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Institute of Computational Mathematics

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Irina Georgieva
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev St., Bl. 8, 1113, Sofia, Bulgaria

Clemens Hofreither
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

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Interpolating Solutions of the Poisson Equation in the Disk based on Radon Projections

I. Georgieva[a], C. Hofreither[b]

[a] *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev St., Bl. 8, 1113, Sofia, Bulgaria.* `irina@math.bas.bg`
[b] *Institute of Computational Mathematics, Johannes Kepler University Linz, Altenbergerstr. 69, 4040 Linz, Austria.* `chofreither@numa.uni-linz.ac.at`

Abstract

We consider an algebraic method for reconstruction of a function satisfying the Poisson equation with a polynomial right-hand side in the unit disk. The given data, besides the right-hand side, is assumed to be in the form of a finite number of values of Radon projections of the unknown function. We first homogenize the problem by finding a polynomial which satisfies the given Poisson equation. This leads to an interpolation problem for a harmonic function, which we solve in the space of harmonic polynomials using a previously established method. For the special case where the Radon projections are taken along chords that form a regular convex polygon, we extend the error estimates from the harmonic case to this Poisson problem. Finally we give some numerical examples.

Keywords: multivariate interpolation, Radon projections, Poisson equation, harmonic polynomials

1. Introduction

The classical approach to interpolation is based on sampling a given function at a finite number of points. This is natural for approximation of univariate functions since a table of function values is a standard type of information in practical problems and processes described by functions in one variable. Moreover, the Lagrange interpolation problem by polynomials is always uniquely solvable.

In the multivariate case, such an approach is met with serious difficulties. For example, the pointwise interpolation by multivariate polynomials is no longer possible for every choice of the nodes. See [1] and the references therein for a survey of multivariate polynomial interpolation. Furthermore, there are many practical problems in which the information about the relevant function comes as a set of functionals which are not point evaluations. For instance, in computer tomography, a table of mean values of a function of d variables on $(d-1)-$dimensional hyperplanes is the data on which the reconstruction is based. Such nondestructive methods have important practical applications in medicine, radiology, geology, etc., and have their theoretical foundation in the work of Johann Radon in the early twentieth century [2].

Mathematically speaking, the problem is to recover or approximate a multivariate function using information given as integrals of the unknown function over a number of hyperplanes. This problem has been intensively studied since the 1960s using different approaches [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and continues to find many applications. Among the developed reconstruction algorithms are filtered backprojection, iterative reconstruction, direct methods, etc., and some are based on the inverse Radon transform.

One class of methods uses direct interpolation by multivariate polynomials [9, 13, 14, 15, 16, 17, 18, 19, 20]. In our work, we follow this general approach. The interpolant is sought in an appropriate polynomial space such that it matches the given Radon projections exactly.

To improve the approximation accuracy and to reduce the amount of input data required as well as the computational effort, it seems natural to incorporate additional knowledge about the function to be recovered into the approximation method. This was first suggested by Borislav Bojanov. Such problem-specific knowledge is often provided in the form of a partial differential equation which the unknown function satisfies.

In the present paper, we concern ourselves with the case where the unknown function satisfies the Poisson equation $\Delta u = u_{xx} + u_{yy} = f$, where f is a polynomial. This elliptic partial differential equation is important both as a model problem as well as in applications.

The present work expands on the earlier articles [21, 22], where the Laplace equation was considered, i.e., the homogeneous case $f = 0$. Therein, first results on interpolation of harmonic functions with harmonic polynomials based on Radon projections along chords of the unit circle were presented. The existence of a unique interpolant in the space of harmonic polynomials was shown for a family of schemes where all chords are chosen at equal distance to the origin. For the special case of chords forming a regular convex polygon, error estimates on the unit circle and in the unit disk were proved.

In the present paper, our main aim is to extend several of these results to the inhomogeneous case, i.e., the Poisson equation, with a polynomial right-hand side, again using Radon projections type of data. Both the Laplace and the Poisson equation have many practical applications such as heat transport, diffusion problems or in Stokes flow of incompressible fluids, making them interesting both as model problems and with a view to applications. The main idea is to reduce the interpolation problem to the harmonic case by finding a suitable homogenizing polynomial. This allows us to prove existence and uniqueness of a polynomial interpolant. We obtain an error estimate under certain more restrictive assumptions.

2. Preliminaries

Let $D \subset \mathbb{R}^2$ denote the open unit disk and ∂D the unit circle. By $I(\theta, t)$ we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Figure 1), parameterized by

$$s \mapsto (t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)^\top, \quad \text{where } s \in (-\sqrt{1-t^2}, \sqrt{1-t^2}).$$

Definition 1. Let $u(x, y)$ be a real-valued bivariate function in the unit disk D. The *Radon projection* $\mathcal{R}_\theta(u; t)$ of u in direction θ is defined by the line integral

$$\mathcal{R}_\theta(u; t) := \int_{I(\theta,t)} u(x, y)\, d\ell = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} u(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

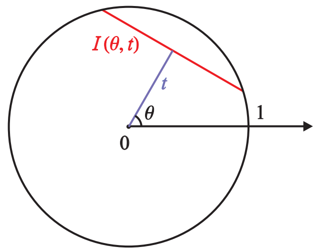

Figure 1: The chord $I(\theta, t)$ of the unit circle.

Johann Radon [2] showed in 1917 that a differentiable function u is uniquely determined by the values of its Radon transform,

$$u \mapsto \left\{\mathcal{R}_\theta(u; t) \,:\, -1 \le t \le 1,\ 0 \le \theta < \pi\right\}.$$

2.1. Interpolation by Harmonic Polynomials

We recall some results on interpolation of harmonic functions by harmonic polynomials using Radon projections from [22]. This corresponds to the special case $f = 0$ of the Poisson equation.

Let Π_n^2 denote the space of real bivariate polynomials of total degree at most n. We consider the subspace

$$\mathcal{H}_n = \left\{p \in \Pi_n^2 : \Delta p = 0\right\}$$

of real bivariate harmonic polynomials of total degree at most n, which has dimension $2n + 1$. We use the basis of the harmonic polynomials

$$\phi_0(x, y) = 1, \quad \phi_{k,1}(x, y) = \mathrm{Re}(x + \mathrm{i}y)^k, \quad \phi_{k,2}(x, y) = \mathrm{Im}(x + \mathrm{i}y)^k,$$

for $k = 1, \dots, n$. In polar coordinates, they have the representation

$$\phi_{k,1}(r, \theta) = r^k \cos(k\theta), \quad \phi_{k,2}(r, \theta) = r^k \sin(k\theta).$$

The following result, which gives a closed formula for Radon projections of the basis harmonic polynomials, is the harmonic form of the famous Marr's formula [9]. A direct proof can be found in [22].

Theorem 1 ([22])**.** *The Radon projections of the basis harmonic polynomials are given by*

$$\int_{I(\theta,t)} \phi_{k,1}(x, y)\, d\ell = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\cos(k\theta),$$
$$\int_{I(\theta,t)} \phi_{k,2}(x, y)\, d\ell = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\sin(k\theta),$$

where $k \in \mathbb{N}$, $\theta \in \mathbb{R}$, $t \in (-1, 1)$, and $U_k(t) = \frac{\sin((k+1)\arccos(t))}{\sin(\arccos(t))}$ is the k-th Chebyshev polynomial of second kind.

We prescribe chords

$$\mathcal{I} := \{I_i = I(\theta_i, t_i) : \theta_i \in [0, \pi),\ t_i \in (-1, 1)\}_{i=1}^{2n+1}$$

of the unit circle and associated given values $\Gamma = \{\gamma_i\}_{i=1}^{2n+1}$, and wish to find a harmonic polynomial $p \in \mathcal{H}_n$ such that

$$\mathcal{R}_{\theta_i}(p, t_i) = \int_{I(\theta_i, t_i)} p(x, y)\, d\ell = \gamma_i, \qquad i = 1, \ldots, 2n+1. \tag{1}$$

Definition 2. A scheme of chords $\mathcal{I}$ is called *regular* if the interpolation problem (1) has a unique solution for arbitrary given values Γ.

Theorem 1 plays a crucial role in proving sufficient conditions for regularity of schemes $\mathcal{I}$ of chords.

Theorem 2 (Existence and uniqueness [21, 22])**.** *The interpolation problem* (1) *has a unique solution for any set of chords* $\mathcal{I} = \{I(\theta_i, t_i)\}_{i=1}^{2n+1}$ *with*

$$0 \le \theta_1 < \theta_2 < \ldots < \theta_{2n+1} < 2\pi$$

and with constant distances $t_i = t \in (-1, 1)$ *such that* t *is not a zero of any Chebyshev polynomial of the second kind* $U_1, \ldots, U_n$*.*

See Figure 2 for some examples of (regular) schemes which satisfy the conditions of the above theorem, and one which does not and is in fact not regular. Note that the theorem is not a characterization, and schemes of many other types can be regular.

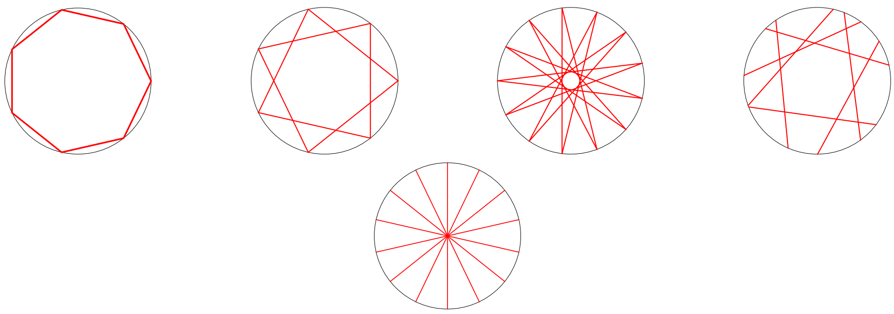

Figure 2: *Top:* Some admissible schemes according to Theorem 2. *Bottom:* A scheme which does not satisfy the assumptions of Theorem 2 since $t = 0$ is a root of every Chebyshev polynomial of odd degree. This scheme is not regular.

For the error estimate, we make the stronger assumption that the chords form a regular convex $(2n+1)$-sided polygon inscribed in the unit circle; cf. Figure 2, first picture. We thus consider the sequence $\mathcal{I}^{(n)} = \{I(\theta_i^{(n)}, t^{(n)}) : i = 1, \ldots, 2n+1\}$ of schemes with the angles and the distances, respectively,

$$\theta_i^{(n)} = \frac{2\pi i}{2n+1}, \quad t^{(n)} = \cos\frac{\pi}{2n+1}, \qquad \text{for } i = 1, \ldots, 2n+1. \tag{2}$$

Then we know the following error estimate.

Theorem 3 ([22]). *Consider the interpolation problem* (1) *with* $\gamma_i = \int_{\mathcal{I}_i} u$ *for a harmonic function* $u \in C^2(D)$. *Assume that* $g = u|_{\partial D}$ *has a uniformly convergent Fourier series*

$$g(\theta) = g_0 + \sum_{k=1}^{\infty} (g_k \cos(k\theta) + g_{-k} \sin(k\theta))$$

and its Fourier coefficients $(g_k)_{k\in\mathbb{Z}}$ *decay like* $|g_k| \leq M|k|^{-s}$ *with* $M > 0$, $s > 1$. *Let* $p^{(n)} \in \mathcal{H}_n$ *be the interpolating polynomial of degree* n *obtained by solving* (1). *Then the approximation error on the unit circle satisfies*

$$\|g - p^{(n)}\|_{L^2(\partial D)} \leq MCn^{-(s-1/2)}$$

with a constant C *which depends only on* s.

Remark 1. Under the assumptions of Theorem 3 above, we have also obtained an L^2-error estimate within the unit disk, namely,

$$\|u - p^{(n)}\|_{L^2(D)} = \mathcal{O}(n^{-(s-1/2)}).$$

This follows immediately from the observation that, for any harmonic function v in the unit disk, we have $\|v\|_{L_2(D)} \leq \|v\|_{L^2(\partial D)}$. See [22, Lemma 4] for a proof.

Remark 2. The condition number of the matrix associated with the interpolation problem (1) has been shown to be uniformly bounded by $2\sqrt{2}$ independently of the degree n of the interpolating polynomial in the regular polygonal case (see [22, Theorem 6]). Hence errors in the input data are not significantly amplified by our interpolation algorithm.

3. Interpolation problem for the Poisson equation

We now consider the following interpolation problem for the Poisson equation:

$$\left.\begin{aligned} \Delta u = f, \quad & f \in \Pi_m^2 \\ \int_{I_i} u(x,y)\, d\ell = \gamma_i, \quad & i = 1, \dots, 2n+1. \end{aligned}\right\} \tag{3}$$

3.1. Construction of the interpolant

We search for a solution to problem (3) in a polynomial space by first homogenizing and then using results for the harmonic case. The first step thus consists in finding a polynomial $u_f \in \Pi_{m+2}^2$ such that

$$\Delta u_f = f. \tag{4}$$

It is relatively easy to see that such a polynomial u_f always exists. We refer to [23, Theorem 1] for a constructive proof and therefore also a possible algorithm for computing it.

Next, we set $u_H := u - u_f$. Since $\Delta u_H = \Delta(u - u_f) = f - f = 0$, we see that u_H is harmonic. We now find a harmonic interpolant for u_H by solving the following problem: find a harmonic polynomial $p^{(n)} \in \mathcal{H}_n$ such that

$$\int_{I_i} p^{(n)}(x,y)\, d\ell = \int_{I_i} u_H(x,y)\, d\ell = \gamma_i - \int_{I_i} u_f(x,y)\, d\ell, \quad i = 1, \dots, 2n+1. \tag{5}$$

We have to assume that this harmonic interpolation problem has a solution, in other words, that the scheme $\mathcal{I}$ is regular. Theorem 2 gives a sufficient (but not necessary) condition for this to be the case.

The interpolant for u, which itself satisfies the Poisson equation, is then given in the form

$$\mathrm{Int}(u) := p^{(n)} + u_f \in \Pi^2_{\max\{m+2,n\}}. \tag{6}$$

3.2. Existence of a unique interpolant

Assume an interpolation problem of type (3) with some functions u and f and regular chords $\mathcal{I}$. Let $u_f^1, u_f^2 \in \Pi^2_{m+2}$ be two different solutions of (4) and let $p_1, p_2 \in \mathcal{H}_n$ be the solutions of the corresponding interpolation problems (5), namely

$$\int_{I_i} p_j(x,y)\, d\ell = \gamma_i - \int_{I_i} u_f^j(x,y)\, d\ell, \quad i = 1,\dots,2n+1, \quad j = 1,2.$$

Let $\mathrm{Int}_1 := u_f^1 + p_1$ and $\mathrm{Int}_2 := u_f^2 + p_2$. Note that

$$\Delta(\mathrm{Int}_1 - \mathrm{Int}_2) = f + 0 - f - 0 = 0, \tag{7}$$

i.e., $\mathrm{Int}_1 - \mathrm{Int}_2 \in \mathcal{H}_{\max\{m+2,n\}}$. We have that for $i = 1,\dots,2n+1$

$$\int_{I_i} (\mathrm{Int}_1 - \mathrm{Int}_2)\, d\ell = \int_{I_i} \big((u_f^1 + p_1) - (u_f^2 + p_2)\big)\, d\ell = 0. \tag{8}$$

In the case where $n \geq m+2$, from Theorem 2, (7),(8) it follows that

$$\mathrm{Int}_1 - \mathrm{Int}_2 \equiv 0,$$

i.e., the interpolant is uniquely determined and does not depend on the choice of u_f.

On the other hand, assume $m+2 > n$ and let us choose $u_f^2 := u_f^1 + d$ with $d \in \mathcal{H}_{m+2} \setminus \mathcal{H}_n$; in particular, $d \neq 0$. If the resulting interpolants were the same, $\mathrm{Int}_1 = \mathrm{Int}_2$, it would follow

$$\mathcal{H}_n \ni p_1 - p_2 = u_f^2 - u_f^1 = d \notin \mathcal{H}_n,$$

a contradiction. This proves the following theorem.

Theorem 4. *Assume that the chords $\mathcal{I}$ are regular; for instance, they satisfy the assumptions of Theorem 2. If $n \geq m+2$, then the interpolant* (6) *is independent of the choice of the homogenizing polynomial u_f. If $n < m+2$, then it is always possible to choose two different homogenizing polynomials such that the resulting interpolants are not equal.*

4. Error estimate

We make the assumption (2), i.e., that the chords form a regular convex $(2n+1)$-sided polygon. Furthermore, we assume that the given data $\{\gamma_i\}$ are the Radon projections of some unknown function $u \in C^2(D)$ satisfying the Poisson problem (3).

In this section, we give error estimates for the interpolating polynomial $\mathrm{Int}(u)$ in terms of the smoothness of the boundary data $g = u|_{\partial D}$. Being defined on the unit circle, g can be written as a periodic function of the angle θ. We will also rely on its Fourier series, i.e., let $(g_k)_{k\in\mathbb{Z}}$ be the Fourier coefficients of g such that

$$g(\theta) = g_0 + \sum_{k=1}^{\infty}(g_k \cos(k\theta) + g_{-k}\sin(k\theta)). \tag{9}$$

For simplicity, we will assume that the Fourier series converges uniformly to g.

The interpolation error is given by

$$u - \mathrm{Int}(u) = u - p^{(n)} - u_f = u_H - p^{(n)}. \tag{10}$$

In the following, we use the notations: let $g_H := u_H|_{\partial D}$ and $(g_{H,k})_{k\in\mathbb{Z}}$ be its Fourier coefficients; let $\tau := u_f|_{\partial D} \in T_{m+2}$ and $(\tau_k)_{|k|\leq m+2}$ be its Fourier coefficients. Here T_m denotes the space of trigonometric polynomials of degree up to m.

To apply Theorem 3 to Problem 2, we need a smoothness condition of the type

$$|g_{H,k}| \leq \overline{M}|k|^{-s} \qquad \forall k \in \mathbb{Z}. \tag{11}$$

However, it seems more natural to pose a smoothness condition on g, the boundary data of the unknown. We thus require $|g_k| \leq M|k|^{-s}$ with $M > 0$, $s > 1$ and analyze smoothness of g_H in dependence of g and u_f.

Since $u = u_H + u_f$, we have

$$|g_{H,k}| \leq |g_k| + |\tau_k|.$$

For $|k| > m+2$, we have

$$g_{H,k} = g_k$$

since τ has degree at most $m+2$, and (11) holds for these k due to the assumption on g_k.

Only for $|k| \leq m+2$, we have to take care to satisfy the assumption. We see that

$$\begin{aligned} |g_{H,k}| &\leq |g_k| + |\tau_k| \leq M|k|^{-s} + |\tau_k| \\ &= |k|^{-s}(M + |\tau_k||k|^s) \leq |k|^{-s}(M + \max_{k\leq m+2} |\tau_k||k|^s), \end{aligned}$$

and thus (11) holds with the choice

$$\overline{M} := M + \max_{|k|\leq m+2} |\tau_k||k|^s.$$

The error estimate follows from (10) and Theorem 3.

$$\|u - \mathrm{Int}(u)\|_{L^2(\partial D)} = \|u_H - p\|_{L^2(\partial D)} \leq (M + \max_{|k|\leq m+2} |\tau_k||k|^s)Cn^{-(s-1/2)}.$$

Thus, we have proved the following theorem.

Theorem 5. *Let $u \in C^2(D)$ be an exact solution of the Poisson problem* (3). *Assume that $g = u|_{\partial D}$ has a uniformly convergent Fourier series* (9) *and its Fourier coefficients $(g_k)_{k\in\mathbb{Z}}$ decay like $|g_k| \leq M|k|^{-s}$ with $M > 0$, $s > 1$. Let $\mathrm{Int}(u) \in \Pi^2_{\max\{m+2,n\}}$ be the interpolant for u according to* (6)*, where the chords are chosen according to* (2).

Then the approximation error on the unit circle satisfies

$$\|u - \mathrm{Int}(u)\|_{L^2(\partial D)} \leq \overline{M} C n^{-(s-1/2)}$$

where $\overline{M} = M + \max_{|k| \leq m+2} |\tau_k||k|^s$ *and* C *is a constant depending only on* s.

Remark 3. Under the assumptions of Theorem 5 above, we immediately obtain also an L^2-error estimate within the unit disk, namely,

$$\|u - \mathrm{Int}(u)\|_{L^2(D)} = \mathcal{O}(n^{-(s-1/2)}).$$

See Remark 1 for the argument which applies here analogously.

Remark 4. The assumptions (2) on the chord distances t can be weakened such that they only have to lie within a certain interval. We refer to [24] for a proof for the harmonic case, which translates directly to the present setting.

5. Numerical examples

In the following examples, we study instances of the problem (3) with known exact solution u and right-hand sides $f = \Delta u$. We then compute the Radon projections $\gamma_i = \mathcal{R}_{\theta_i}(u, t)$ of u taken along the edges of a regular $(2n + 1)$-sided convex polygon (Figure 2, first picture), i.e., $I_i = I(\theta_i, t)$ as in (2). Finally we compute the interpolant $\mathrm{Int}(u) \in \Pi^2_{\max\{m+2,n\}}$ as given in (6) and compute the error between the exact solution u and its interpolant.

Of course, only f and the (γ_i) serve as input to the interpolation algorithm. The exact solution u itself is only used to compute the interpolation errors.

5.1. Example 1

We interpolate the exact solution of the equation $\Delta u = 1$ $(m = 0)$,

$$u(x, y) = \exp(x) \cos(y) + \frac{y^2}{2} + x,$$

by $\mathrm{Int}(u) \in \Pi^2_{\max\{2,n\}}$ given the Radon projections $\gamma_i = \mathcal{R}_{\theta_i}(u, t)$ of u. In Figure 3, we display the graphs of the function u, its interpolant and of the error function $u - \mathrm{Int}(u)$ for $n = 7$.

For Figure 4, we vary the degree of the interpolating polynomial and plot the resulting relative L^2-errors. We see that the error decreases exponentially with n, indicating that the smooth function u is being approximated with optimal order.

5.2. Example 2

In order to study the behavior of the method for functions with less smoothness, we consider the Poisson equation $\Delta u = y^2 + xy + 1$ $(m = 2)$ with the exact solution

$$u = u_H + y^4/12 + x^3 y/6 + x^2/2 + 1. \tag{12}$$

Here u_H is chosen as the harmonic extension of the boundary function $g_H(\theta) = \theta^2$ on the unit circle in radial coordinates, with the argument θ in the interval $[-\pi, \pi]$. The

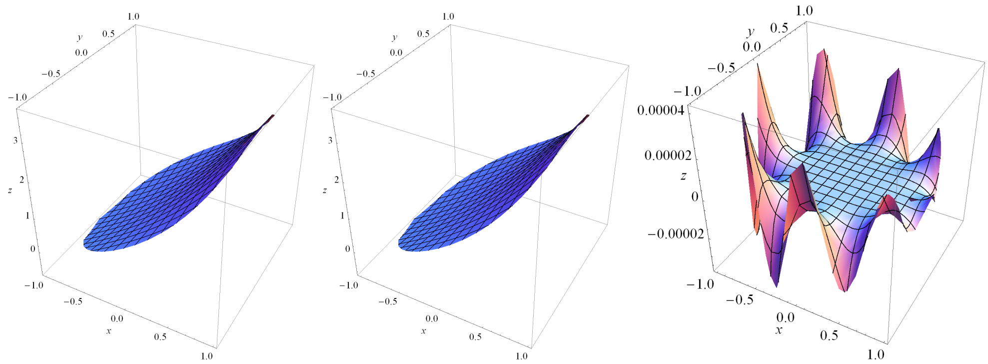

Figure 3: Example 1: $n = 7$: function u, interpolant $\mathrm{Int}(u)$, error $u - \mathrm{Int}(u)$ using 15 values of Radon projections.

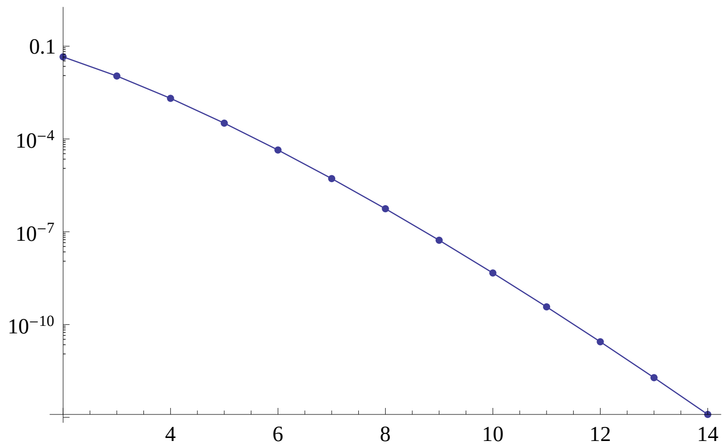

Figure 4: Example 1: errors. x-axis: degree of interpolating polynomial. y-axis: relative L^2-error

function u_H, and hence also u, is only C^0 on the unit circle, but analytic within the unit disk. By expanding the boundary data g_H into its Fourier series, we find that the corresponding harmonic function has the representation

$$u_H(x, y) = \mathrm{Re}\left(\frac{\pi^2}{3} + 2(\mathrm{Li}_2(-x - \mathrm{i}y) + \mathrm{Li}_2(-x + \mathrm{i}y))\right),$$

where

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

is the dilogarithm or Spence's function.

In Figure 5, we display the graphs of the function u, its interpolant and of the error function $u - \mathrm{Int}(u)$ for $n = 10$.

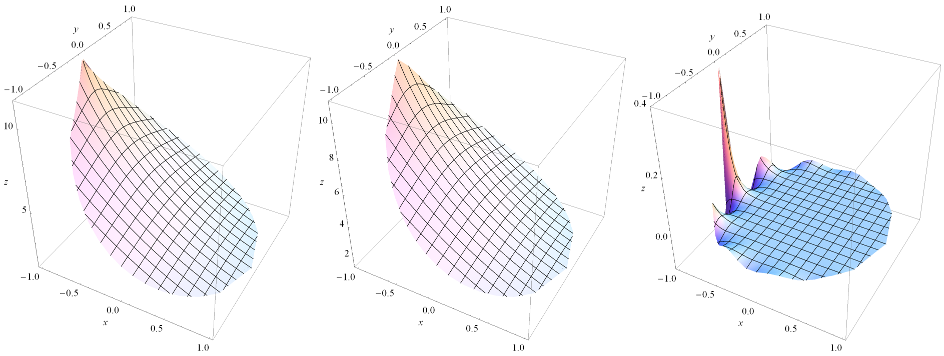

Figure 5: Example 2: $n = 10$: function u, interpolant $\mathrm{Int}(u)$, error $u - \mathrm{Int}(u)$ using 21 values of Radon projections.

The harmonic function u_H and hence also u satisfy the smoothness assumption from Theorem 5 with a parameter $s = 2$. Hence the boundary function $g = u|_{\partial D}$ satisfies the smoothness assumption with a parameter $s = 2$ also. The following error estimate follows from Remark 1

$$\|u - \mathrm{Int}(u)\|_{L^2(D)} = \|u_H - p^{(n)}\|_{L^2(D)} = \mathcal{O}(n^{-(s-1/2)}).$$

For Figure 6, we vary the degree of the interpolating polynomial and plot the resulting relative L^2-errors. We observe that the error decreases like $\mathcal{O}(n^{-1.8})$, indicating that the smooth function u is being approximated with slightly better order than the one predicted by the theory, $\mathcal{O}(n^{-3/2})$.

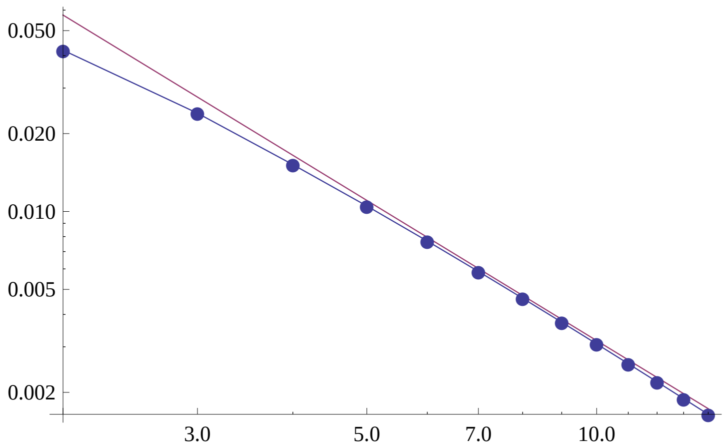

Figure 6: Example 2: errors and reference line $0.2n^{-1.8}$. x-axis: degree of interpolating polynomial. y-axis: relative L^2-error

5.3. Example 3

Consider the Poisson equation $\Delta u = 1$, with the exact solution $u = u_H + y^2/2 + x + 1$, where u_H which is given by the harmonic extension of the quadratic spline $g_H(\theta)$ on the unit circle,

$$g_H(\theta) = \begin{cases} -\frac{1}{2}(\theta + \frac{\pi}{2})(\theta + \frac{3}{2}\pi), & -\pi \le \theta < -\frac{\pi}{2}, \\ \frac{1}{2}(\theta - \frac{\pi}{2})(\theta + \frac{\pi}{2}), & -\frac{\pi}{2} \le \theta < \frac{\pi}{2}, \\ -\frac{1}{2}(\theta - \frac{\pi}{2})(\theta - \frac{3}{2}\pi), & \frac{\pi}{2} \le \theta < \pi. \end{cases}$$

In Figure 7, we display the graph of the function $g_H(\theta)$.

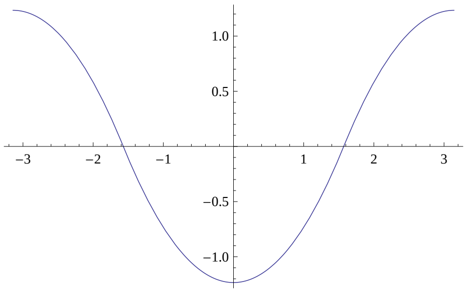

Figure 7: Example 3: function $g_H(\theta)$

Note that $g_H(\theta)$ is a periodic C^1-function with discontinuous second derivative. The resulting harmonic function u_H has the series representation (in polar coordinates)

$$u_H(r, \theta) = \sum_{k=1}^{\infty} (-1)^k r^{2k-1} \frac{4\cos((2k-1)\theta)}{(2k-1)^3 \pi}.$$

The harmonic function u_H and hence also u satisfy the smoothness assumption from Theorem 5 with a parameter $s = 3$.

In Figure 8, we display the graphs of the function u, its interpolant and of the error function $u - \mathrm{Int}(u)$ for $n = 10$.

For Figure 9, we vary the degree of the interpolating polynomial and plot the resulting relative L^2-errors. We observe that the error decreases like $\mathcal{O}(n^{-2.8})$, indicating that the smooth function u is being approximated slightly better than the rate $\mathcal{O}(n^{-5/2})$ predicted by the theory.

5.4. Example 4

We consider the same problem as in Example 1, but with artificially added measurement noise in the Radon projections. For this, we add to the exact values (γ_i) of the Radon projections random numbers from a normal distribution with zero mean and standard deviation ϵ. We perform three experiments with error levels $\epsilon \in \{10^{-3}, 10^{-6}, 10^{-9}\}$. The resulting relative errors in the reconstructed function are plotted in Figure 10. We

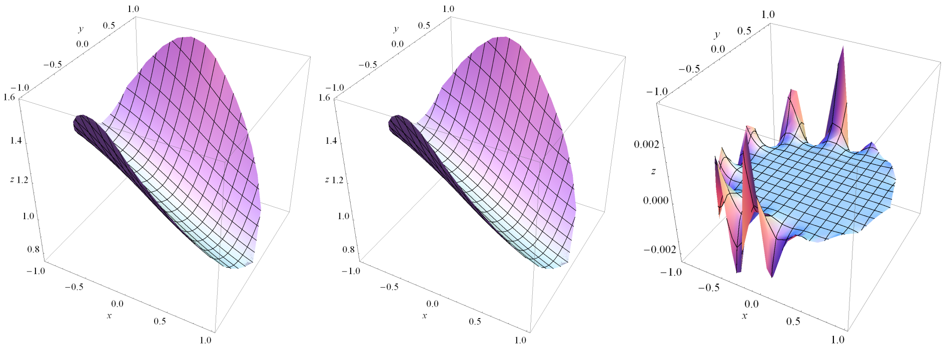

Figure 8: Example 3: $n = 10$: function u, interpolant $\mathrm{Int}(u)$, error $u - \mathrm{Int}(u)$ using 21 values of Radon projections.

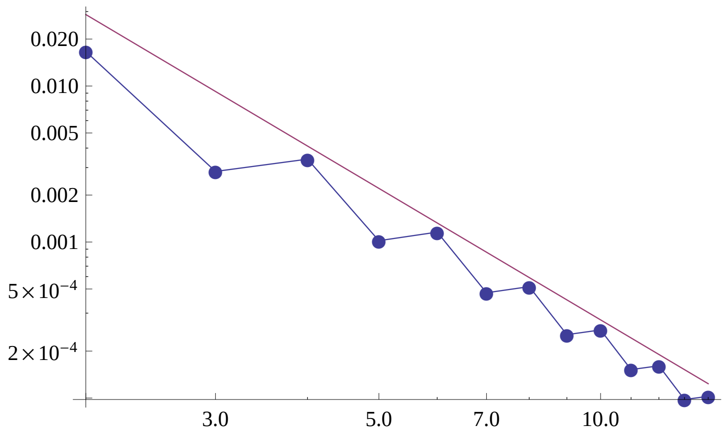

Figure 9: Example 3: errors and reference line $0.2n^{-2.8}$. x-axis: degree of interpolating polynomial. y-axis: relative L^2-error

see that the input function is reconstructed to the accuracy limit given by the noise level. No amplification of the noise or instabilities are observed which agrees with Remark 2. The computation of u_f does not depend on the values (γ_i) and therefore does not introduce any additional errors.

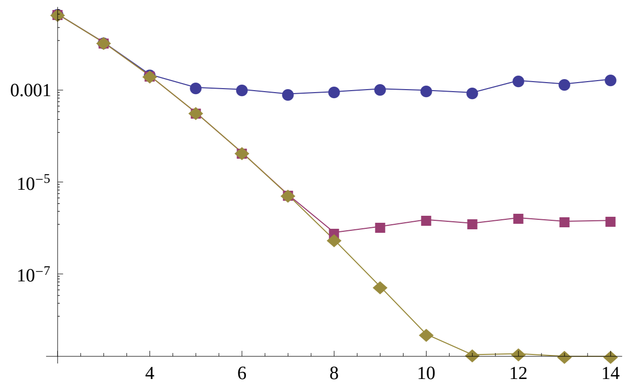

Figure 10: Example 4: errors with noisy data. Displayed are three experiments with noise levels of 10^{-3} (circles), 10^{-6} (squares), 10^{-9} (diamonds). x-axis: degree of interpolating polynomial. y-axis: relative L^2-error

6. Conclusions

We have constructed an interpolation method for functions in the unit disk which satisfy a Poisson equation with a polynomial right-hand side of degree m, and where Radon projections serve as the input data. The method proceeds by finding a suitable homogenizing polynomial of degree $m+2$ and then interpolating in the space of harmonic polynomials of degree up to n. Here $2n+1$ distinct Radon projections are the given data. We have shown that, if $n \geq m+2$, the interpolant does not depend on the choice of the homogenizing polynomial.

For a particular choice of chords, we have proved an interpolation error estimate which depends on the smoothness of the unknown function on the boundary. The error estimate also depends on the choice of homogenizing polynomial, but only by a constant factor which does not interfere with the asymptotic behavior with respect to n.

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