\Gamma = -\int_\Gamma p\ \partial_n v\ ds \quad \text{for all } v \in W, \tag{5}$$

where $\partial_t \phi = (\operatorname{Curl} \phi) n \in (H^{-\frac{1}{2}}(\Gamma))^2$ with $H^{-\frac{1}{2}}(\Gamma) = (H^{\frac{1}{2}}(\Gamma))^*$. Here the symmetric Curl is defined as $\operatorname{symCurl} \psi = \frac{1}{2}(\operatorname{Curl} \psi + (\operatorname{Curl} \psi)^T)$ with

$$\operatorname{Curl} \psi = \begin{pmatrix} \partial_2 \psi_1 & -\partial_1 \psi_1 \\ \partial_2 \psi_2 & -\partial_1 \psi_2 \end{pmatrix}.$$

The analogous representation for the test functions associated to $\boldsymbol{M}$ in the mixed formulation leads to the following equivalent formulation of (3): For $F \in Q^*$, find $(p, \phi) \in V$ and $w \in Q$ such that

$$\begin{aligned} (p\boldsymbol{I} + \operatorname{symCurl} \phi, q\boldsymbol{I} + \operatorname{symCurl} \psi)_{\mathcal{C}^{-1}} - (\nabla w, \nabla q) &= 0, \\ -(\nabla p, \nabla v) &= -\langle F, v \rangle, \end{aligned} \tag{6}$$

for all $v \in Q = H^1_{0,\Gamma_c \cup \Gamma_s}(\Omega)$ and $(q, \psi) \in V$, where the function space V is given as the subset of $(q, \psi) \in Q \times (H^1(\Omega))^2$ satisfying

$$\langle \partial_t \psi, \nabla v \rangle_\Gamma = -\int_\Gamma q \, \partial_n v \, ds \quad \text{for all } v \in W. \tag{7}$$

Here, we use the notation $(\boldsymbol{M}, \boldsymbol{N})_{\mathcal{C}^{-1}} = (\mathcal{C}^{-1}\boldsymbol{M}, \boldsymbol{N})$.

2.1 Coupling condition as standard boundary conditions for ϕ

In [4] we obtain for polygonal domains Ω an equivalent formulation of the Kirchhoff plate bending problem (3) in terms of three (consecutively to solve) second-order elliptic problems:

1. The p-problem: Find $p \in Q$ such that
$$(\nabla p, \nabla v) = \langle F, v \rangle \quad \text{for all } v \in Q.$$
2. The ϕ-problem: For given $p \in Q$, find $\phi \in \Psi_p = \psi[p] + \Psi_0$ such that
$$(\operatorname{symCurl} \phi, \operatorname{symCurl} \psi_0)_{\mathcal{C}^{-1}} = -(p\boldsymbol{I}, \operatorname{symCurl} \psi_0)_{\mathcal{C}^{-1}} \quad \text{for all } \psi_0 \in \Psi_0.$$
3. The w-problem: For given $\boldsymbol{M} = p\boldsymbol{I} + \operatorname{symCurl} \phi$, find $w \in Q$ such that
$$(\nabla w, \nabla q) = (\boldsymbol{M}, q\boldsymbol{I} + \operatorname{symCurl} \psi[q])_{\mathcal{C}^{-1}} \quad \text{for all } q \in Q.$$

The second and the third problem require the construction of a particular function $\psi[q]$ satisfying the coupling condition (7) for given $q \in Q$, for details see [4]. The space Ψ_0 consists of all functions in $(H^1(\Omega))^2$ satisfying (7) for $q = 0$.

The approach presented in [4] is to characterize Ψ_0 as space of functions ψ with standard boundary conditions available in $(H^1(\Omega))^2$. Originally, the boundary conditions for $\psi \in \Psi_0$ are, roughly speaking, conditions for tangential derivatives of ψ of the form

$$\partial_t \psi \cdot n = 0 \qquad \text{on } \Gamma_s, \tag{8}$$

$$\partial_t^2 \psi \cdot t = 0, \quad \partial_t \psi \cdot n = 0 \qquad \text{on } \Gamma_f. \tag{9}$$

For polygonal domains we obtain from (9) a Dirichlet boundary condition for ψ. Moreover, (8) yields a Dirichlet boundary condition for the normal component $\psi \cdot n$. However, the considerations heavily rely on a polygonal domain and it is not clear how to obtain standard boundary conditions in the curved case. This is our main motivation to investigate an alternative approach to incorporate the coupling condition (7) based on Lagrange multipliers, which we introduce in the next section.

In [4] we propose a discretization method for the above introduced formulation using a Nitsche method to incorporate the boundary conditions in the ϕ-problem and present a numerical analysis of the method.

3 Coupling condition via Lagrange multipliers

We consider a domain Ω, whose boundary is a curvilinear polygon of class C^∞. This means that $\Gamma = \bigcup_{k=1}^{K} \overline{E}_k$, where the edges E_k are C^∞ curves for $k = 1, 2, \dots, K$ and $\overline{E}_k$ denotes the closure of E_k. The edges are numbered consecutively in counterclockwise direction. We denote the vertex at the endpoint of $\overline{E}_k$ by a_k and the interior angle at a_k by ω_k. Note, since we consider a closed boundary curve, the index $k = 0$ is in the following always identified with $k = K$.

Furthermore, we assume that each edge E_k is contained in exactly one of the sets Γ_c, Γ_s, Γ_f, and the edges are maximal in the sense that two edges with the same boundary condition do not meet at an angle of π.

By using the representation $\nabla v = (\partial_n v)\, n + (\partial_t v)\, t$ and incorporating the boundary conditions for $v \in W$, the coupling condition (5) reads

$$(\partial_t \phi \cdot n + p, \partial_n v)_{\Gamma_s \cup \Gamma_f} + (\partial_t \phi \cdot t, \partial_t v)_{\Gamma_f} = 0 \quad \text{for all } v \in W,$$

provided $\partial_t \phi \in L^2(\Gamma)$. We can rewrite the condition as follows

$$\sum_{E_k \subset \Gamma_s \cup \Gamma_f} (\partial_t \phi \cdot n \; + \; p, \mu_n^k)_{E_k} + \sum_{E_k \subset \Gamma_f} (\partial_t \phi \cdot t, \mu_t^k)_{E_k} = 0, \tag{10}$$

for all $\mu = ((\mu_t^1, \mu_t^2, \dots, \mu_t^K), (\mu_n^1, \mu_n^2, \dots, \mu_n^K)) \in \Lambda$ where

$$\Lambda = \{(\partial_t v, \partial_n v) \colon \text{for } v \in W\},$$

with

$$\partial_t v = (\partial_t v|_{E_1}, \partial_t v|_{E_2}, \dots, \partial_t v|_{E_K}), \quad \partial_n v = (\partial_n v|_{E_1}, \partial_n v|_{E_2}, \dots, \partial_n v|_{E_K}).$$

We view the original formulation (6) as optimality system with constraint $(\nabla p, \nabla v) = \langle F, v \rangle$ and replace the space V by $Q \times (H^1(\Omega))^2$ and add (10) as additional constraint. The corresponding optimality system is the starting point for the discretization method we introduce in Sect. 3.2.

3.1 Characterization of Λ

In this subsection we provide an explicit characterization of Λ. Let us consider $\mu = ((\mu_t^1, \mu_t^2, \dots, \mu_t^K), (\mu_n^1, \mu_n^2, \dots, \mu_n^K))$, where μ_t^k and μ_n^k for $k = 1, 2, \dots, K$ are Lipschitz continuous functions on $\overline{E}_k$. Then $\mu \in \Lambda$ if and only if the following three conditions are satisfied:

1. The boundary conditions $\mu_t^k = 0$ on edges $E_k \subset \Gamma_s \cup \Gamma_c$ and $\mu_n^k = 0$ on edges $E_k \subset \Gamma_c$ have to hold.
2. On each connected component C of Γ_f the compatibility condition
$$\sum_{E_k \subset C} \int_{E_k} \mu_t^k \, ds = 0$$
has to be satisfied.
3. The four corner values $\mu_t^{k-1}(a_k), \mu_t^k(a_k), \mu_n^{k-1}(a_k), \mu_n^k(a_k)$ have to be coupled appropriately. For the case $\omega_k \neq \pi$, the conditions are given by
$$\begin{aligned} \mu_t^{k-1}(a_k) + \cos\omega_k \ \mu_t^k(a_k) - \sin\omega_k \ \mu_n^k(a_k) &= 0, \\ \cos\omega_k \ \mu_t^{k-1}(a_k) + \sin\omega_k \ \mu_n^{k-1}(a_k) + \mu_t^k(a_k) &= 0, \end{aligned} \tag{11}$$
for all $k = 1, 2, \dots, K$. These conditions follow as special case from [3, Theorem 1.5.2.8].

   Remark 1. In order to describe a change of boundary condition we may also consider an interior angle ω_k of π. A corresponding adaption of the conditions (11) can be found in [3].

In the following we fix a corner a_k and work out the relation implied by the corresponding boundary conditions and the conditions (11) for the four involved quantities $\mu_t^{k-1}(a_k)$, $\mu_t^k(a_k)$, $\mu_n^{k-1}(a_k)$, $\mu_n^k(a_k)$, where we skip in the following the argument a_k for better readability. We distinguish three situations:

1. Let a_k be an interior corner point of Γ_f. Then the conditions (11) lead to
$$\mu_n^{k-1} = -\frac{1}{\sin w_k}(\cos w_k \ \mu_t^{k-1} + \mu_t^k), \quad \mu_n^k = \frac{1}{\sin w_k}(\mu_t^{k-1} + \cos w_k \ \mu_t^k),$$
for arbitrary μ_t^{k-1} and μ_t^k.
2. Let a_k be a corner point on the interface of $E_{k-1} \subset \Gamma_s$ and $E_k \subset \Gamma_f$. Then the conditions (11) provide
$$\mu_t^{k-1} = 0, \quad \mu_n^{k-1} = -\frac{1}{\sin w_k}\mu_t^k, \quad \mu_n^k = \frac{1}{\sin w_k}\cos w_k \ \mu_t^k,$$
where μ_t^k can be freely chosen. For the reverse case $E_{k-1} \subset \Gamma_f$ and $E_k \subset \Gamma_s$ an analogous result holds.
3. In all other cases, we obtain $\mu_t^{k-1} = \mu_t^k = \mu_n^{k-1} = \mu_n^k = 0$.

3.2 The discretization method

Let $\mathcal{S}_h(\Omega)$ be a finite dimensional subspace of $H^1(\Omega)$ of piecewise polynomials (with respect to a subdivision of Ω) and we set $\mathcal{S}_{h,0}(\Omega) = \mathcal{S}_h(\Omega) \cap H^1_{0,\Gamma_c \cup \Gamma_s}(\Omega)$. The restriction of functions from $\mathcal{S}_h(\Omega)$ to E_k is defined as

$$\mathcal{S}_h(E_k) = \{v|_{E_k} \colon v \in \mathcal{S}_h(\Omega)\}.$$

The discrete space Λ_h consists of all $\mu_h = ((\mu_t^1, \mu_t^2, \ldots, \mu_t^K), (\mu_n^1, \mu_n^2, \ldots, \mu_n^K))$, where $\mu_t^k \in \mathcal{S}_h(E_k)$ and $\mu_n^k \in \mathcal{S}_h(E_k)$ for $k = 1, 2, \ldots, K$, subject to the constraints derived in Sect. 3.1.

In the discrete setting the original formulation (6) is equivalent to three (consecutively to solve) second-order problems:

1. The discrete p-problem: Find $p_h \in \mathcal{S}_{h,0}(\Omega)$ such that
$$(\nabla p_h, \nabla v_h) = \langle F, v_h \rangle \quad \text{for all } v_h \in \mathcal{S}_{h,0}(\Omega).$$
2. The discrete $(\phi,\, \lambda)$-problem:
For given $p_h \in \mathcal{S}_{h,0}(\Omega)$, find $\phi_h \in (\mathcal{S}_h(\Omega))^2/RT_0$ and $\lambda_h \in \Lambda_h$ such that
$$\begin{aligned} (\operatorname{symCurl}\phi_h, \operatorname{symCurl}\psi_h)_{\mathcal{C}^{-1}} + l_\phi(\psi_h, \lambda_h) &= -(p_h \boldsymbol{I}, \operatorname{symCurl}\psi_h)_{\mathcal{C}^{-1}} \\ l_\phi(\phi_h, \mu_h) &= -l_p(p_h, \mu_h), \end{aligned}$$
for all $\psi_h \in (\mathcal{S}_h(\Omega))^2/RT_0$ and $\mu_h \in \Lambda_h$, where
$$\begin{aligned} l_\phi(\phi, \mu) &= \sum_{E_k \subset \Gamma_s \cup \Gamma_f} (\partial_t \phi \cdot n, \mu_n^k)_{E_k} + \sum_{E_k \subset \Gamma_f} (\partial_t \phi \cdot t, \mu_t^k)_{E_k}, \\ l_p(p, \mu) &= \sum_{E_k \subset \Gamma_f} (p, \mu_n^k)_{E_k}, \end{aligned}$$
for $\mu = ((\mu_t^1, \mu_t^2, \ldots, \mu_t^K), (\mu_n^1, \mu_n^2, \ldots, \mu_n^K))$. Here, we use the notation $RT_0 = \{ax + b \colon a \in \mathbb{R},\ b \in \mathbb{R}^2\}$.
3. The discrete w-problem: For given $\boldsymbol{M}_h = p_h \boldsymbol{I} + \operatorname{symCurl}\phi_h$ and $\lambda_h \in \Lambda_h$, find $w_h \in \mathcal{S}_{h,0}(\Omega)$ such that
$$(\nabla w_h, \nabla q_h) = (\boldsymbol{M}_h, q_h \boldsymbol{I})_{\mathcal{C}^{-1}} + l_p(q_h, \lambda_h) \quad \text{for all } q_h \in \mathcal{S}_{h,0}(\Omega).$$

In comparison with the decoupled formulation in Sect. 2.1, here the second problem, the $(\phi,\, \lambda)$-problem, is a saddle point problem.

4 Numerical tests

As discretization space $\mathcal{S}_h(\Omega)$ we consider B-splines of degree $p \geq 1$ with maximum smoothness; see, e.g, [2,6] for more information on this space in the context of isogeometric analysis (IGA). A sparse direct solver is used for each of the three sub-problems. The implementation is done in the framework of the object-oriented C++ library G+Smo (”Geometry + Simulation Modules”) [1].

[1] `https://ricamsvn.ricam.oeaw.ac.at/trac/gismo/wiki/WikiStart`

4.1 Square plate with clamped, simply supported and free boundary

We consider a square plate $\Omega = (-1,1)^2$ with simply supported north and south boundaries, clamped west boundary and free east boundary. The material tensor $\mathcal{C}$ is given as in (2) with $D = 1$, $\nu = 0$ and the load is $f(x,y) = 4\pi^4 \sin(\pi x)\sin(\pi y)$. The exact solution is written in the form

$$w(x,y) = \big((a+bx)\cosh(\pi x) + (c+dx)\sinh(\pi x) + \sin(\pi x)\big)\sin(\pi y),$$

which automatically satisfies the boundary conditions on the simply supported boundary parts. The constants a, b, c and d are chosen such that the four remaining boundary conditions (on the clamped and free boundary parts) are satisfied, for details, see [5]. In Table 1 and Table 2 the discretization errors for $p = 1, 3$ are presented. The first column shows the refinement level L, the next three pairs of columns show the respective discretization error and the error reduction relative the previous level. The results show optimal convergence rates for w and $\boldsymbol{M}$.

Table 1. Discretization errors for square plate, $p = 1$

L	$\|w - w_h\|_0$	order	$\|w - w_h\|_1$	order	$\|\boldsymbol{M} - \boldsymbol{M}_h\|_0$	order
4	$2.82 \cdot 10^{-2}$	1.909	$6.83 \cdot 10^{-1}$	0.992	$2.83 \cdot 10^{0}$	0.975
5	$7.17 \cdot 10^{-3}$	1.976	$3.42 \cdot 10^{-1}$	0.998	$1.42 \cdot 10^{0}$	0.993
6	$1.80 \cdot 10^{-3}$	1.994	$1.71 \cdot 10^{-1}$	0.999	$7.13 \cdot 10^{-1}$	0.998
7	$4.50 \cdot 10^{-4}$	1.998	$8.55 \cdot 10^{-2}$	0.999	$3.56 \cdot 10^{-1}$	0.999

Table 2. Discretization errors for square plate, $p = 3$

L	$\|w - w_h\|_0$	order	$\|w - w_h\|_1$	order	$\|\boldsymbol{M} - \boldsymbol{M}_h\|_0$	order
4	$5.47 \cdot 10^{-5}$	4.147	$2.75 \cdot 10^{-3}$	3.070	$1.10 \cdot 10^{-2}$	3.104
5	$3.41 \cdot 10^{-6}$	4.001	$3.46 \cdot 10^{-4}$	2.993	$1.38 \cdot 10^{-3}$	2.989
6	$2.15 \cdot 10^{-7}$	3.984	$4.37 \cdot 10^{-5}$	2.985	$1.75 \cdot 10^{-4}$	2.978
7	$1.35 \cdot 10^{-8}$	3.989	$5.50 \cdot 10^{-6}$	2.989	$2.22 \cdot 10^{-5}$	2.985

4.2 Circular plate with simply supported boundary

As a second example, we consider the simply supported circular plate with radius $r = 1$ and uniform loading $f = 1$. The material tensor $\mathcal{C}$ is given as in (2) with $D = 1$ and $\nu = 0.3$. The exact solution is given by $w(x) =$

$c_1 + c_2 r^2 + c_3 r^4$ where $r^2 = x_1^2 + x_2^2$, $c_3 = 1/64$ and c_1, c_2 are determined from the boundary conditions. For this test reproducing the exact geometry is essential, see the discussion of the so-called Babuška paradox in [1]. Therefore, we use an exact geometry representation by means of non-uniform rational B-splines (NURBS). In Table 3 and Table 4 the discretization errors for $p = 1, 3$ are presented. The results show optimal convergence rates for w and $\boldsymbol{M}$.

Table 3. Discretization errors for circular plate, $p = 1$

L	$\|w - w_h\|_0$	order	$\|w - w_h\|_1$	order	$\|\boldsymbol{M} - \boldsymbol{M}_h\|_0$	order
4	$3.58 \cdot 10^{-4}$	1.984	$8.37 \cdot 10^{-3}$	1.002	$8.79 \cdot 10^{-3}$	1.020
5	$8.98 \cdot 10^{-5}$	1.996	$4.18 \cdot 10^{-3}$	1.000	$4.38 \cdot 10^{-3}$	1.005
6	$2.24 \cdot 10^{-5}$	1.999	$2.09 \cdot 10^{-3}$	1.000	$2.18 \cdot 10^{-3}$	1.001
7	$5.62 \cdot 10^{-6}$	1.999	$1.04 \cdot 10^{-3}$	1.000	$1.09 \cdot 10^{-3}$	1.000

Table 4. Discretization errors for circular plate, $p = 3$

L	$\|w - w_h\|_0$	order	$\|w - w_h\|_1$	order	$\|\boldsymbol{M} - \boldsymbol{M}_h\|_0$	order
4	$4.05 \cdot 10^{-7}$	4.319	$1.93 \cdot 10^{-5}$	3.163	$2.01 \cdot 10^{-5}$	3.206
5	$2.38 \cdot 10^{-8}$	4.083	$2.35 \cdot 10^{-6}$	3.034	$2.42 \cdot 10^{-6}$	3.054
6	$1.47 \cdot 10^{-9}$	4.019	$2.93 \cdot 10^{-7}$	3.004	$3.00 \cdot 10^{-7}$	3.013
7	$9.17 \cdot 10^{-11}$	4.004	$3.67 \cdot 10^{-8}$	2.999	$3.74 \cdot 10^{-8}$	3.003

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