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Approximation Error Estimates and Inverse Inequalities for B-Splines of Maximum Smoothness

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Approximation error estimates and inverse inequalities for B-splines of maximum smoothness

Stefan Takacs, Thomas Takacs

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Abstract In this paper, we will give approximation error estimates as well as corresponding inverse inequalities for B-splines of maximum smoothness, where both the function to be approximated and the approximation error are measured in standard Sobolev norms and semi-norms. The presented approximation error estimates do not depend on the polynomial degree of the splines but only on the mesh size.

We will see that the approximation lives in a subspace of the classical B-spline space. We show that for this subspace, there is an inverse inequality which is also independent of the polynomial degree. As the approximation error estimate and the inverse inequality show complementary behavior, the results shown in this paper can be used to construct fast iterative methods for solving problems arising from isogeometric discretizations of partial differential equations.

1 Introduction

The objective of this paper is to prove approximation error estimates as well as corresponding inverse estimates for B-splines of maximum smoothness. The presented approximation error estimates do not depend on the degree of the splines but only on the mesh size. All bounds are given in terms of classical Sobolev norms and semi-norms.

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In approximation theory, B-splines have been studied for a long time and many properties are already well known. We do not go into the details of the existing results but present the results of importance for our study throughout this paper.

The emergence of Isogeometric Analysis, introduced in [12], sparked new interest in the theoretical properties of B-splines. Since isogeometric Galerkin methods are aimed at solving variational formulations of differential equations, approximation properties measured in Sobolev norms need to be studied.

The results presented in this paper improve the results given in [14,8,1] by explicitly studying the dependence on the polynomial degree p. Such an analysis was done in [2]. However, the results there, do not cover (for $p > 3$) the most important case of B-splines of maximum smoothness $k = p - 1$. Unlike the aforementioned papers we only consider approximation with B-splines in the parameter domain within the framework of Isogeometric Analysis. A generalization of the results to NURBS as well as the introduction of a geometry mapping, as presented in [1], is straightforward and does not lead to any additional insight.

Note that a detailed study of direct and inverse estimates may lead to a deeper understanding of isogeometric multigrid methods and give insight to suitable preconditioning methods. We refer to [11,9], where similar techniques were used.

We now go through the main results of this paper. For simplicity, we consider the case of one dimension first. Here, without loss of generality, we assume that $\Omega = (0, 1)$.

For this domain we can introduce for any $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ a uniform grid $\mathcal{M}_n$ by subdividing Ω into subintervals (elements) of length $h_n := \frac{1}{n}$. On these grids we can introduce spaces of spline functions.

Definition 1 $S_{p,k,n}(\Omega)$ is the space of all spline functions in $C^k(\Omega)$, which are piecewise polynomials of degree p on the mesh $\mathcal{M}_n$, i.e. polynomials of degree p on each element of the mesh.

Here and in what follows, $C^0(\Omega)$ is the space of all continuous functions mapping $\Omega \to \mathbb{R}$. For $k > 0$, $C^k(\Omega)$ is the space of all k times continuously differentiable functions. For completeness, we have to define also $C^{-1}(\Omega)$ as the space of all Riemann-integrable functions $\Omega \to \mathbb{R}$.

The main result of this paper is the following.

Theorem 1 *For each $u \in H^1(\Omega)$, each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$, there is a spline approximation $u_{p,p-1,n} \in S_{p,p-1,n}(\Omega)$ such that*

$$\|u - u_{p,p-1,n}\|_{L^2(\Omega)} \leq 2\sqrt{2}\, h_n |u|_{H^1(\Omega)}$$

is satisfied.

Here and in what follows, L^2 is the standard Lebesgue space of square integrable function and H^r denotes the standard Sobolev space of order $r \geq 0$ with standard norms and semi-norms.

Remark 1 Obviously $S_{p,k,n}(\Omega) \supseteq S_{p,p-1,n}(\Omega)$ for all $0 \leq k \leq p-1$. So, Theorem 1 is also valid in that case. However, for this case there might be better estimates for these larger B-spline spaces. Moreover, Theorem 1 is also satisfied in the case of having repeated knots, as this is just a local reduction of the continuity (which just enlarges the corresponding space of spline functions).

As we are mostly interested in the case $k = p-1$, here and in what follows, we will use $S_{p,n}(\Omega) := S_{p,p-1,n}(\Omega)$.

Below, we will introduce a subspace $\widetilde{S}_{p,n}(\Omega) \subseteq S_{p,n}(\Omega)$ (cf. Definition 3) and show that the spline approximation is even in the space $\widetilde{S}_{p,n}(\Omega)$ (cf. Corollary 1). Note that for $\widetilde{S}_{p,n}(\Omega)$ there is also a corresponding inverse inequality.

Theorem 2 *For each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$,*

$$|u_{p,n}|_{H^1(\Omega)} \leq 2\sqrt{3} h_n^{-1} \|u_{p,n}\|_{L^2(\Omega)} \tag{1}$$

is satisfied for all $u_{p,n} \in \widetilde{S}_{p,n}(\Omega)$.

Remark 2 This inverse inequality does not extend to the whole space $S_{p,n}(0,1)$. Here it is easy to find a counterexample: The function $u_{p,n}$, given by

$$u_{p,n}(x) = \begin{cases} (1 - x/h_n)^p & \text{for } x \in [0, h_n) \\ 0 & \text{for } x \in [h_n, 1], \end{cases}$$

is a member of the space $S_{p,n}(0,1)$. Straight-forward computations yield

$$\frac{|u_{p,n}|_{H^1(0,1)}}{\|u_{p,n}\|_{L^2(0,1)}} = \sqrt{\frac{2p+1}{2p-1}}\, p\, h_n^{-1},$$

which cannot be bounded from above by a constant times h_n^{-1} uniformly in p.

Will moreover show that both the approximation error estimate and the inverse inequality are sharp up to constants (Corollaries 2 and 3).

The remainder of this paper is organized as follows. In Section 2, we give the necessary definitions and briefly discuss the known approximation error results. The Sections 3 and 4 are dedicated to the proof of the approximation error estimate (Theorem 1). The inverse inequality (Theorem 2) will be proven in Section 5. In the following two sections, we generalize those results: In Section 6 we consider higher Sobolev indices and in Section 7, the results are generalized to two or more dimensions.

2 Known results and spline spaces

In this section, we first recall some known results on spline approximation. In the second subsection, we give the definitions that are needed throughout the paper and, moreover, we extend some known results to the periodic case.

2.1 Known approximation results

We start with a well-known approximation error estimate which relates – for a fixed polynomial degree p and a fixed smoothness k – the approximation error and the grid size. The result is well-known in literature, cf. [14], Theorem 6.25 or [8], Theorem 7.3. In the framework of Isogeometric Analysis, such results have been used, e.g., in [1], Lemma 3.3.

Theorem 3 *For each $r \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$, each $k \in \mathbb{N}_0$, each $q \in \mathbb{N}$ and each $p \in \mathbb{N}$, with $0 \le r \le q \le p+1$ and $r-1 \le k < p$, there is for each $u \in H^q(\Omega)$ a spline approximation $u_{p,k,n} \in S_{p,k,n}(\Omega)$ such that*

$$|u - u_{p,k,n}|_{H^r(\Omega)} \le C(p,k,r,q) h_n^{q-r} |u|_{H^q(\Omega)}$$

is satisfied, where $C(p,k,r,q)$ is a constant that might depend on p, k, r and q. $C(p,k,r,q)$ is independent of the gird size h_n.

This lemma is valid for tensor-product spaces in any dimension and gives a local bound for locally quasi-uniform knot vectors. However, the dependence of the constant on the polynomial degree has not been derived.

A major step towards p-dependent estimates was presented in [2], Theorem 2, where an estimate with an explicit dependence on p, k, r and q was given. However, there the continuity is limited by the upper bound $\frac{1}{2}(p-1)$. In our notation, the theorem reads as follows.

Theorem 4 *For each $r \in \mathbb{N}_0$, each $k \in \mathbb{N}_0$, each $q \in \mathbb{N}$ and each $p \in \mathbb{N}$ with $0 \le r \le k+1 \le q \le p+1$ and $k \le \frac{1}{2}(p-1)$, there is for each $u \in H^q(\Omega)$ a spline approximation $u_{p,k,n} \in S_{p,k,n}(\Omega)$ such that*

$$|u - u_{p,k,n}|_{H^r(\Omega)} \le C h_n^{q-r} (p-k)^{-(q-r)} |u|_{H^q(\Omega)}$$

is satisfied, where C is a constant that is independent of p, k, r, q and the grid size h_n.

Remark 3 The requirement $q \ge k+1$ can be dropped easily. First note that $\Pi^{(r)}_{p,k,n}$, the H^r-orthogonal projection operator from $H^r(\Omega)$ to $S_{p,k,n}(\Omega)$, is the optimal interpolation and satisfies both,

$$|u - \Pi^{(r)}_{p,k,n} u|_{H^r(\Omega)} \le C h_n^{q-r} (p-k)^{-(q-r)} |u|_{H^q(\Omega)} \text{ and}$$
$$|u - \Pi^{(r)}_{p,k,n} u|_{H^r(\Omega)} \le |u|_{H^r(\Omega)}.$$

Due to the Hilbert space interpolation theorem (Theorem 3.2.23 in [6]), we obtain that

$$|u - \Pi^{(r)}_{p,k,n} u|_{H^r(\Omega)} \le \hat{C} C^{(\hat{q}-r)/(q-r)} h_n^{\hat{q}-r} (p-k)^{-(\hat{q}-r)} |u|_{H^{\hat{q}}(\Omega)}$$

holds for any (real) $\hat{q}$ that satisfies $r \le \hat{q} \le q$. Here, $\hat{C}$ only depends on $\hat{q}$.

Again, the original result was stated for locally quasi-uniform knots. For any $p > 3$ the relevant case $k = p - 1$, which we consider, is not covered by this theorem.

Similar results to Theorem 1 are known in approximation theory, cf. [13]. There, however, different norms have been discussed. Hence we do not go into the details.

In [10], it was suggested and confirmed by numerical experiments that Theorem 1 is satisfied. A proof was however not given.

2.2 Spline spaces

For any function $u \in H^1(0,1)$, the symmetric function $w : (-1,1) \to \mathbb{R}$, given by $w(x) := u(|x|)$, also satisfies $w \in H^1(-1,1)$. Because $w(-1) = w(1)$, the function w can be extended in a periodic way to $\mathbb{R}$.

The periodic extension is still symmetric, i.e. $w(-x) = w(x)$ for all $x \in \mathbb{R}$, and can be approximated by a periodic and symmetric spline. Here and in what follows, the term *symmetric* spline refers to a spline that is an even function, i.e., to a spline satisfying $w_{p,n}(x) = w_{p,n}(-x)$. The space of periodic splines is given by the following definition.

Definition 2 $S^{per}_{p,n}(-1,1)$ is the space of all $w_{p,n} \in S_{p,n}(-1,1)$ that satisfy the periodicity condition

$$\frac{\partial^l}{\partial x^l} w_{p,n}(-1) = \frac{\partial^l}{\partial x^l} w_{p,n}(1) \text{ for all } l \in \mathbb{N}_0 \text{ with } l < p. \tag{2}$$

The restriction of a symmetric spline function $w_{p,n} \in S^{per}_{p,n}(-1,1)$ to the interval $(0,1)$ is again a function in $u_{p,n} \in S_{p,n}(0,1)$. However, we know more: As $w_{p,n}$ is assumed to be a symmetric spline, i.e. an even function, we have

$$\frac{\partial^l}{\partial x^l} w_{p,n}(x) = (-1)^l \frac{\partial^l}{\partial x^l} w_{p,n}(-x) \text{ for all } l \in \mathbb{N}_0.$$

By plugging $x = 0$ into this condition, we obtain that all odd derivatives vanish for $x = 0$. By plugging $x = 1$ into the condition, we obtain together with (2) that also for $x = 1$ all odd derivatives vanish.

Concluding, the restricted function $u_{p,n}$ is the member of some space $\widetilde{S}_{p,n}(0,1)$, which can be characterized by the following definition.

Definition 3 $\widetilde{S}_{p,n}(0,1)$ is the space of all $u_{p,n} \in S_{p,n}(0,1)$ that satisfy the following symmetry condition:

$$\frac{\partial^{2l+1}}{\partial x^{2l+1}} u_{p,n}(0) = \frac{\partial^{2l+1}}{\partial x^{2l+1}} u_{p,n}(1) = 0 \text{ for all } l \in \mathbb{N}_0 \text{ with } 2l+1 < p.$$

We can extend Theorem 3 for $k = p - 1$ to the following lemma stating that the approximation error estimate is still satisfied if we restrict ourselves to periodic splines. Let $H^{q,per}(-1,1)$ be the space of all $w \in H^q(-1,1)$ that satisfy the periodicity condition

$$\frac{\partial^l}{\partial x^l} w(-1) = \frac{\partial^l}{\partial x^l} w(1) \text{ for all } l \in \mathbb{N}_0 \text{ with } l < q.$$

The following lemma holds.

Lemma 1 *For each $r \in \mathbb{N}_0$, each $q \in \mathbb{N}$ and each $p \in \mathbb{N}$ with $0 \le r \le q \le p+1$ and $r-1 < p$, there is for each $w \in H^{q,per}(-1,1)$ a spline approximation $w_{p,n} \in S_{p,n}^{per}(-1,1)$ such that*

$$|w - w_{p,n}|_{H^r(-1,1)} \le C(p,r,q) h_n^{q-r} |w|_{H^q(-1,1)}$$

is satisfied, where $C(p,r,q)$ is a constant that might depend on p, r and q. $C(p,r,q)$ is independent of the gird size h_n.

Proof We make use of the fact that the proof in [14] uses local projections. So, there is a local approximation error estimate available, cf. [14], Theorem 6.24: The value of the approximation $Q_{p,n}w$ of a function w at a certain element $I_i := (x_i, x_{i+1})$ only depends on the values of the function to be approximated in a certain neighborhood $\widetilde{I}_i := (x_i - p\,h_n, x_{i+1} + p\,h_n)$. We assume that the grid is fine enough such that $\widetilde{I}_i \subseteq (x_i - \frac{1}{2}, x_i + \frac{1}{2})$. Due to [14], Theorem 6.24, the following local estimate

$$|w - Q_{p,k}w|_{H^r(I_i)} \le \widetilde{C}(p,r,q) h_n^{q-r} |w|_{H^q(\widetilde{I}_i)}. \tag{3}$$

is satisfied for a constant $\widetilde{C}(p,r,q)$ which is independent of the grid size h_n.

The function w can be extended periodically to $\mathbb{R}$. For this function, we can construct a spline approximation $w_{p,n} := Q_{p,k}w$, defined on $\mathbb{R}$. Based on the local error estimates (3) and using $\widetilde{I}_i \subseteq (x_i - \frac{1}{2}, x_i + \frac{1}{2})$, we obtain

$$|w - w_{p,n}|_{H^r(-1,1)} \le \widetilde{C}(p,r,q)\; p\; h_n^{q-r} |w|_{H^q(-3/2,3/2)}.$$

Because w is periodic, we obtain

$$|w - w_{p,n}|_{H^r(-1,1)} \le 2\; \widetilde{C}(p,r,q)\; p\; h_n^{q-r} |w|_{H^q(-1,1)}.$$

This finishes the proof. □

The next step is to introduce bases for the B-spline spaces which we have defined in this section to make it easier to work with them and to make the reader more familiar with the function spaces. For the construction of the basis we assume that $n > p$, i.e., that the grid is fine enough not to have basis functions that interact with both end points of the grid. Note that we do not need this requirement for the proofs of the main theorems.

On $\mathbb{R}$, B-spline basis functions (cardinal B-splines) are typically defined as follows, cf. [14], (4.22).

Definition 4 The B-splines of degree $p = 0$ are given for any $n \in \mathbb{N}$ by

$$\psi_{0,n}^{(i)}(x) = \begin{cases} 1 & \text{for } x \in (ih_n, (i+1)h_n], \\ 0 & \text{else,} \end{cases}$$

where $i \in \mathbb{Z}$. The B-splines of degree $p > 0$ are for any $n \in \mathbb{N}$ given by the recurrence formula

$$\psi_{p,n}^{(i)}(x) = \frac{x - i\, h_n}{p\, h_n} \psi_{p-1,n}^{(i)}(x) + \frac{(p+i+1)\, h_n - x}{p\, h_n} \psi_{p-1,n}^{(i+1)}(x), \tag{4}$$

where $i \in \mathbb{Z}$.

We obtain by construction that supp $\psi_{p,n}^{(i)} = [ih_n, (i+1+p)h_n]$ or, equivalently, that for $i \in \{-p, n\}$ the intersection of the support with $\Omega = (0,1)$ is non-empty.

For $S_{p,n}^{per}(0,1)$, we introduce a B-spline basis as follows:

$$\{\varphi_{p,n}^{(i)} := \psi_{p,n}^{(i)} + \psi_{p,n}^{(i-n)} \; : \; i = 1, \ldots, n\}. \tag{5}$$

To show that this is actually a basis, we observe that the functions $\varphi_{p,n}^{(i)}$ are linear independent because the cardinal B-spline functions $\psi_{p,n}^{(i)}$ are linear independent and each function $\psi_{p,n}^{(i)}$ only contributes to one of the functions $\varphi_{p,n}^{(i)}$. The number of functions in (5) is n, so it coincides with the dimension of $S_{p,n}(0,1)$ minus the number of periodicity conditions. Hence we conclude that (5) is a basis. If we omit the functions which vanish in $(0,1)$, we obtain

$$\{\varphi_{p,n}^{(i)} = \psi_{p,n}^{(i)} \; : \; i = 1, \ldots, n-p-1\} \cup \{\varphi_{p,n}^{(i)} = \psi_{p,n}^{(i)} + \psi_{p,n}^{(i-n)} \; : \; i = n-p, \ldots, n\}.$$

For $\widetilde{S}_{p,n}(0,1)$, we may introduce a B-spline basis as follows:

$$\{\psi_{p,n}^{(-i-p-1)} + \psi_{p,n}^{(i)} + \psi_{p,n}^{(-i+2n-p+1)} \; : \; i = -\left\lceil \frac{p}{2} \right\rceil, \ldots, n - \left\lfloor \frac{p}{2} \right\rfloor\}.$$

Again, after omitting the vanishing terms and eliminating duplicates, we derive for odd p the basis

$$\begin{aligned} &\{\psi_{p,n}^{(-(p+1)/2)}\} \\ &\quad \cup \{\psi_{p,n}^{(i)} + \psi_{p,n}^{(-i-p-1))} \; : \; i = -(p-1)/2, \ldots, -1\} \\ &\quad \cup \{\psi_{p,n}^{(i)} \; : \; i = 0, \ldots, n-p\} \\ &\quad \cup \{\psi_{p,n}^{(i)} + \psi_{p,n}^{(-i+2n-p+1))} \; : \; i = n-p+1, \ldots, n-(p+1)/2\} \\ &\quad \cup \{\psi_{p,n}^{(n-(p-1)/2)}\} \end{aligned}$$

and for even p the basis

$$\begin{aligned} &\{\psi_{p,n}^{(i)} + \psi_{p,n}^{(-i-(p+1))} \; : \; i = -p/2, \ldots, -1\} \\ &\quad \cup \{\psi_{p,n}^{(i)} \; : \; i = 0, \ldots, n-p\} \\ &\quad \cup \{\psi_{p,n}^{(i)} + \psi_{p,n}^{(n-(p-1)-i)} \; : \; i = n-p+1, \ldots, n-p/2\}. \end{aligned}$$

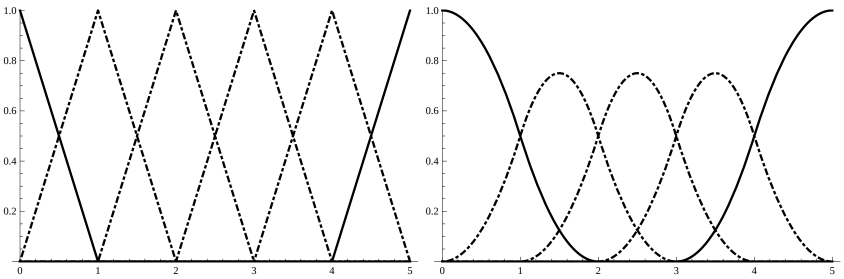

Fig. 1 B-spline basis functions for $\widetilde{S}_{1,n}(0,1)$ and $\widetilde{S}_{2,n}(0,1)$

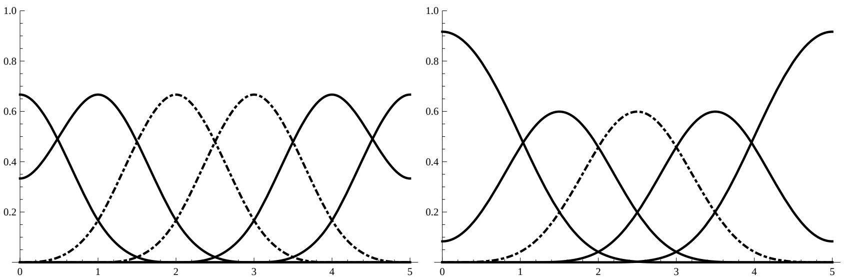

Fig. 2 B-spline basis functions for $\widetilde{S}_{3,n}(0,1)$ and $\widetilde{S}_{4,n}(0,1)$

Again, we observe that these functions are linear independent. Moreover, we see that the dimension of the space corresponds to the dimension of $S_{p,n}$ minus the number of $2\lfloor p/2 \rfloor$ boundary conditions. This again indicates that the presented sets are bases.

Remark 4 By construction, the latter bases for $\widetilde{S}_{p,n}(0,1)$ form a partition of unity. Moreover, all basis functions are obviously non-negative.

Fig. 1 and 2 depict the B-spline basis functions that span $\widetilde{S}_{p,n}(0,1)$. Here, the basis functions that have an influence at the boundary are plotted with solid lines. The basis functions that have zero derivatives up to order $p-1$ at the boundary coincide with standard B-spline functions. They are plotted with dashed lines.

If we compare the pictures of the B-spline basis functions in $\widetilde{S}_{p,n}(0,1)$ (Fig. 1 and 2) with the standard B-spline basis functions for $S_{p,n}(0,1)$ (Fig. 3 and 4) obtained from a classical open knot vector, we see that the latter ones have more basis functions that are associated with the boundary. This can also be seen by counting the number of degrees of freedom, cf. Table 1.

3 An estimate for two consecutive grids

In this section we will show an approximation error estimate for two consecutive grids for the periodic case. To be able to do a proper telescoping argument,

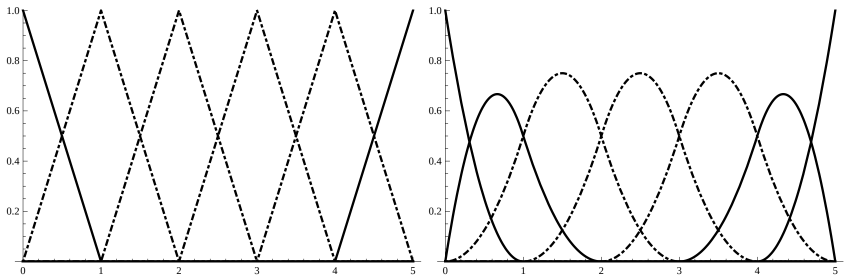

Fig. 3 B-spline basis functions for $S_{1,n}(0,1)$ and $S_{2,n}(0,1)$

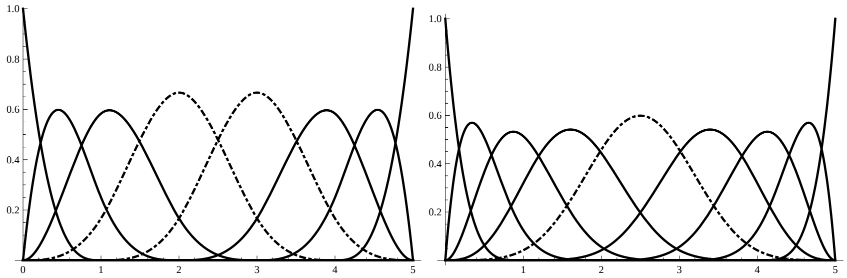

Fig. 4 B-spline basis functions for $S_{3,n}(0,1)$ and $S_{4,n}(0,1)$

	dim $S_{p,n}(0,1)$	dim $S^{per}_{p,n}(0,1)$	dim $\widetilde{S}_{p,n}(0,1)$
p even	$n+p$	n	n
p odd	$n+p$	n	$n+1$

Table 1 Degrees of freedom, where n is the number of elements and $n+1$ is the number of nodes.

we have to analyze a fixed interpolation operator. So, we show that

$$\|(I - \Pi_{p,n})w_{p,2n}\|_{L^2(-1,1)} \leq \sqrt{2}\, h_n |w_{p,2n}|_{H^1(-1,1)} \tag{6}$$

holds for all $w_{p,2n} \in S^{per}_{p,2n}(-1,1)$, where I is the identity and $\Pi_{p,n}$ is the H^1-orthogonal projection operator, given by the following definition.

Definition 5 For each $w \in H^{1,per}(-1,1)$, $\Pi^{per}_{p,n} w$ is the solution of the following problem. Find $w_{p,n} \in S^{per}_{p,n}(-1,1)$ such that

$$(w_{p,n}, \widetilde{w}_{p,n})_{\widetilde{H}^1(-1,1)} = (w, \widetilde{w}_{p,n})_{\widetilde{H}^1(-1,1)}$$

for all $\widetilde{w}_{p,n} \in S^{per}_{p,n}(-1,1)$, where

$$(u,v)_{\widetilde{H}^1(-1,1)} := (u', v')_{L^2(-1,1)} + \left(\int_{-1}^{1} u(x)\mathrm{d}x\right)\left(\int_{-1}^{1} v(x)\mathrm{d}x\right).$$

The proof will be done using Fourier analysis. For this purpose, we have to introduce a matrix-vector formulation of (6). Here, we follow the classical approaches that are used in finite element (FEM) approaches as well as in Isogeometric Analysis (IGA). So we first introduce the standard mass matrix and give some results. Then we introduce standard stiffness matrix and intergrid transfer matrices as it is common for standard multigrid methods.

3.1 Some results for the standard mass matrix

Using the B-spline basis, we can introduce a standard mass matrix $M_{p,n} = (m_{p,n}^{(i,j)})_{i,j=0}^{2n-1}$, where

$$m_{p,k}^{(i,j)} = (\varphi_{p,k}^{(i)}, \varphi_{p,k}^{(j)})_{L^2(-1,1)}.$$

For an equidistant grid, we can derive the coefficients of the mass matrix explicitly. Due to [17], we have

$$m_{p,n}^{(i,j)} = h_n \frac{E(2p+1, p+\xi)}{(2p+1)!},$$

where $\xi \in \{-n/2, \ldots, n/2-1\}$ such that $i - j \equiv \xi \bmod n$. Here, $E(n,k)$ are the Eulerian numbers, which satisfy the recurrence relation

$$E(n,k) = (n-k)E(n-1,k-1) + (k+1)E(n-1,k)$$

and the initial condition

$$E(0,j) = \begin{cases} 1 \text{ for } j = 0 \\ 0 \text{ for } j \neq 0 \end{cases}.$$

The following lemma relates the mass matrices for two polynomial degrees.

Lemma 2 *For all $p \in \mathbb{N}$, all $n \in \mathbb{N}$ and all vectors $\underline{w}_n \in \mathbb{R}^{2n}$, the inequality*

$$\|\underline{w}_n\|_{M_{p,n}} \leq 2\|\underline{w}_n\|_{M_{p-1,n}} \tag{7}$$

is satisfied.

Proof Take $\sigma := \min\{0, \min \underline{w}_n\}$ and define $\underline{u}_n := \underline{w}_n - \sigma \underline{e}_n$, where $\underline{e}_n := (1,1,1,\ldots)^T \in \mathbb{R}^{2n}$. Then we have for $q \in \{p-1, p\}$

$$\|\underline{w}_n\|_{M_{q,n}}^2 = \|\underline{u}_n\|_{M_{q,n}}^2 - 2\sigma(\underline{u}_n, M_{q,n}\underline{e}_n)_{\ell^2} + \sigma^2(\underline{e}_n, M_{q,n}\underline{e}_n)_{\ell^2}.$$

As the B-splines form a partition of unity (Theorem 4.20 in [14]), we have $M_{q,n}\underline{e}_n = h_n^{-1}\underline{e}_n$, so

$$\|\underline{w}_n\|_{M_{q,n}}^2 = \|\underline{u}_n\|_{M_{q,n}}^2 - 2\sigma h_n^{-1}(\underline{u}_n, \underline{e}_n)_{\ell^2} + \sigma^2 h_n^{-1}(\underline{e}_n, \underline{e}_n)_{\ell^2}.$$

Note that $\sigma \leq 0$ and $\underline{u}_n \geq 0$. So, $-2\sigma h_n^{-1}(\underline{u}_n, \underline{e}_n)_{\ell^2} + \sigma^2 h_n^{-1}(\underline{e}_n, \underline{e}_n)_{\ell^2} \geq 0$. Hence (7) is a consequence of

$$\|\underline{u}_n\|_{M_{p,n}} \leq 2\|\underline{u}_n\|_{M_{p-1,n}}. \tag{8}$$

For $\underline{u}_n = (u_i)_{i=0}^{2n-1}$ and $q \in \{p-1, p\}$, we have

$$\|\underline{u}_n\|^2_{M_{q,n}} = \int_0^1 f_q(x)^2 dx, \text{ with } f_q(x) := \sum_{i=0}^{n} u_i \varphi^{(i)}_{q,n}(x),$$

where the basis function $\varphi^{(i)}_{q,n}$ are as specified in (5). Using the recurrence formula (4), we obtain

$$\begin{aligned} f_p(x) &= \sum_{i=0}^{2n-1} u_i \varphi^{(i)}_{p,n}(x) \\ &= \sum_{i=0}^{2n-1} u_i \Big(\frac{x - ih_n}{ph_n} \varphi^{(i)}_{p-1,n}(x) + \frac{(p+i+1)h_n - x}{ph_n} \varphi^{(i-1)}_{p-1,n}(x) \Big) \\ &= \sum_{i=0}^{2n-1} \Big(\underbrace{\frac{x - ih_n}{ph_n}}_{A_i :=} u_i + \underbrace{\frac{(p+i)h_n - x}{ph_n}}_{B_i :=} u_{i-1} \Big) \varphi^{(i)}_{p-1,n}(x). \end{aligned}$$

We have (by construction) that $\varphi^{(i)}_{p-1,n}(x) \geq 0$, $u_i \geq 0$ and $u_{i-1} \geq 0$. Moreover, on the support of $\varphi^{(i)}_{p-1,n}(x)$ we have $A_i \leq 1$ and $B_i \leq 1$. So we obtain

$$f_p(x) \leq \sum_{i=0}^{2n-1} (u_i + u_{i-1}) \varphi^{(i)}_{p-1,n}(x) = \sum_{i=0}^{2n-1} (\underline{u}_n + S_n \underline{u}_n)_i \varphi^{(i)}_{p-1,n}(x),$$

where S_n is a shift operator, mapping $(u_0, u_1, \ldots, u_{n-1})$ to $(u_{n-1}, u_0, \ldots, u_{n-2})$. Hence we conclude

$$\begin{aligned} \|\underline{u}_n\|_{M_{p,n}} = \left(\int_0^1 f_p^2(x) dx \right)^{1/2} &\leq \|\underline{u}_n + S_n \underline{u}_n\|_{M_{p-1,n}} \\ &\leq \|\underline{u}_n\|_{M_{p-1,n}} + \|S_n \underline{u}_n\|_{M_{p-1,n}} = 2 \|\underline{u}_n\|_{M_{p-1,n}}, \end{aligned}$$

where the last equation is due to the fact that $M_{p-1,n}$ is a simple, periodic band-matrix. This finishes the proof. □

3.2 Fourier analysis and the symbol of the mass matrix

A. Brandt has proposed to use Fourier series to analyze multigrid methods, cf. [5]. Fourier analysis provides a framework to determine sharp bounds for the convergence rates of multigrid methods and other iterative solvers for problems arising from partial differential equations. This is different to classical analysis, which typically yields qualitative statements only. For a detailed introduction into Fourier analysis, see, e.g., [16].

Besides the analysis of multigrid solvers, the idea of Fourier analysis can be carried over to the computation of approximation error estimates, as we will see in the remainder of this section.

The key idea of Fourier analysis is to represent the vectors $\underline{u}_n \in \mathbb{R}^n$ in terms of a Fourier series

$$\underline{u}_n = \sum_{\theta\in\Theta_n} u_{n,\theta} \underbrace{(\mathrm{e}^{\mathrm{i}\theta l})_{l=0}^{2n-1}}_{\underline{\phi}_n(\theta) :=},$$

where $\Theta_n := \{0, h_n\pi, 2h_n\pi, 3h_n\pi, \dots, (2n-1)h_n\pi\}$. We observe that

$$M_{p,n}\underline{\phi}_n(\theta) = \underbrace{h_n \sum_{l=-p}^{p} \frac{E(2p+1,p+l)}{(2p+1)!} \mathrm{e}^{\mathrm{i}\theta l}}_{\widehat{M_{p,n}}(\theta) :=} \underline{\phi}_n(\theta) \tag{9}$$

is satisfied, i.e., that $\underline{\phi}_n(\theta)$ is an eigenvector of $M_{p,n}$. In Fourier analysis, the eigenvalue $\widehat{M_{p,n}}(\theta)$ is called the symbol of $M_{p,n}$.

As the factor $\frac{E(2p+1,p+l)}{(2p+1)!}$ in (9) is symmetric in l, we have

$$\widehat{M_{p,n}}(\theta) = h_n \sum_{l=-p}^{p} \frac{E(2p+1,p+l)}{(2p+1)!} \cos(l\theta). \tag{10}$$

The symbol is better characterized by the following lemma.

Lemma 3 *$\widehat{M_{p,n}}(\theta) \geq 0$ for all θ. Moreover, $\widehat{M_{p,n}}(\theta_1) \leq \widehat{M_{p,n}}(\theta_2)$ for all θ_1 and θ_2 where $\cos\theta_1 \leq \cos\theta_2$.*

Proof For $c \in [0,2]$, we define

$$f_{p,n}(c) := h_n^{-1}\widehat{M_{p,n}}(\arccos(c-1)).$$

The statement of the lemma is now equivalent to

– $h_n^{-1}\widehat{M_{p,n}}(\arccos(-1)) = f_{p,n}(0) > 0$ and
– $h_n^{-1}\widehat{M_{p,n}}(\arccos(c-1)) = f_{p,n}(c)$ is monotonically increasing for $c > 0$.

Since we can express $\cos(l\arccos(c))$ as the l-th Chebyshev polynomial, $f_{p,n}$ is a polynomial function in c. Using the recurrence relation for the Eulerian numbers, we can derive the following recurrence formula for $f_{p,n}$:

$$f_{p,n}(c) = \frac{1+cp}{1+2p} f_{p-1,n}(c) + \frac{(2-c)(1+c(-1+2p))}{p(1+2p)} f'_{p-1,n}(c) + \frac{(-2+c)^2 c}{p(1+2p)} f''_{p-1,n}(c).$$

We can make an ansatz

$$f_{p,n}(c) = \sum_{j=0}^{p} a_{p,j} c^j,$$

and derive the recurrence formula

$$a_{p,j} = \underbrace{\frac{(1-j+p)^2}{p+2p^2}}_{A_{p,j}:=} a_{p-1,j-1} + \underbrace{\frac{4j(p-j)+j+p}{p+2p^2}}_{B_{p,j}:=} a_{p-1,j} + \underbrace{\frac{2+6j+4j^2}{p+2p^2}}_{C_{p,j}:=} a_{p-1,j+1}$$

for the coefficients $a_{p,j}$. For $p=1$, we obtain

$$a_{1,j} = \begin{cases} \frac{1}{3} & \text{for } j \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

As $A_{p,j} \geq 0$, $B_{p,j} \geq 0$ and $C_{p,j} \geq 0$ for $0 \leq j \leq p$, one can show using induction in p that for all $p \geq 1$:

$$\begin{cases} a_{p,j} > 0 & \text{for } j \in \{0,1,\ldots,p\} \\ a_{p,j} = 0 & \text{otherwise.} \end{cases}$$

This immediately implies that $f_{p,n}(0) > 0$ and that $f_{p,n}(c)$ is monotonically increasing for $c > 0$, which concludes the proof. □

3.3 The symbol of the stiffness matrix

The next step is to derive the symbol of the stiffness matrix $K_{p,n} = (k_{p,n}^{(i,j)})_{i,j=0}^{2n-1}$, where

$$k_{p,n}^{(i,j)} = \left(\varphi_{p,n}^{(i)}, \varphi_{p,n}^{(j)}\right)_{\widetilde{H}^1(-1,1)}.$$

Since the basis functions $\varphi_{p,n}^{(i)}$ form a partition of unity, $\int_{-1}^{1} \varphi_{p,n}^{(i)}(x)\mathrm{d}x = h_n$ holds for all i. So, we obtain

$$k_{p,n}^{(i,j)} = \left(\frac{\partial}{\partial x}\varphi_{p,n}^{(i)}, \frac{\partial}{\partial x}\varphi_{p,n}^{(j)}\right)_{L^2(-1,1)} + h_n^2.$$

Note that for uniform knot vectors the identity

$$\frac{\partial}{\partial x}\varphi_{p,n}^{(j)}(x) = \frac{1}{h_n}\left(\varphi_{p-1,n}^{(j-1)}(x) - \varphi_{p-1,n}^{(j)}(x)\right)$$

holds, see e.g. (5.36) in [14]. Hence we directly obtain

$$k_{p,n}^{(i,j)} = \frac{1}{h_n^2}\left(m_{p-1,n}^{(i,j)} - m_{p-1,n}^{(i,j-1)} - m_{p-1,n}^{(i-1,j)} + m_{p-1,n}^{(i-1,j-1)}\right) + h_n^2.$$

Because the grid is equidistant, this reduces to

$$k_{p,n}^{(i,j)} = \frac{1}{h_n^2}\left(2m_{p-1,n}^{(i,j)} - m_{p-1,n}^{(i,j-1)} - m_{p-1,n}^{(i-1,j)}\right) + h_n^2.$$

As $\sum_{l=0}^{2n-1} \mathrm{e}^{\mathrm{i}\theta l} h_n^2 = 0$ for all $\theta \in \Theta_n \backslash \{0\}$, the term "$+h_n^2$" does not have any effect in this case. So, the symbol of the stiffness matrix $K_{p,n}$ is given via

$$\widehat{K_{p,n}}(\theta) = \frac{2 - \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{h_n^2}\widehat{M_{p-1,n}}(\theta) = 2\frac{1-\cos\theta}{h_n^2}\widehat{M_{p-1,n}}(\theta)$$

for $\theta \in \Theta_n \backslash \{0\}$. For $\theta = 0$, the part representing the derivatives vanishes and we obtain $\widehat{K_{p,n}}(0) = h_n^2 \sum_{l=0}^{2n-1} \mathrm{e}^0 = 2nh_n^2 = 2h_n$.

3.4 Representing intergrid transformations in Fourier analysis

As we have been mentioning in the beginning of this section, we are interested in showing that

$$\|(I - \Pi_{p,n}^{per}) u_{p,2n}\|_{L^2(-1,1)} \le \sqrt{2} h_n |u_{p,2n}|_{H^1(-1,1)}, \tag{11}$$

is satisfied for any $u_{p,2n} \in S_{p,2n}^{per}(-1,1)$. Using mass matrix M_n and stiffness matrix K_n, this can be rewritten as

$$\|(I - P_{p,2n} K_{p,n}^{-1} P_{p,2n}^T K_{p,2n}) \underline{u}_{2n}\|_{M_{p,2n}} \le \sqrt{2} h_n \|\underline{u}_{2n}\|_{K_{p,2n}}$$

for all $\underline{u}_{2n} \in \mathbb{R}^{4n}$. Here, $P_{p,2n}$ is the matrix representing the canonical embedding of $S_{p,n}^{per}(-1,1)$ into $S_{p,2n}^{per}(-1,1)$. Before we can analyze (11), we have to determine the symbol for $P_{p,2n}$.

The following lemma is rather well-known in literature, cf. [7] equation (4.3.4), and can be easily shown by induction in p.

Lemma 4 *For all $p \in \mathbb{N}$, all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$,*

$$\varphi_{p,2n}^{(j)}(x) = 2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \varphi_{p,n}^{(2j+l)}(x) \tag{12}$$

is satisfied.

This allows to introduce the symbol of the prolongation operator. Here, the symbol cannot be understood as eigenvalue anymore as the prolongation operator is a rectangular matrix. However, we have

$$\begin{aligned}
\underline{\phi}_{p,2n}(2\theta) &= \sum_{j=0}^{2n-1} \mathrm{e}^{2\theta \mathrm{i} j} \varphi_{p,2n}^{(j)} = \sum_{j} \mathrm{e}^{2\theta \mathrm{i} j} 2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \varphi_{p,n}^{(2j+l)} \\
&= 2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \mathrm{e}^{-\theta \mathrm{i} l} \sum_{j=0}^{2n-1} \mathrm{e}^{\theta \mathrm{i}(2j+l)} \varphi_{p,n}^{(2j+l)} \\
&= 2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \mathrm{e}^{-\theta \mathrm{i} l} \sum_{j=0}^{2n-1} \mathrm{e}^{\theta \mathrm{i} j} \varphi_{p,n}^{j} \\
&\qquad + 2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \mathrm{e}^{-(\theta+\pi) \mathrm{i} l} \sum_{j=0}^{2n-1} \mathrm{e}^{(\theta+\pi) \mathrm{i} j} \varphi_{p,n}^{(j)} \\
&= \left(2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \mathrm{e}^{-\mathrm{i}\theta l} \right) \underline{\phi}_{p,n}(\theta) \\
&\qquad + \left(2^{-p} \sum_{l=0}^{p+1} \binom{p+1}{l} \mathrm{e}^{-\mathrm{i}(\theta+\pi) l} \right) \underline{\phi}_{p,n}(\theta+\pi).
\end{aligned}$$

We observe that the prolongation operator $P_{p,2n}$ maps the linear span, spanned by

$$\underline{\phi}_{p,2n}(2\theta) \tag{13}$$

to the linear span, spanned by

$$\underline{\phi}_{p,n}(\theta) \quad \text{and} \quad \underline{\phi}_{p,n}(\theta+\pi). \tag{14}$$

The restriction operator $P_{p,2n}^T$ maps the linear span, spanned by (14), to the linear span, spanned by (13). So, we can introduce the symbol as a 2×1-matrix

$$\widehat{P_{p,2n}}(\theta) = \frac{1}{2^p}\begin{pmatrix} \sum_{l=0}^{p+1}\binom{p+1}{l}\mathrm{e}^{-\mathrm{i}\theta l} \\ \sum_{l=0}^{p+1}\binom{p+1}{l}\mathrm{e}^{-\mathrm{i}(\theta+\pi)l}\end{pmatrix} = \frac{1}{2^p}\begin{pmatrix}(1+\mathrm{e}^{-\mathrm{i}\theta})^{p+1} \\ (1+\mathrm{e}^{-\mathrm{i}(\theta+\pi)})^{p+1}\end{pmatrix}.$$

Now we have all the ingredients needed to derive the symbol of the projection operator in (11).

3.5 Analysis of the projection operator

The symbol $\widehat{\Pi_{p,n}^{per}}(\theta)$ of the projection operator $\Pi_{p,n}^{per}$ is represented in the basis (14), i.e., it is a 2×2-matrix with

$$\widehat{\Pi_{p,n}^{per}}(\theta) = I - \widehat{P_{p,2n}}(\theta)\widehat{K_{p,n}}(\theta)^{-1}\widehat{P_{p,2n}}^{*}(\theta)\widehat{\mathcal{K}_{p,2n}}(\theta), \tag{15}$$

where A^* is the conjugate complex of the transpose of a matrix A. Here, $\widehat{\mathcal{K}_{p,2n}}(\theta)$ is the representation of the symbol of the stiffness matrix corresponding to the basis (14). Since we know that the multiplication with the stiffness matrix preserves any frequency, we obtain

$$\widehat{\mathcal{K}_{p,2n}}(\theta) = \begin{pmatrix}\widehat{K_{p,2n}}(\theta) & \\ & \widehat{K_{p,2n}}(\theta+\pi)\end{pmatrix},$$

where $\widehat{K_{p,2n}}(\theta)$ is the representation of the symbol of the stiffness matrix corresponding to the basis (13).

The basis (14) naturally splits the domain of frequencies $\Theta = [0, 2\pi)$ into subdomains $\Theta^{(high)} := [\pi/2, 3\pi/2)$ and $\Theta^{(low)} := \Theta\backslash\Theta^{(high)}$. Here, for any $\theta \in \Theta^{(low)}$, we obtain $\theta+\pi \in \Theta^{(high)}$. We observe that $\cos\theta \geq 0$ for $\theta \in \Theta^{(low)}$ and $\cos\theta \leq 0$ for $\theta \in \Theta^{(high)}$ and vice versa.

Next, we show the following lemma.

Lemma 5 *For any* $u_{p,2n} \in S_{p,2n}^{per}(-1,1)$

$$\|(I - \Pi_{p,n}^{per})u_{p,2n}\|_{L^2(-1,1)} \leq \sqrt{2}h_n|u_{p,2n}|_{H^1(-1,1)}, \tag{16}$$

is satisfied.

Proof Rewriting (16) in matrix notation, we obtain

$$\|M_{p,2n}^{1/2}(I - P_{p,2n}K_{p,n}^{-1}P_{p,2n}^{T}K_{p,2n})K_{p,2n}^{-1/2}\| \leq \sqrt{2}h_n,$$

where $K_{p,n} = P_{p,2n}K_{p,2n}P_{p,2n}^{T}$ (Galerkin projection). Here and in what follows, $\|\cdot\|$ is the Euclidean norm. Using Lemma 2 we get

$$\begin{aligned}
&\|M_{p,2n}^{1/2}(I - P_{p,2n}K_{p,n}^{-1}P_{p,2n}^{T}K_{p,2n})K_{p,2n}^{-1/2}\| \\
&\quad \leq \|M_{p,2n}^{1/2}M_{p-1,2n}^{-1/2}\|\|M_{p-1,2n}^{1/2}(I - P_{p,2n}K_{p,n}^{-1}P_{p,2n}^{T}K_{p,2n})K_{p,2n}^{-1/2}\| \\
&\quad \leq 2\|M_{p-1,2n}^{1/2}(I - P_{p,2n}K_{p,n}^{-1}P_{p,2n}^{T}K_{p,2n})K_{p,2n}^{-1/2}\|.
\end{aligned}$$

So it suffices to show

$$\|M_{p-1,2n}^{1/2}(I - P_{p,2n}K_{p,n}^{-1}P_{p,2n}^{T}K_{p,2n})K_{p,2n}^{-1/2}\| \leq \frac{1}{2}\sqrt{2}h_n,$$

which is equivalent to

$$\rho(M_{p-1,2n}(I - P_{p,2n}K_{p,n}^{-1}P_{p,2n}^{T}K_{p,2n})K_{p,2n}^{-1}) \leq \frac{1}{2}h_n^2, \tag{17}$$

where ρ denotes the spectral radius. This statement is equivalent to

$$\underbrace{\rho(\widehat{\mathcal{M}_{p-1,2n}}(\theta)\widehat{\mathit{\Pi}_{p,n}^{per}}(\theta)\widehat{\mathcal{K}_{p,2n}}(\theta)^{-1})}_{q(\theta)h_n^2 :=} \leq \frac{1}{2}h_n^2, \tag{18}$$

for all $\theta \in \mathit{\Theta}_n$, where $\widehat{\mathit{\Pi}_{p,n}^{per}}(\theta)$ is as defined in (15) and

$$\widehat{\mathcal{M}_{p-1,2n}}(\theta) = \begin{pmatrix} \widehat{M_{p-1,2n}}(\theta) & \\ & \widehat{M_{p-1,2n}}(\theta+\pi) \end{pmatrix}.$$

We observe that replacing θ by $\theta+\pi$ has the same effect as switching both the rows and the columns of the symbol, therefore $q(\theta) = q(\theta+\pi)$. So, we can restrict ourselves to showing (18) for all $\theta \in \mathit{\Theta}_n^{(low)} = \mathit{\Theta}_n \cap \mathit{\Theta}^{(low)}$.

As the matrix in (18) is a 2-by-2 matrix with rank 1, the spectral radius is equal to its trace.

From Lemma 3, we obtain that $0 \leq \widehat{M_{p-1,2n}}(\theta+\pi) \leq \widehat{M_{p-1,2n}}(\theta)$ for all $\theta \in \mathit{\Theta}_n^{(low)}$, so we can substitute $\widehat{M_{p-1,2n}}(\theta+\pi)$ by $\xi\widehat{M_{p-1,2n}}(\theta)$, where $\xi \in [0,1]$. By doing this substitution, the term $\widehat{M_{p-1,2n}}(\theta+\pi)$ cancels out. We obtain for $\theta \in \mathit{\Theta}_n^{(low)}\backslash\{0\}$

$$q(\theta) = \frac{-(1+c)^p + c(1+c)^p - (1-c)^p\xi - (1-c)^p c\xi}{2(c^2-1)((1+c)^p + (1-c)^p\xi)},$$

where $c := \cos\theta$ and $\xi = \widehat{M_{p-1,2n}}(\theta+\pi)/\widehat{M_{p-1,2n}}(\theta)$. The case $\theta = 0$ will be dealt with at the end of the proof. To finalize the proof we need to show

$$q(\theta) \leq \frac{1}{2}$$

for all $\theta \in \Theta_n^{(low)} \backslash \{0\}$. It suffices to show

$$\frac{-(1+c)^p + c(1+c)^p - (1-c)^p\xi - (1-c)^p c\xi}{2(c^2-1)((1+c)^p + (1-c)^p\xi)} \le \frac{1}{2}$$

for all $c \in]0,1[$, all $\xi \in [0,1]$ and all $p \in \mathbb{N}$, i.e., to show the inequality for the whole range of all of these variables ignoring their dependence on θ. We observe that

$$0 \le \left(\frac{1-c}{1+c}\right)^p \le \frac{1-c}{1+c},$$

so there is some $\omega \in [0,1]$ such that $((1-c)/(1+c))^p = (1-c)/(1+c)\omega$. After substituting $(1-c)^p$ by $(1-c)(1+c)^{p-1}\omega$ it suffices to show

$$\frac{-(1+c)^p + c(1+c)^p - (1-c)(1+c)^{p-1}\omega\xi - (1-c)(1+c)^{p-1}\omega c\xi}{2(c^2-1)((1+c)^p + (1-c)(1+c)^{p-1}\omega\xi)} \le \frac{1}{2}$$

for all $c \in]0,1[$, all $\xi \in [0,1]$, all $\omega \in [0,1]$ and all $p \in \mathbb{N}$. This can be simplified to

$$\frac{-(1+c) + c(1+c) - (1-c)\omega\xi - (1-c)\omega c\xi}{2(c^2-1)((1+c) + (1-c)\omega\xi)} \le \frac{1}{2}$$

for all $c \in]0,1[$, all $\xi \in [0,1]$ and all $\omega \in [0,1]$. Here the denominator is always negative. So we can multiply with the denominator and obtain

$$c(1-c^2)(1-\omega\xi) \ge 0,$$

which is obviously true for all $c \in]0,1[$, all $\xi \in [0,1]$ and all $\omega \in [0,1]$.

The case $\theta = 0$ has the be considered separately. Here, we have to use $\widehat{K_{p,n}}(0) = 2h_n$ and obtain – by straight-forward computation – that $q(0) = \frac{1}{4}$. The inequality $q(\theta) \le \frac{1}{2}$ still holds in this case, which finishes the proof. □

4 The proof of the approximation error estimate (Theorem 1)

First, we show the following lemma.

Lemma 6 *For each $w \in H^{1,per}(-1,1)$, each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$*

$$\|(I - \Pi_{p,n}^{per})w\|_{L^2(-1,1)} \le 2\sqrt{2}\, h_n |w|_{H^1(-1,1)}$$

is satisfied.

Proof Using the triangular inequality, we obtain for any $q \in \mathbb{N}$

$$\|(I - \Pi_{p,n}^{per})w\|_{L^2(-1,1)} \le \|(I - \Pi_{p,2^q n}^{per})w\|_{L^2(-1,1)} + \sum_{l=0}^{q-1} \|(I - \Pi_{p,2^l n}^{per})\Pi_{p,2^{l+1}n}^{per} w\|_{L^2(-1,1)}.$$

We use Lemma 1 and a standard Aubin-Nitsche duality argument to estimate $\|(I-\Pi^{per}_{p,N})w\|_{L^2(-1,1)}$ from above. Using [4], Lemma 7.6, and Lemma 1 for $r=1$ and $q=2$, we immediately obtain

$$\|(I-\Pi^{per}_{p,N})w\|_{L^2(-1,1)} \le \widetilde{C}(p) h_N \|w\|_{H^1(-1,1)}, \tag{19}$$

where $\widetilde{C}(p)$ is independent of the grid size. Using (19) and Lemma 5, we obtain

$$\begin{aligned}\|(I-\Pi^{per}_{p,n})w\|_{L^2(-1,1)} \le{}& \widetilde{C}(p)\; 2^{-q} h_n \|w\|_{H^1(-1,1)} \\ &+ \sum_{l=0}^{q-1} \sqrt{2}\; 2^{-l} h_n |\Pi^{per}_{p,2^{l+1}n} w|_{H^1(-1,1)}.\end{aligned}$$

Because $\Pi^{per}_{p,2^{l+1}n}$ is H^1-orthogonal, we further obtain

$$\|(I-\Pi^{per}_{p,n})w\|_{L^2(-1,1)} \le \widetilde{C}(p)\; 2^{-q} h_n \|w\|_{H^1(-1,1)} + \sum_{l=0}^{q-1} \sqrt{2}\; 2^{-l} h_n |w|_{H^1(-1,1)}.$$

and using the summation formula for the geometric series gives

$$\|(I-\Pi^{per}_{p,n})w\|_{L^2(-1,1)} \le \widetilde{C}(p)\; 2^{-q} h_n \|w\|_{H^1(-1,1)} + 2\sqrt{2} h_n |w|_{H^1(-1,1)}.$$

As this is true for all $q \in \mathbb{N}$, we can take the limit $q \to \infty$ and obtain the desired result. □

Theorem 1 is the extension of Lemma 6 to the non-periodic case.

Proof of Theorem 1 First, we observe that any $u \in H^1(0,1)$ can be extended to a $w \in H^{1,per}(-1,1)$ by defining $w(x) := u(|x|)$. Using Lemma 6, we can find a function $w_{p,n} \in S^{per}_{p,n}(-1,1)$ such that

$$\|w - w_{p,n}\|_{L^2(-1,1)} \le 2\sqrt{2}\; h_n |w|_{H^1(-1,1)}.$$

This function is not necessarily symmetric, i.e., $w_{p,n}(x) = w_{p,n}(-x)$ might not be true. However, $\widetilde{w}_{p,n}(x) := \frac{1}{2}(w_{p,n}(x) + w_{p,n}(-x))$ is symmetric and still satisfies the error estimate

$$\|w - \widetilde{w}_{p,n}\|_{L^2(-1,1)} \le 2\sqrt{2}\; h_n |w|_{H^1(-1,1)}.$$

By restricting $\widetilde{w}_{p,n}$ to $(0,1)$, we obtain a function $u_{p,n} \in S_{p,n}(0,1)$. This function satisfies the desired approximation error estimate since both $|w|_{H^1(-1,1)} = \sqrt{2}|u|_{H^1(0,1)}$ and $\|w-\widetilde{w}_{p,n}\|_{L^2(-1,1)} = \sqrt{2}\|u-u_{p,n}\|_{L^2(0,1)}$ hold due to the symmetry of w. □

In the proof, we have defined $u_{p,n}$ to be the restriction of a symmetric and periodic spline $\widetilde{w}_{p,n}$ to $(0,1)$. So, $u_{p,n} \in \widetilde{S}_{p,n}(0,1)$ is satisfied, i.e., we have shown the following result.

Corollary 1 *For each $u \in H^1(0,1)$, each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$, there is a spline approximation $u_{p,n} \in \widetilde{S}_{p,n}(0,1)$ such that*

$$\|u - u_{p,n}\|_{L^2(0,1)} \le 2\sqrt{2}\; h_n |u|_{H^1(0,1)}$$

is satisfied.

5 The proof of the inverse inequality (Theorem 2)

The proof of Theorem 2 is rather easy.

Proof of Theorem 2 We can extend every $u_{p,n} \in \widetilde{S}_{p,n}(0,1)$ to $(-1,1)$ by defining $w_{p,n}(x) := u_{p,n}(|x|)$ and obtain $w_{p,n} \in S^{per}_{p,n}(-1,1)$. The inverse inequality (1) is equivalent to

$$|w_{p,n}|_{H^1(-1,1)} \leq 2\sqrt{3}h_n^{-1}\|w_{p,n}\|_{L^2(-1,1)}. \tag{20}$$

This is shown using induction in p for all $u \in \widetilde{S}_{p,n}(-1,1)$. For $p = 1$, (20) is known, cf. [15], Theorem 3.91.

Now, we show that the constant does not increase for larger p. So assume $p > 1$ to be fixed. Due to the periodicity and due to the Cauchy-Schwarz inequality,

$$\begin{aligned} |w_{p,n}|^2_{H^1(-1,1)} &= \int_{-1}^{1} (w'_{p,n})^2 dx = -\int_{-1}^{1} w''_{p,n} w_{p,n} dx \\ &\leq \|w''_{p,n}\|_{L^2(-1,1)}\|w_{p,n}\|_{L^2(-1,1)} = |w'_{p,n}|_{H^1(-1,1)}\|w_{p,n}\|_{L^2(-1,1)} \end{aligned}$$

is satisfied. Using the induction assumption (and $w'_{p,n} \in S^{per}_{p-1,n}(-1,1)$, cf. [14], Theorem 5.9), we know that

$$|w'_{p,n}|_{H^1(-1,1)} \leq 2\sqrt{3}h_n^{-1}\|w'_{p,n}\|_{L^2(-1,1)} = Ch_n^{-1}|w_{p,n}|_{H^1(-1,1)}.$$

Combining these results, we obtain

$$|w_{p,n}|^2_{H^1(-1,1)} \leq 2\sqrt{3}h_n^{-1}|w_{p,n}|_{H^1(-1,1)}\|w_{p,n}\|_{L^2(-1,1)}$$

and further

$$|w_{p,n}|_{H^1(-1,1)} \leq 2\sqrt{3}h_n^{-1}\|w_{p,n}\|_{L^2(-1,1)}.$$

This shows (20), which concludes the proof. □

Remark 5 The proof of Theorem 2 does not require the grid to be equidistant. Having a general grid, the following estimate is satisfied:

$$|u_{p,n}|_{H^1(0,1)} \leq 2\sqrt{3}\; h^{-1}\|u_{p,n}\|_{L^2(0,1)}$$

for all splines $u_{p,n}$ on $(0,1)$ with vanishing odd derivatives at the boundary, where h is the size of the *smallest* element.

As we have proven both an approximation error estimate and a corresponding inverse inequality, both of them are sharp (up to constants independent of p and h_n):

Corollary 2 *For each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$, there is a function $u \in H^1(0,1)$ such that*

$$\inf_{u_{p,n}\in\widetilde{S}_{p,n}(0,1)} \|u - u_{p,n}\|_{L^2(0,1)} \geq \frac{1}{4\sqrt{3}}\; h_n|u|_{H^1(0,1)} > 0.$$

Proof Let $u \in S_{p,n+1}(0,1)$ be a non-constant function with $(u, \tilde{u}_{p,n})_{L^2(0,1)} = 0$ for all $\tilde{u}_{p,n} \in S_{p,n}$. In this case, the infimum is taken for $u_{p,n} = 0$. So, we obtain using Theorem 2 $\inf_{u_{p,n} \in \widetilde{S}_{p,n}(0,1)} \|u - u_{p,n}\|_{L^2(0,1)} = \|u\|_{L^2(0,1)} \geq \frac{1}{2\sqrt{3}}\, h_{n+1} |u|_{H^1(0,1)}$. As $h_{n+1} \geq h_n/2$ for all $n \in \mathbb{N}$, this finishes the proof. □

Corollary 3 *For each $n \in \mathbb{N}$ (with $n \geq 2$) and each $p \in \mathbb{N}$, there is a function $u_{p,n} \in \widetilde{S}_{p,n}(0,1) \backslash \{0\}$ such that*

$$|u_{p,n}|_{H^1(0,1)} \geq \frac{1}{4\sqrt{2}}\, h_n^{-1} \|u_{p,n}\|_{L^2(0,1)}.$$

Proof Let $u_{p,n} \in S_{p,n}(0,1)\backslash\{0\}$ be such that $(u_{p,n}, \tilde{u}_{p,n-1})_{L^2(0,1)} = 0$ for all $\tilde{u}_{p,n-1} \in S_{p,n-1}$. For this case $\|u_{p,n}\|_{L^2(0,1)} = \inf_{u_{p,n-1} \in \widetilde{S}_{p,n-1}(0,1)} \|u_{p,n} - u_{p,n-1}\|_{L^2(0,1)} \leq 2\sqrt{2} h_{n-1}^{-1} |u_{p,n}|_{H^1(0,1)}$. As $h_n \geq h_{n-1}/2$ for all $n \geq 2$, this finishes the proof. □

6 An extension to higher Sobolev indices

We can easily lift Theorem 1 (Corollary 1) up to higher Sobolev indices.

Theorem 5 *For each $q \in \mathbb{N}$, each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$ with $0 < q \leq p+1$, there is for each $u \in H^q(0,1)$, a spline approximation $u_{p,n} \in \widetilde{S}^{(q)}_{p,n}(0,1)$ such that*

$$|u - u_{p,n}|_{H^{q-1}(0,1)} \leq 2\sqrt{2}\, h_n |u|_{H^q(0,1)},$$

where $\widetilde{S}^{(q)}_{p,n}(0,1)$ is the space of all $u_{p,n} \in S_{p,n}(0,1)$ that satisfy the following symmetry condition:

$$\frac{\partial^{2l+q}}{\partial x^{2l+q}} u_{p,n}(0) = \frac{\partial^{2l+q}}{\partial x^{2l+q}} u_{p,n}(1) = 0 \text{ for all } l \in \mathbb{N}_0 \text{ with } 2l + q < p.$$

Proof The proof is done by induction. From Corollary 1, we know the estimate for $q = 1$ (as $\widetilde{S}^{(1)}_{p,n}(0,1) = \widetilde{S}_{p,n}(0,1)$) and all $p > q - 1 = 0$. For $q = 1$ and $p = q - 1 = 0$, the estimate is a well-known result, cf. [14], Theorem 6.1, (6.7), where (in our notation) $|u - u_{0,n}|_{L^2(0,1)} \leq h_n |u|_{H^1(0,1)}$ has been shown.

So, now we assume to know the estimate for some $q - 1$ and show it for q.

As $u \in H^q(0,1)$, we know that $u' \in H^{q-1}(0,1)$, so we can apply the induction hypothesis and obtain that there is some $u_{p-1,n} \in \widetilde{S}^{(q-1)}_{p-1,n}(0,1)$ such that

$$|u' - u_{p-1,n}|_{H^{q-1}(0,1)} \leq 2\sqrt{2}\, h_n |u'|_{H^{q-1}(0,1)}.$$

Define

$$u_{p,n}(x) := c + \int_0^x u_{p-1,n}(\xi) d\xi. \tag{21}$$

Note that $u_{p,n} \in S_{p,n}(0,1)$ as integrating increases both the polynomial degree and the differentiability by 1, cf. [14], Theorem 5.16. Because by integrating the boundary conditions on the l-th derivative become conditions on the $l+1$-st derivative, we further have $u_{p,n} \in \widetilde{S}^{(q)}_{p,n}(0,1)$.

Therefore, we have

$$|u' - u'_{p,n}|_{H^{q-1}(0,1)} \leq 2\sqrt{2}\, h_n |u'|_{H^{q-1}(0,1)},$$

which is the same as

$$|u - u_{p,n}|_{H^{q}(0,1)} \leq 2\sqrt{2}\, h_n |u|_{H^{q}(0,1)}.$$

This finishes the proof. □

Remark 6 The integration constant (integration constants for $q > 2$) in (21) can be used to guarantee that

$$\int_0^1 \frac{\partial^l}{\partial x^l}(u(x) - u_{p,n}(x))\mathrm{d}x = 0$$

for all $l \in \{0, 1, \dots, q-1\}$.

For the spaces $\widetilde{S}^{(q)}_{p,n}(0,1)$ there is again an inverse inequality.

Theorem 6 *For each $n \in \mathbb{N}$, each $q \in \mathbb{N}$ and each $p \in \mathbb{N}$ with $0 < q \leq p+1$,*

$$|u_{p,n}|_{H^q(0,1)} \leq 2\sqrt{3} h_n^{-1} |u_{p,n}|_{H^{q-1}(0,1)} \tag{22}$$

is satisfied for all $u_{p,n} \in \widetilde{S}^{(q)}_{p,n}(0,1)$, where $\widetilde{S}^{(q)}_{p,n}(0,1)$ is as defined in Theorem 5.

Proof First note that (22) is equivalent to

$$\left| \frac{\partial^{q-1}}{\partial x^{q-1}} u_{p,n} \right|_{H^1(0,1)} \leq 2\sqrt{3} h_n^{-1} \left\| \frac{\partial^{q-1}}{\partial x^{q-1}} u_{p,n} \right\|_{L^2(0,1)}. \tag{23}$$

As $\frac{\partial^{q-1}}{\partial x^{q-1}} u_{p,n} \in \widetilde{S}^{(1)}_{p-q+1,n}(0,1) = \widetilde{S}_{p-q+1,n}(0,1)$, cf. [14], Theorem 5.9, the estimate (23) follows directly from Theorem 2. □

Again, as we have both an approximation error estimate and an inverse inequality, we know that both of them are sharp (cf. Corollaries 2 and 3).

The following theorem is directly obtained from telescoping.

Theorem 7 *For each $q \in \mathbb{N}_0$, each $n \in \mathbb{N}$, each $p \in \mathbb{N}$, each $r \in \mathbb{N}$ with $0 \leq r \leq q \leq p+1$, there is for each $u \in H^q(0,1)$ a spline approximation $u_{p,n} \in S_{p,n}(0,1)$ such that*

$$|u - u_{p,n}|_{H^r(0,1)} \leq (2\sqrt{2}\, h_n)^{q-r} |u|_{H^q(0,1)}$$

is satisfied.

Proof Theorem 5 states the desired result for $r = q - 1$. For $r < q - 1$, the statement is shown by induction in r. So, we assume to know the desired result for some r, i.e., there is a spline approximation $u_{p,n} \in S_{p,n}(0,1)$ such that

$$|u - u_{p,n}|_{H^r(0,1)} \leq (2\sqrt{2}\, h_n)^{q-r} |u|_{H^q(0,1)}. \tag{24}$$

Now, we show that there is some $\widetilde{u_{p,n}} \in S_{p,n}(0,1)$ such that

$$|u - \widetilde{u_{p,n}}|_{H^{r-1}(0,1)} \leq (2\sqrt{2}\, h_n)^{q-(r-1)} |u|_{H^q(0,1)}. \tag{25}$$

Theorem 5 states that there is a function $\widetilde{u_{p,n}} \in S_{p,n}(0,1)$ such that

$$|u - \widetilde{u_{p,n}}|_{H^{r-1}(0,1)} \leq 2\sqrt{2}\, h_n |u - u_{p,n}|_{H^r(0,1)},$$

which shows together with the induction assumption (24) the induction hypothesis (25). □

Here, it is not known to the authors how to choose a proper subspace of $S_{p,n}(0,1)$ such that a complementary inverse inequality can be shown.

7 Extension to two and more dimensions and application in Isogeometric Analysis

We can extend Theorem 1 (and also Corollary 1) to the following theorem for a tensor-product structured grid on $\Omega := (0,1)^d$. Here, we can introduce $\widetilde{W}_{p,n}(\Omega) = \otimes_{i=1}^d \widetilde{S}_{p,n}(0,1)$. Assuming that $(\phi_{p,n}^{(0)}, \ldots, \phi_{p,n}^{(N)})$ is a basis of $\widetilde{S}_{p,n}(0,1)$, the space $\widetilde{W}_{p,n}(\Omega)$ is given by

$$\widetilde{W}_{p,n}(\Omega) = \left\{ w \,:\, w(x_1,\ldots,x_d) = \sum_{i_1,\ldots,i_d=0}^{N} w_{i_1,\ldots,i_d} \phi_{p,n}^{(i_1)}(x_1) \cdots \phi_{p,n}^{(i_d)}(x_d) \right\}.$$

Theorem 8 *Let $\Omega = (0,1)^d$. For each $u \in H^1(\Omega)$, each $n \in \mathbb{N}$ and each $p \in \mathbb{N}_0$ there is a spline approximation $w_{p,n} \in \widetilde{W}_{p,n}(\Omega)$ such that*

$$\|u - w_{p,n}\|_{L^2(\Omega)} \leq 2\sqrt{2d}\, h_n |u|_{H^1(\Omega)}$$

is satisfied.

The proof is similar to the proof in [3], Section 4, for the two dimensional case. To keep the paper self-contained we give a proof of this theorem.

Proof of Theorem 8 For sake of simplicity, we restrict ourselves to $d = 2$. The extension to more dimensions is completely analogous. Here

$$\widetilde{W}_{p,n}(\Omega) = \widetilde{S}_{p,n}(0,1) \otimes \widetilde{S}_{p,n}(0,1) = \left\{ w \,:\, w(x,y) = \sum_{i,j=0}^{N} w_{i,j} \phi_{p,n}^{(i)}(x) \phi_{p,n}^{(j)}(y) \right\}.$$

We assume $u \in C^\infty(\Omega)$ and show the desired result using a standard density argument. Using Theorem 1, we can introduce for each $x \in (0,1)$ a function $v(x,\cdot) \in \widetilde{S}_{p,n}(0,1)$ with

$$\|u(x,\cdot) - v(x,\cdot)\|_{L^2(0,1)} \leq 2\sqrt{2}\, h_n |u(x,\cdot)|_{H^1(0,1)}.$$

By squaring and taking the integral over x, we obtain

$$\|u - v\|_{L^2(\Omega)} \leq 2\sqrt{2}\, h_n \left\| \frac{\partial}{\partial y} u \right\|_{L^2(\Omega)}. \tag{26}$$

By choosing $v(x,\cdot)$ to be the L^2-orthogonal projection, we also have

$$\|v(x,\cdot)\|_{L^2(0,1)} \leq \|u(x,\cdot)\|_{L^2(0,1)}$$

for all $x \in (0,1)$ and consequently

$$\left\| \frac{\partial}{\partial x} v(x,\cdot) \right\|_{L^2(0,1)} \leq \left\| \frac{\partial}{\partial x} u(x,\cdot) \right\|_{L^2(0,1)}. \tag{27}$$

As $v(x,\cdot) \in \widetilde{S}_{p,n}(0,1)$, there are coefficients $v_j(x)$ such that

$$v(x,y) = \sum_{j=0}^{N} v_j(x) \phi_{p,n}^{(j)}(y).$$

Using Corollary 1, we can introduce for each $j \in \{0,\ldots,N\}$ a function $w_j \in \widetilde{S}_{p,n}(0,1)$ with

$$\|v_j - w_j\|_{L^2(0,1)} \leq 2\sqrt{2}\, h_n |v_j|_{H^1(0,1)}. \tag{28}$$

Next, we introduce a function w by defining

$$w(x,y) := \sum_{j=0}^{N} w_j(x) \phi_{p,n}^{(j)}(y),$$

which is obviously a member of the space $\widetilde{W}_{p,n}(\Omega)$. By squaring (28), multiplying it with $\phi_{p,n}^{(j)}(y)^2$, summing over j and taking the integral, we obtain

$$\int_0^1 \sum_{j=0}^{N} \|v_j - w_j\|_{L^2(0,1)}^2 \phi_{p,n}^{(j)}(y)^2 \mathrm{d}y \leq 8\, h_n^2 \int_0^1 \sum_{j=0}^{N} |v_j|_{H^1(0,1)}^2 \phi_{p,n}^{(j)}(y)^2 \mathrm{d}y.$$

Using the definition of the norms, we obtain

$$\int_0^1 \int_0^1 \sum_{j=0}^{N} (v_j(x) - w_j(x))^2 \phi_{p,n}^{(j)}(y)^2 \mathrm{d}x\mathrm{d}y \leq 8\, h_n^2 \int_0^1 \int_0^1 \sum_{j=0}^{N} v_j'(x)^2 \phi_{p,n}^{(j)}(y)^2 \mathrm{d}x\mathrm{d}y$$

and further

$$\|v - w\|_{L^2(\Omega)} \leq 2\sqrt{2}\, h_n \left\| \frac{\partial}{\partial x} v \right\|_{L^2(\Omega)}.$$

Using (27), we obtain

$$\|v-w\|_{L^2(\Omega)} \le 2\sqrt{2}\, h_n \left\| \frac{\partial}{\partial y} u \right\|_{L^2(\Omega)}. \tag{29}$$

Using (26) and (29), we obtain

$$\begin{aligned}\|u-w\|_{L^2(\Omega)} &\le \|u-v\|_{L^2(\Omega)} + \|v-w\|_{L^2(\Omega)} \\ &\le 2\sqrt{2}\, h_n \left\| \frac{\partial}{\partial y} u \right\|_{L^2(\Omega)} + 2\sqrt{2}\, h_n \left\| \frac{\partial}{\partial x} u \right\|_{L^2(\Omega)} \\ &\le 4\, h_n |u|_{H^1(\Omega)},\end{aligned}$$

which finishes the proof. □

The extension of Theorem 2 to two or more dimensions is rather easy.

Theorem 9 *Consider $\Omega := (0,1)^d$. For each $n \in \mathbb{N}$ and each $p \in \mathbb{N}$,*

$$|u_{p,n}|_{H^1} \le 2\,\sqrt{3d}\, h_n^{-1}\, \|u_{p,n}\|_{L^2}$$

is satisfied for all $u_{p,n} \in \widetilde{W}_{p,n}(\Omega)$.

Proof For sake of simplicity, we restrict ourselves to $d = 2$. The generalization to more dimensions is completely analogous.

We have obviously

$$\begin{aligned}|u_{p,n}|^2_{H^1} &= \left\| \frac{\partial}{\partial x} u_{p,n} \right\|^2_{L^2} + \left\| \frac{\partial}{\partial y} u_{p,n} \right\|^2_{L^2} \\ &= \int_0^1 |u_{p,n}(\cdot,y)|^2_{H^1} \mathrm{d}y + \int_0^1 |u_{p,n}(x,\cdot)|^2_{H^1} \mathrm{d}x\end{aligned}$$

This can be bounded from above using Theorem 2 by

$$= 12 h_n^{-2} \left(\int_0^1 \|u_{p,n}(\cdot,y)\|^2_{L^2} \mathrm{d}y + \int_0^1 \|u_{p,n}(x,\cdot)\|^2_{L^2} \mathrm{d}x \right) = 24 h_n^{-2} \|u_{p,n}\|^2_{L^2},$$

which finishes the proof. □

The extension to isogeometric spaces can be done following the approach presented in [1], Section 3.3. In Isogeometric Analysis, we have a geometry parameterization $\mathbf{F} : (0,1)^d \to \Omega$. An isogeometric function on Ω is then given as the composition of a B-spline on $(0,1)^d$ with the inverse of $\mathbf{F}$. The following result can be shown using a standard chain rule argument.

There exists a constant $C = C(\mathbf{F}, q)$ such that

$$C^{-1}\, \|f\|_{H^q(\Omega)} \le \|f \circ \mathbf{F}\|_{H^q((0,1)^d)} \le C\, \|f\|_{H^q(\Omega)} \tag{30}$$

for all $f \in H^q(\Omega)$.

See [1], Lemma 3.5, or [3], Corollary 5.1, for related results. In both papers the statements are slightly more general, [1] gives a more detailed dependence on the parameterization $\mathbf{F}$ whereas [3] establishes bounds for anisotropic meshes.

Using this equivalence of norms, we can transfer all results from the parameter domain $(0,1)^d$ to the physical domain Ω. However, we need to point out that this equivalence is not valid for seminorms. Hence, in Theorem 1 (and follow-up Theorems 5, 7 and 8) the seminorms on the right hand side of the equations need to be replaced by the full norms. Moreover, the bounds depend on the geometry parameterization via the constant C in (30).

A similar strategy can be followed when extending the results to NURBS. We do not go into the details here but refer to [1,3] for a more detailed study. In the case of NURBS the seminorms again have to be replaced by the full norms due to the quotient rule of differentiation. In that case the constants of the bounds additionally depend on the given denominator of the NURBS parameterization.

References

1. Yuri Bazilevs, Lourenço Beirão da Veiga, J Austin Cottrell, Thomas JR Hughes, and Giancarlo Sangalli, *Isogeometric analysis: approximation, stability and error estimates for h-refined meshes*, Mathematical Models and Methods in Applied Sciences **16** (2006), no. 07, 1031–1090.
2. Lourenço Beirão da Veiga, Annalisa Buffa, J Rivas, and Giancarlo Sangalli, *Some estimates for h–p–k-refinement in isogeometric analysis*, Numerische Mathematik **118** (2011), no. 2, 271–305.
3. Lourenço Beirão da Veiga, Durkbin Cho, and Giancarlo Sangalli, *Anisotropic nurbs approximation in isogeometric analysis*, Computer Methods in Applied Mechanics and Engineering **209** (2012), 1–11.
4. Dietrich Braess, *Finite elements: Theory, fast solvers, and applications in solid mechanics (second editin)*, Cambridge University Press, 2001.
5. Achi Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp. **31** (1977), 333 – 390.
6. Paul Butzer and Hubert Berens, *Semi-Groups of Operators and Approximation*, Springer-Verlag, Berlin Heidelberg New York, 1967.
7. Charles K Chui, *An introduction to wavelets, wavelet analysis and its applications*, Academic Press, Boston (1992).
8. Ronald A DeVore and George G Lorentz, *Constructive approximation*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Springer, 1993.
9. Marco Donatelli, Carlo Garoni, Carla Manni, Stefano Serra-Capizzano, and Hendrik Speleers, *Robust and optimal multi-iterative techniques for iga galerkin linear systems*, Computer Methods in Applied Mechanics and Engineering **284** (2015), no. 0, 230 – 264, Isogeometric Analysis Special Issue.
10. John A Evans, Yuri Bazilevs, Ivo Babuška, and Thomas JR Hughes, *n-widths, sup–infs, and optimality ratios for the k-version of the isogeometric finite element method*, Computer Methods in Applied Mechanics and Engineering **198** (2009), no. 21, 1726–1741.
11. Carlo Garoni, Carla Manni, Francesca Pelosi, Stefano Serra-Capizzano, and Hendrik Speleers, *On the spectrum of stiffness matrices arising from isogeometric analysis*, Numerische Mathematik **127** (2014), no. 4, 751–799 (English).
12. Thomas JR Hughes, J Austrin Cottrell, and Yuri Bazilevs, *Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement*, Computer Methods in Applied Mechanics and Engineering **194** (2005), no. 39-41, 4135 – 4195.

13. Nicolai Korneichuk, *Exact constants in approximation theory*, vol. 38, Cambridge University Press, 1991.
14. Larry L Schumaker, *Spline functions: basic theory*, Cambridge University Press, 1981.
15. Christoph Schwab, *p- and hp- finite element methods: Theory and applications in solid and fluid mechanics*, Numerical Mathematics and Scientific Computation, Clarendon Press, 1998.
16. Ulrich Trottenberg, Cornelius Oosterlee, and Anton Schüller, *Multigrid*, Academic Press, London, 2001.
17. Ren-Hong Wang, Yan Xu, and Zhi-Qiang Xu, *Eulerian numbers: a spline perspective*, Journal of Mathematical Analysis and Applications **370** (2010), no. 2, 486–490.

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