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Quantitative Stability of Optimization Problems and Generalized Equations

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Quantitative stability of optimization problems and generalized equations

Helmut Gfrerer* Diethard Klatte†

Abstract

This paper studies stability aspects of solutions of parametric mathematical programs and generalized equations, respectively, with disjunctive constraints. We present sufficient conditions that, under some constraint qualifications ensuring metric subregularity of the constraint mapping, continuity results of upper Lischitz and upper Hölder type, respectively, hold. Furthermore, we apply the above results to parametric mathematical programs with equilibrium constraints and demonstrate, how some classical results for the nonlinear programming problem can be recovered and even improved by our theory.

Key words. Mathematical programs with disjunctive constraints, stationarity, metric subregularity, variational analysis, upper Lipschitz stability, upper Hölder stability.
AMS subject classification. 49K40, 90C31, 90C33.

1 Introduction

Consider the optimization problem

$$P(\omega) \qquad \min_x f(x,\omega) \quad \text{subject to} \quad q(x,\omega)\in P, \tag{1}$$

depending on the parameter vector ω belonging to some topological space Ω. In (1), $f:\mathbb{R}^n\times\Omega\to\mathbb{R}$ and $q:\mathbb{R}^n\times\Omega\to\mathbb{R}^m$ are continuous mappings and $P\subset\mathbb{R}^m$ is the union of finitely many convex polyhedra $P_i, i=1,\ldots,p$ having the representation

$$P_i=\{y\,|\,a_{ij}^Ty\le b_{ij},\, j=1,\ldots,m_i\} \tag{2}$$

with $a_{ij}\in\mathbb{R}^m$ and $b_{ij}\in\mathbb{R}$.

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Of course, the parameter dependent nonlinear programming problem

$$NLP(\omega) \qquad \min f(x,\omega) \quad \text{subject to} \quad g(x,\omega) \le 0,\ h(x,\omega) = 0,$$

where $f(x,\omega) : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I}$ and $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_E}$, is a special case of (1) with $q(x,\omega) := (g(x,\omega), h(x,\omega))$ and $P := \mathbb{R}^{m_I}_- \times \{0\}^{m_E}$. Let us consider some more involved examples.

Example 1. *Consider the parameter dependent MPEC*

$$MPEC(\omega) \qquad \min f(x,\omega) \quad \textit{subject to} \quad \begin{aligned} g(x,\omega) \le 0,\ h(x,\omega) = 0,& \\ \left.\begin{aligned} G_i(x,\omega) \ge 0,\ H_i(x,\omega) \ge 0 \\ G_i(x,\omega)H_i(x,\omega) = 0 \end{aligned}\right\}& \ i = 1,\ldots,m_C, \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I}$, $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_E}$ *and* $G,H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_C}$.

The problem $MPEC(\omega)$ *fits into our setting* (1) *with*

$$q := (g, h, -(G_1,H_1), \ldots -(G_{m_C}, H_{m_C})),$$

$$P := \mathbb{R}^{m_I}_- \times \{0\}^{m_E} \times Q^{m_C}_{EC},$$

where $Q_{EC} := \{(a,b) \in \mathbb{R}^2_- \,|\, ab = 0\}$. *Since* Q_{EC} *is the union of the convex polyhedra* $\mathbb{R}_- \times 0$ *and* $\{0\} \times \mathbb{R}_-$, P *is the union of* 2^{m_C} *polyhedra.*

Example 2. *Another prominent example is the mathematical program with vanishing constraints (MPVC)*

$$MPVC(\omega) \qquad \min f(x,\omega) \quad \textit{subject to} \quad \begin{aligned} g(x,\omega) \le 0,\ h(x,\omega) = 0,& \\ \left.\begin{aligned} G_i(x,\omega) \ge 0 \\ G_i(x,\omega)H_i(x,\omega) \le 0 \end{aligned}\right\}& \ i = 1,\ldots,m_V, \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I}$, $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_E}$ *and* $G,H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_V}$. *For more details on MPVCs we refer the reader to [1, 12].*

Again, the problem $MPVC(\omega)$ *can be written in the form* (1) *with*

$$q := (g, h, (G_1,H_1), \ldots (G_{m_V}, H_{m_V})),$$

$$P := \mathbb{R}^{m_I}_- \times \{0\}^{m_E} \times Q^{m_V}_{VC},$$

where $Q_{VC} := \{(a,b) \in \mathbb{R}_+ \times \mathbb{R} \,|\, ab \le 0\}$ *is the union of the two convex polyhedra* $\mathbb{R}_+ \times \mathbb{R}_-$ *and* $\{0\} \times \mathbb{R}_+$.

If f is partially differentiable with respect to x and setting $F(x,\omega) := \nabla_x f(x,\omega)$, then the first order optimality conditions at a local minimizer x for $P(\omega)$ can be written as a generalized equation

$$GE(\omega) \qquad 0 \in F(x,\omega) + \hat{N}(x; \mathscr{F}(\omega)), \tag{3}$$

where $\hat{N}(x;\mathscr{F}(\omega))$ stands for the *Fréchet normal cone* to the set $\mathscr{F}(\omega)$ at x and

$$\mathscr{F}(\omega) := \{x \in \mathbb{R}^n \,|\, q(x,\omega) \in P\} \tag{4}$$

denotes the feasible region of the problem $P(\omega)$. We also consider the generalized equation (3) for arbitrary continuous mappings $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$.

Throughout this paper we will make the following assumption:

Assumption 1. *There are neighborhoods U of $\bar{x}$ and W of $\bar{\omega}$ such that f and q are twice partially differentiable with respect to x, F is partially differentiable with respect to x on $U \times W$, $F(x,\omega)$, $q(x,\omega)$, $\nabla_x f(x,\omega)$, $\nabla_x q(x,\omega)$, $\nabla_x F(x,\omega)$, $\nabla_x^2 f(x,\omega)$ and $\nabla_x^2 q(x,\omega)$ are continuous at $(\bar{x},\bar{\omega})$ and $\nabla_x^2 f(\cdot,\omega)$, $\nabla_x^2 q(\cdot,\omega)$ are continuous on U for every $\omega \in W$.*

Given a fixed parameter $\bar{\omega}$ and a solution $\bar{x}$ of $P(\bar{\omega})$ respectively $GE(\bar{\omega})$, we are interested in quantitative estimates of the distance of solutions x of problem $P(\omega)$ respectively $GE(\omega)$ to $\bar{x}$ for parameters ω belonging to some neighborhood of $\bar{\omega}$.

We will present such estimates in terms of the mappings $e_l, \tau_l, \hat{\tau}_l : \Omega \to \mathbb{R}$, $l = 1,2$ given by

$$e_l(\omega) = \|\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega})\| + \|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\|^{\frac{1}{l}}$$

and

$$\tau_l(\omega) := \|\nabla_x f(\bar{x},\omega) - \nabla_x f(\bar{x},\bar{\omega})\| + e_l(\omega), \quad \hat{\tau}_l(\omega) := \|F(\bar{x},\omega) - F(\bar{x},\bar{\omega})\| + e_l(\omega).$$

$\tau_l(\omega)$ respectively $\hat{\tau}_l(\omega)$ are measures how much the problem data at the reference point $\bar{x}$ for the perturbed problem $P(\omega)$ respectively $GE(\omega)$ differ from that for the unperturbed problem $P(\bar{\omega})$ respectively $GE(\bar{\omega})$.

In case that we can bound the distance of a solution x of $P(\omega)$ ($GE(\omega)$) to $\bar{x}$ by the estimate $L\tau_1(\omega)$ ($L\hat{\tau}_1(\omega)$), where L denotes some constant, we speak of *upper Lipschitz stability* of the solutions. We speak of *upper Hölder stability* when a bound of the form $\|x - \bar{x}\| \leq L\tau_2(\omega)$ respectively $\|x - \bar{x}\| \leq L\hat{\tau}_2(\omega)$ is available. This notation is motivated by the situation that Ω is a metric space equipped with the metric d and $q(\bar{x},\cdot)$, $\nabla_x q(\bar{x},\cdot)$ and $\nabla_x f(\bar{x},\cdot)$ respectively $F(\bar{x},\cdot)$ are Lipschitz near $\bar{\omega}$, because in this circumstance the bounds are of the form $\|x-\bar{x}\| \leq Ld(\omega,\bar{\omega})$ and $\|x - \bar{x}\| \leq L\sqrt{d(\omega,\bar{\omega})}$, respectively.

Many quantitative stability results are known for the parameter dependent nonlinear programming problem $NLP(\omega)$. We refer to the monographs [4, 5, 17] and the references therein. Compared with the huge amount of stability results for *NLP*, very little research has been done with the stability of $P(\omega)$. Most of the results are known for $MPEC(\omega)$, see e.g. [3, 13, 15, 23, 25]. Sensitivity and stability results for MPVC are given in [14]. In the recent paper [11], Guo, Lin and Ye presented various stability results for more general problems. In particular, they proved upper Lipschitz stability for stationary pairs consisting of stationary solutions and associated multipliers under the structural assumption, that the graph of the limiting normal cone mapping to the set P is the union of finitely many convex polyhedra.

In contrary to the stability results of [11], we focus our interest on the stability of solutions of (3) on its own and not of stationary pairs. This has the advantage that our theory is also applicable in case when multipliers do not exist or the multipliers do not behave continuous.

Our results are mainly based on characterizations of metric subregularity as introduced in [7, 8, 9, 10]. The main constraint qualifications used in this paper are that at the reference point $\bar{x}$ either the first order or the second order sufficient conditions for metric subregularity are fulfilled for the problem $P(\bar{\omega})$. Although the property of metric subregularity is not stable in general, we will see that the sufficient conditions of order l, $l = 1,2$, for metric subregularity guarantee some stability. In particular, we will prove that that there is some constant γ such that for all points x feasible for the problem $P(\omega)$ and satisfying $\|x-\bar{x}\| > \gamma e_l(\omega)$, the constraints of $P(\omega)$ are metrically regular near $(x,0)$ with some uniform modulus. This result allows us to divide the solution sets of $P(\omega)$ respectively $GE(\omega)$ into two parts: one part is contained in a ball around $\bar{x}$ with radius $\gamma e_l(\omega)$ and behaves upper Lipschitz ($l = 1$) or Hölder ($l = 2$) stable by the definition, whereas the other part is outside this ball and we can assume metric regularity. Moreover, we can show that locally optimal solutions for the perturbed problems $P(\omega)$ exist, provided ω is sufficiently close to $\bar{\omega}$. The obtained results are partially new even in case of $NLP(\omega)$.

The rest of the paper is organized as follows: In section 2 we recall the basic definitions of metric (sub)regularity and their directional versions together with the characterization of these properties by objects from generalized differentiation. In section 3 we give some stability results for the feasible point mapping $\mathscr{F}$. Sections 4 and 5 are devoted to the stability behavior of solutions of the generalized equation $GE(\omega)$ and the optimization problem $P(\omega)$, respectively. In section 6 we apply the obtained results to the special problem $MPEC(\omega)$ by explicitly calculating all objects from generalized differentiation. Moreover we present some examples.

2 Preliminaries

We start by recalling several definitions and results from variational analysis: Let $\Gamma \subset \mathbb{R}^n$ be an arbitrary closed set and $x \in \Gamma$. The *contingent* (also called *tangent* or *Bouligand*) *cone* to Γ at x, denoted by $T(x;\Gamma)$, is given by

$$T(x;\Gamma) := \{u \in \mathbb{R}^n \,|\, \exists (u_k) \to u, (t_k) \downarrow 0 : x + t_k u_k \in \Gamma\}.$$

We denote by

$$\hat{N}(x;\Gamma) = \{\xi \in \mathbb{R}^n \,|\, \limsup_{x' \xrightarrow{\Gamma} x} \frac{\xi^T(x'-x)}{\|x'-x\|} \le 0\} \tag{5}$$

the *regular* (or *Fréchet*) *normal cone* to Γ. Finally, the *limiting* (or *basic/Mordukhovich*) *normal cone* to Γ at x is defined by

$$N(x;\Gamma) := \{\xi \,|\, \exists (x_k) \xrightarrow{\Gamma} x,\ (\xi_k) \to \xi : \xi_k \in \hat{N}(x_k;\Gamma) \forall k\}.$$

If $x \notin \Gamma$ we put $T(x;\Gamma) = \emptyset$, $\hat{N}(x;\Gamma) = \emptyset$ and $N(x;\Gamma) = \emptyset$.

The limiting normal cone is generally nonconvex whereas the regular normal cone is always convex. In the case of a convex set Γ, both the regular normal cone and the limiting normal cone coincide with the standard normal cone from convex analysis and moreover, the contingent cone is equal to the tangent cone in the sense of convex analysis.

Note that $\xi \in \hat{N}(x;\Gamma) \Leftrightarrow \xi^T u \le 0\ \forall u \in T(x;\Gamma)$, i.e. $\hat{N}(x;\Gamma)$ is the polar cone of $T(x;\Gamma)$.

Given a multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} M := \{(x,y) \in X \times Y \,|\, y \in M(x)\}$ from its graph, the *coderivative* of M at $(\bar{x},\bar{y})$ is a multifunction $D^*M(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values $D^*M(\bar{x},\bar{y})(\eta) := \{\xi \in \mathbb{R}^n \,|\, (\xi,-\eta) \in N((\bar{x},\bar{y});\operatorname{gph} M)\}$, i.e. $D^*M(\bar{x},\bar{y})(\eta)$ is the collection of all $\xi \in \mathbb{R}^n$ for which there are sequences $(x_k,y_k) \to (\bar{x},\bar{y})$ and $(\xi_k,\eta_k) \to (u,v)$ with $(\xi_k,-\eta_k) \in \hat{N}((x_k,y_k);\operatorname{gph} M)$.

For more details we refer to the monographs [19, 22]

The following directional versions of these limiting constructions were introduced in [8], see also [10] for the finite dimensional setting. Given a direction $u \in \mathbb{R}^n$, the limiting normal cone to a subset $\Gamma \subset \mathbb{R}^n$ in direction u at $x \in \Gamma$ is defined by

$$N(x;\Gamma;u) := \{\xi \in \mathbb{R}^n \,|\, \exists (t_k) \downarrow 0,\ (u_k) \to u,\ (\xi_k) \to \xi : \xi_k \in \hat{N}(x + t_k u_k;\Gamma) \forall k\}.$$

For a multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a direction $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$, the coderivative of M in direction (u,v) at $(\bar{x},\bar{y}) \in \operatorname{gph} M$ is defined as the multifunction $D^*M((\bar{x},\bar{y});(u,v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by $D^*M((\bar{x},\bar{y});(u,v))(\eta) := \{\xi \in \mathbb{R}^n \,|\, (\xi,-\eta) \in N((\bar{x},\bar{y});\operatorname{gph} M;(u,v))\}$.

Note that by the definition we have $N(x;\Gamma;0) = N(x;\Gamma)$ and $D^*M((\bar{x},\bar{y});(0,0)) = D^*M(\bar{x},\bar{y})$. Further $N(x;\Gamma;u) \subset N(x;\Gamma)$ for all u and $N(x;\Gamma;u) = \emptyset$ if $u \notin T(x;\Gamma)$.

We now turn our attention to the set P from problem (1). For every $y \in P$ we denote by $\mathscr{P}(y) := \{i \in \{1,\ldots,p\} \,|\, y \in P_i\}$ the index set of polyhedra containing y and for each $i \in \mathscr{P}(y)$ we denote by $\mathscr{A}_i(y) := \{j \in \{1,\ldots,m_i\} \,|\, a_{ij}^T y = b_{ij}\}$ the index set of active constraints. Then for every $y \in P$ we have

$$T(y;P) = \bigcup_{i\in\mathscr{P}(y)} T(y;P_i) = \bigcup_{i\in\mathscr{P}(y)} \{z \in \mathbb{R}^m \,|\, a_{ij}^T z \le 0, j \in \mathscr{A}_i(y)\},$$

$$\hat{N}(y;P) = \bigcap_{i\in\mathscr{P}(y)} \hat{N}(y;P_i) = \bigcap_{i\in\mathscr{P}(y)} \{\sum_{j\in\mathscr{A}_i(y)} \mu_{ij} a_{ij} \,|\, \mu_{ij} \ge 0, j \in \mathscr{A}_i(y)\}.$$

Some formulas for the limiting normal cone respectively its directional counterpart can be found in [9].

The following lemma will be useful for applications.

Lemma 1. *Let $\Gamma = \Gamma_1 \times \ldots \times \Gamma_l \subset \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_l}$ be the Cartesian product of the closed sets Γ_i and $y = (y_1,\ldots,y_l) \in \Gamma$. Then*

$$T(y;\Gamma) \subset T(y_1;\Gamma_1) \times \ldots \times T(y_l;\Gamma_l) \tag{6}$$

and for every $u = (u_1,\ldots,u_l) \in T(y;\Gamma)$ one has

$$N(y;\Gamma;u) \subset N(y_1;\Gamma_1;u_1) \times \ldots \times N(y_l;\Gamma_l;u_l). \tag{7}$$

Furthermore, equality holds in both inclusions if Γ is the union of finitely many convex polyhedra.

Proof. The inclusion (6) can be found in [22, Proposition 6.41] and the inclusion (7) follows immediately from the formula for the regular normal cone from this proposition and the definition of the directional normal cone. To show equality, assume that Γ coincides with the set P from (1) and let $(u_1,\ldots,u_l) \in T(y_1;\Gamma_1)\times\ldots\times T(y_l;\Gamma_l)$ and $(\xi_1,\ldots,\xi_l)\in N(y_1;\Gamma_1;u_1)\times\ldots\times N(y_l;\Gamma_l;u_l)$. Then there are sequences $(t_{ik}\downarrow 0$, $(u_{ik})\to u_i$, $(\xi_{ik})\to\xi_i$, $i=1,\ldots,l$ with $z_k := (y_1+t_{1k}u_{1k},\ldots,y_l+t_{lk}u_{lk})\in\Gamma_1\times\ldots\times\Gamma_l$ and $(\xi_{1k},\ldots,\xi_{lk})\in\hat{N}(y_1+t_{1k}u_{1k};\Gamma_1)\times\ldots\times\hat{N}(y_l+t_{lk}u_{lk};\Gamma_l)$ for all k. By passing to subsequences we can assume that there are index sets $\mathscr{P}\subset\{1,\ldots,p\}$, $\mathscr{A}_i$, $i\in\mathscr{P}$ such that $\mathscr{P}=\mathscr{P}(z_k)$ and $\mathscr{A}_i=\mathscr{A}_i(z_k)$, $i\in\mathscr{P}$ holds for all k. Furthermore we can assume that for each $i\notin\mathscr{P}$ there is an index j_i with $a_{ij_i}z_k > b_{ij_i}$ for all k. Since the convex polyhedra P_j are closed, for each $j\in\mathscr{P}$ we also have $\lim_k z_k = y\in P_j$ and therefore $(1-\alpha)y+\alpha z_k\in P_j$ $\forall\alpha\in[0,1]$, $\forall k$. Further, for every $\alpha\in(0,1)$ and every k we have $\mathscr{A}_i=\mathscr{A}_i((1-\alpha)y+\alpha z_k)$ and $a_{ij_i}((1-\alpha)y+\alpha z_k)>b_{ij_i}$, $i\notin\mathscr{P}$ showing $\hat{N}((1-\alpha)y+\alpha z_k;\Gamma)=\hat{N}(z_k;\Gamma)$ and, together with [22, Proposition 6.41]

$$\begin{aligned}\hat{N}(z_k;\Gamma) &= \hat{N}(y_1+t_{1k}u_{1k};\Gamma_1)\times\ldots\times\hat{N}(y_l+t_{lk}u_{lk};\Gamma_l)\\ &= \hat{N}((1-\alpha)y+\alpha z_k;\Gamma)=\hat{N}(y_1+\alpha t_{1k}u_{1k};\Gamma_1)\times\ldots\times\hat{N}(y_l+\alpha t_{lk}u_{lk};\Gamma_l).\end{aligned}$$

Now let $t_k := \min_i t_{ik}$. Then for each $i=1,\ldots,l$ we have

$$(1-\frac{t_k}{t_{ik}})y+\frac{t_k}{t_{ik}}z_k=(y_1+t_{1k}\frac{t_k}{t_{ik}}u_{1k},\ldots,y_i+t_ku_{ik},\ldots,y_l+t_{lk}\frac{t_k}{t_{ik}}u_{mk})\in\Gamma_1\times\ldots\times\Gamma_l,$$

$$(\xi_{1k},\ldots,\xi_{lk})\in\hat{N}(z_k,\Gamma)=\hat{N}(y_1+t_{1k}\frac{t_k}{t_{ik}}u_{1k};\Gamma_1)\times\ldots\times\hat{N}(y_i+t_ku_{ik};\Gamma_i)\times\ldots\times\hat{N}(y_l+t_{lk}\frac{t_k}{t_{ik}}u_{mk};\Gamma_l)$$

showing $y_i+t_ku_{ik}\in\Gamma_i$ and $\xi_{ik}\in\hat{N}(y_i+t_ku_{ik};\Gamma_i)$. Hence $\tilde{z}_k := (y_1+t_ku_{1k},\ldots,y_l+t_ku_{lk})\in\Gamma_1\times\ldots\times\Gamma_l=\Gamma$ and, by using [22, Proposition 6.41] again, $(\xi_{1k},\ldots,\xi_{lk})\in\hat{N}(y_1+t_ku_{1k};\Gamma_1)\times\ldots\times\hat{N}(y_l+t_ku_{lk};\Gamma_l)=\hat{N}(\tilde{z}_k;\Gamma)$, showing $(u_1,\ldots,u_l)\in T(x;\Gamma)$ and $(\xi_1,\ldots,\xi_l)\in N(y;\Gamma;u)$. $\square$

In this paper we are mainly concerned with multifunctions $M:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ of the form $M(x)=q(x)-\Gamma$, where $q:\mathbb{R}^n\to\mathbb{R}^m$ is continuously differentiable and $\Gamma\subset\mathbb{R}^m$ is a closed set. Since for all x and all $\gamma\in\Gamma$

$$\hat{N}((x,q(x)-\gamma);\mathrm{gph}M)=\{(-\nabla q(x)^T\lambda,\lambda)\,|\,-\lambda\in\hat{N}(\gamma;\Gamma)\},$$

we obtain for every $\bar{x}\in M^{-1}(0)$

$$\begin{aligned}D^*M((\bar{x};0);(u,v))(\lambda) &= \{x^*\,|\,(x^*,-\lambda)\in N((\bar{x},0);\mathrm{gph}M;(u,v))\}\\ &= \{\nabla q(\bar{x})^T\lambda\,|\,\lambda\in N(q(\bar{x});\Gamma;\nabla q(\bar{x})u-v))\}.\end{aligned}\qquad(8)$$

Now we consider the notions of metric regularity and subregularity, respectively, and its characterization by coderivatives and limiting normal cones.

Definition 1. *Let $M:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ be a multifunction, let $(\bar{x},\bar{y})\in\mathrm{gph}M$ and let $\kappa>0$*

1. *M is called* metrically regular with modulus κ *around* $(\bar{x},\bar{y})$ *if there are neighborhoods* U *of* $\bar{x}$ *and* V *of* $\bar{y}$ *such that*

$$d(x, M^{-1}(y)) \le \kappa d(y, M(x)) \ \forall (x,y) \in U \times V. \tag{9}$$

2. *M is called* metrically subregular with modulus κ *at* $(\bar{x},\bar{y})$ *if there is a neighborhood* U *of* $\bar{x}$ *such that*

$$d(x, M^{-1}(\bar{y})) \le \kappa d(\bar{y}, M(x)) \ \forall x \in U. \tag{10}$$

It is well known that metric regularity of the multifunction M around $(\bar{x},\bar{y})$ is equivalent to the Aubin property of the inverse multifunction M^{-1}. A multifunction $S : R^m \rightrightarrows R^n$ has the *Aubin property* with modulus $L \ge 0$ around some point $(\bar{y},\bar{x}) \in \mathrm{gph}\, S$, if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$S(y_1) \cap U \subset S(y_2) + L\|y_1 - y_2\|\mathscr{B}_{\mathbb{R}^n} \quad \forall y_1, y_2 \in V,$$

where $\mathscr{B}_{\mathbb{R}^n}$ denotes the unit ball in $\mathbb{R}^n$ in the underlying norm.

Metric subregularity of M at $(\bar{x},\bar{y})$ is equivalent with the property of *calmness* of the inverse multifunction M^{-1}. A multifunction $S : R^m \rightrightarrows R^n$ is called *calm* with modulus $L \ge 0$ at $(\bar{y},\bar{x}) \in \mathrm{gph}\, S$, if there is some neighborhood U of $\bar{x}$ such that

$$S(y) \cap U \subset S(\bar{y}) + L\|y - \bar{y}\|\mathscr{B}_{\mathbb{R}^n} \quad \forall y.$$

To introduce directional versions of metric (sub)regularity it is convenient to define the following neighborhoods of a direction: Given a direction $u \in \mathbb{R}^n$ and positive numbers $\rho, \delta > 0$, the set $V_{\rho,\delta}(u)$, is given by

$$V_{\rho,\delta}(u) := \{z \in \rho\mathscr{B}_{\mathbb{R}^n} \mid \big\| \|u\|z - \|z\|u \big\| \le \delta\|z\| \, \|u\|\}. \tag{11}$$

This can also be written in the form

$$V_{\rho,\delta}(u) = \begin{cases} \{0\} \cup \{z \in \rho\mathscr{B}_{\mathbb{R}^n} \setminus \{0\} \mid \left\| \frac{z}{\|z\|} - \frac{u}{\|u\|} \right\| \le \delta\} & \text{if } u \neq 0, \\ \rho\mathscr{B}_{\mathbb{R}^n} & \text{if } u = 0. \end{cases}$$

Definition 2. *Let* $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *be a multifunction, let* $(\bar{x},\bar{y}) \in \mathrm{gph}\, M$ *and let* $\kappa > 0$, $u \in \mathbb{R}^n$ *and* $v \in \mathbb{R}^m$.

1. *M is called* metrically regular with modulus κ in direction $w := (u,v)$ *at* $(\bar{x},\bar{y})$ *if there are positive real numbers* ρ *and* δ *such that*

$$d(x, M^{-1}(y)) \le \kappa d(y, M(x)) \tag{12}$$

holds for all $(x,y) \in (\bar{x},\bar{y}) + V_{\rho,\delta}(w)$ *with* $\|w\| d((x,y), \mathrm{gph}\, M) \le \delta\|w\| \|(x,y) - (\bar{x},\bar{y})\|$.

2. *M is called* metrically subregular with modulus κ in direction u *at* $(\bar{x},\bar{y})$ *if there are positive real numbers* ρ *and* δ *such that*

$$d(x, M^{-1}(\bar{y})) \le \kappa d(\bar{y}, M(x)) \ \forall x \in \bar{x} + V_{\rho,\delta}(u). \tag{13}$$

Note that metric regularity in direction $(0,0)$ respectively metric subregularity in direction 0 is equivalent with the property of metric regularity respectively metric subregularity. Further, metric regularity in direction $(u,0)$ implies metric subregularity in direction u.

Theorem 1. *Let the multifunction $M:\mathbb{R}^n \rightrightarrows R^m$ be given by $M(x):=q(x)-\Gamma$, where $q:R^n\to\mathbb{R}^m$ is continuously differentiable and $\Gamma\subset\mathbb{R}^m$ is a closed set. Further let $(\bar{x},0)\in \operatorname{gph} M$, $u\in\mathbb{R}^n$ and $v\in\mathbb{R}^m$ be given.*

1. *(Mordukhovich criterion): M is metrically regular around $(\bar{x},0)$ if and only if*

$$\nabla q(\bar{x})^T\lambda = 0, \lambda \in N(q(\bar{x});\Gamma) \implies \lambda = 0.$$

2. *M is metrically regular in direction (u,v) at $(\bar{x},0)$ if and only if*

$$\nabla q(\bar{x})^T\lambda = 0, \lambda \in N(q(\bar{x});\Gamma;\nabla q(\bar{x})u-v) \implies \lambda = 0.$$

3. *Assume that q is twice Fréchetdifferentiable at $\bar{x}$, that Γ is the union of finitely many convex polyhedra and the condition*

$$\nabla q(x)^T\lambda = 0,\ \lambda \in N(q(\bar{x});\Gamma;\nabla q(\bar{x})u),\ u^T\nabla^2(\lambda^T q)(\bar{x})u \geq 0 \implies \lambda = 0$$

is fulfilled. Then M is metrically subregular in direction u at $(\bar{x},0)$.

Proof. The first statement is a specialization of the more general statement [19, Theorem 4.18] and can be found e.g. in [22, Example 9.44]. Similarly, the second statement follows from [8, Theorem 5] by taking into account that the involved spaces are finite dimensional and (8). Finally, the last statement follows from [10, Theorem 2.6] □

Taking into account that in finite dimensions a multifunction is metrically subregular if and only if it is metrically subregular in every nonzero direction, see [10, Lemma 2.7], and $N(q(\bar{x});\Gamma;\nabla q(\bar{x})u)=\emptyset$ if $\nabla q(\bar{x})u\notin T(q(\bar{x});\Gamma)$, we obtain the following corollary:

Corollary 1. *Let the multifunction $M:\mathbb{R}^n \rightrightarrows R^m$ be given by $M(x):=q(x)-\Gamma$, where $q:R^n\to\mathbb{R}^m$ is continuously differentiable and $\Gamma\subset\mathbb{R}^m$ is a closed set. Then M is metrically subregular at $(\bar{x},0)$ if one of the following conditions is fulfilled:*

1. First order sufficient condition for metric subregularity (FOSCMS)*: For every $0\neq u\in\mathbb{R}^n$ with $\nabla q(\bar{x})u\in T(q(\bar{x});\Gamma)$ one has*

$$\nabla q(\bar{x})^T\lambda = 0, \lambda \in N(q(\bar{x});\Gamma;\nabla q(\bar{x})u) \implies \lambda = 0.$$

2. Second order sufficient condition for metric subregularity (SOSCMS)*: q is twice Fréchet differentiable at $\bar{x}$, Γ is the union of finitely many convex polyhedra and for every $0\neq u\in\mathbb{R}^n$ with $\nabla q(\bar{x})u\in T(q(\bar{x});\Gamma)$ one has*

$$\nabla q(x)^T\lambda = 0,\ \lambda \in N(q(\bar{x});\Gamma;\nabla q(\bar{x})u),\ u^T\nabla^2(\lambda^T q)(\bar{x})u \geq 0 \implies \lambda = 0.$$

Next we consider optimality conditions for the problem

$$\min_x f(x) \quad \text{subject to} \quad q(x) \in \Gamma \tag{14}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $q : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable and $\Gamma \subset \mathbb{R}^m$ is closed. We denote the feasible region of (14) by $\mathscr{F}$. Given a feasible point $\bar{x} \in \mathscr{F}$, we define the *linearized cone* by

$$T^{\text{lin}}(\bar{x}) := \{u \in \mathbb{R}^n \,|\, \nabla q(\bar{x})u \in T(q(\bar{x});\Gamma)\}$$

and the *critical cone* by

$$\mathscr{C}(\bar{x}) = \{u \in T^{\text{lin}}(\bar{x}) \,|\, \nabla f(\bar{x})u \le 0\}.$$

Definition 3. *Let $\bar{x} \in \mathscr{F}$ be feasible for the problem* (14).

1. *We say that $\bar{x}$ is* B-stationary*, if*

$$0 \in \nabla f(\bar{x}) + \hat{N}(\bar{x};\mathscr{F}).$$

2. *We say that $\bar{x}$ is* M-stationary*, if*

$$0 \in \nabla f(\bar{x}) + \nabla q(\bar{x})^T N(q(\bar{x});\Gamma).$$

Since $\hat{N}(\bar{x};\mathscr{F})$ is the polar cone of $T(\bar{x};\mathscr{F})$, B-stationarity can be equivalently written as $\nabla f(\bar{x})u \ge 0 \; \forall u \in T(\bar{x};\mathscr{F})$. Hence B-stationarity means that there does not exist feasible descent directions at $\bar{x}$, which is a first-order necessary condition for $\bar{x}$ being a local minimizer.

Usually B-stationarity is not very useful in practice, since the regular normal cone of $\mathscr{F}$ at $\bar{x}$ is difficult to compute in general. Hence, M-stationarity conditions are used as first-order necessary condition, which, however, are only valid under some constraint qualification condition. Indeed, as the following lemma shows, under some weak constraint qualification M-stationarity is not only necessary for local minimizers, but also for B-stationarity.

Lemma 2. *Let $\bar{x} \in \mathscr{F}$ be B-stationary for the problem* (14) *and assume that either $\mathscr{C}(\bar{x}) = \{0\}$ and the multifunction $\tilde{M}(u) := \nabla q(\bar{x})u - T(q(\bar{x});\Gamma)$ is metrically subregular at $(0,0)$ or there exists $\bar{u} \in \mathscr{C}(\bar{x})$ such that the mapping $M(x) := q(x) - \Gamma$ is metrically subregular in direction $\bar{u}$ at $(\bar{x},0)$. Then $\bar{x}$ is M-stationary.*

Proof. If $\mathscr{C}(\bar{x}) = \{0\}$ then 0 is solution of the problem

$$\min_{u \in \mathbb{R}^n} \nabla f(\bar{x})u \quad \text{subject to} \quad 0 \in \nabla q(\bar{x})u - T(q(\bar{x});\Gamma). \tag{15}$$

Since the mapping $\tilde{M}(u)$ is assumed to be metrically subregular at $(0,0)$, by [8, Corollary 2] there is some λ such that $0 \in \nabla f(\bar{x}) + D^*\tilde{M}(0,0)(\lambda)$. By [22, Example 9.44, Proposition 6.27] we conclude $-\nabla f(\bar{x}) \in D^*\tilde{M}(0,0)(\lambda) = \{\nabla q(\bar{x})^T\lambda\}$ and $\lambda \in N(0;T(q(\bar{x});\Gamma)) \subset N(q(\bar{x});\Gamma)$ showing M-stationarity of $\bar{x}$. In the second case, since $-\nabla f(\bar{x}) \in \hat{N}(\bar{x};\mathscr{F})$, by [22, Theorem 6.11] there is some smooth function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla \tilde{f}(\bar{x}) = \nabla f(\bar{x})$ and $\bar{x}$ is a global minimizer of the problem

$$\min \tilde{f}(x) \quad \text{subject to} \quad x \in \mathscr{F} = \{x \,|\, 0 \in M(x)\}.$$

Now, again by [8, Corollary 2] we obtain that there is some multiplier λ such that $-\nabla f(\bar{x}) = -\nabla \tilde{f}(\bar{x}) \in D^*M(\bar{x};0)(\lambda)$ and from [22, Example 9.44] we obtain $D^*M(\bar{x};0)(\lambda) = \{\nabla q(\bar{x})^T \lambda\}$ and $\lambda \in N(q(\bar{x});\Gamma)$. □

Remark 1. *If Γ is the union of finitely many convex polyhedra, then so is $T(q(\bar{x});\Gamma)$ and hence the multifunction $\tilde{M}$ is a polyhedral multifunction and consequently metrically subregular by Robinson's result [21].*

Given any element $g \in \hat{N}(\bar{x};\mathscr{F})$, then by the definition $\bar{x}$ is a B-stationary solution of the problem

$$\min -g^T x \quad \text{subject to} \quad q(x) \in \Gamma.$$

If M is metrically subregular in $(\bar{x},0)$ then it is also metrically subregular in every direction and further $\tilde{M}$ is metrically subregular by [7, Proposition 2.1]. Hence $\bar{x}$ is also M-stationary and we obtain $g \in \nabla q(\bar{x})^T N(q(\bar{x});\Gamma)$. We summarize this observation in the following formula:

Corollary 2. *Let $q : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable, $\Gamma \subset \mathbb{R}^m$ be closed and let $\bar{x} \in q^{-1}(\Gamma)$. If the multifunction $x \rightrightarrows q(x) - \Gamma$ is metrically subregular at $(\bar{x},0)$ then*

$$\hat{N}(\bar{x};q^{-1}(\Gamma)) \subset \nabla q(\bar{x})^T N(q(\bar{x});\Gamma).$$

3 Stability properties of the feasible set mapping $\mathscr{F}$

In this section, we study conditions for metric regularity as well as Hölder and Lipschitz stability of the feasible set mapping $\mathscr{F}$. In particular, we introduce and discuss two regularity properties which imply metric regularity around points *near* some reference point.

We start with the following technical lemma, where the functions q and F satisfy the basic Assumption 1:

Lemma 3. *For every $\varepsilon > 0$ there are neighborhoods $\hat{U}_\varepsilon$ of $\bar{x}$ and $\hat{W}_\varepsilon$ of $\bar{\omega}$ such that*

$$\|q(x,\omega) - q(x,\bar{\omega})\| + \|\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega})\| \le e_1(\omega) + \varepsilon\|x - \bar{x}\|, \tag{16}$$

$$\|F(x,\omega) - F(x,\bar{\omega})\| + \|q(x,\omega) - q(x,\bar{\omega})\| + \|\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega})\| \le \hat{\tau}_1(\omega) + \varepsilon\|x - \bar{x}\|, \tag{17}$$

$$\begin{aligned} \|q(x,\omega) - q(x,\bar{\omega})\| &\le \|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\| + e_2(\omega)\|x - \bar{x}\| + \frac{\varepsilon}{2}\|x - \bar{x}\|^2 \\ &\le e_2(\omega)^2 + e_2(\omega)\|x - \bar{x}\| + \frac{\varepsilon}{2}\|x - \bar{x}\|^2 \end{aligned} \tag{18}$$

hold for all $(x,\omega) \in \hat{U}_\varepsilon \times \hat{W}_\varepsilon$.

Proof. Let $\varepsilon > 0$ be fixed and choose a ball $\hat{U}_\varepsilon$ around $\bar{x}$ and a neighborhood $\hat{W}_\varepsilon$ of $\bar{\omega}$ such that

$$\|\nabla_x F(x,\omega) - \nabla_x F(\bar{x},\bar{\omega})\| + \|\nabla_x q(x,\omega) - \nabla_x q(\bar{x},\bar{\omega})\| + \|\nabla_x^2 q(x,\omega) - \nabla_x^2 q(\bar{x},\bar{\omega})\| \le \frac{\varepsilon}{2}$$

and consequently

$$\|\nabla_x F(x,\omega) - \nabla_x F(x,\bar{\omega})\| + \|\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega})\| + \|\nabla_x^2 q(x,\omega) - \nabla_x^2 q(x,\bar{\omega})\| \le \varepsilon$$

hold for all $(x,\omega) \in \hat{U}_\varepsilon \times \hat{W}_\varepsilon$.

For arbitrarily fixed $(x,\omega) \in \hat{U}_\varepsilon \times \hat{W}_\varepsilon$ we can find $u,\mu \in \mathscr{B}_{\mathbb{R}^n}$, $\xi,\lambda \in \mathscr{B}_{\mathbb{R}^m}$ such that

$$\begin{aligned}
&\|F(x,\omega) - F(x,\bar{\omega})\| = \mu^T(F(x,\omega) - F(x,\bar{\omega}))\\
&\|q(x,\omega) - q(x,\bar{\omega})\| = \xi^T(q(x,\omega) - q(x,\bar{\omega}))\\
&\|\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega})\| = \lambda^T(\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega}))u.
\end{aligned}$$

Then there is a point z belonging to the line segment $[\bar{x},x] \subset U_\varepsilon$ such that

$$\begin{aligned}
&\xi^T(q(x,\omega) - q(x,\bar{\omega})) + \lambda^T(\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega}))u\\
&\quad = \xi^T(q(\bar{x},\omega) - q(\bar{x},\bar{\omega})) + \lambda^T(\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega}))u + \big(\xi^T(\nabla_x q(z,\omega) - \nabla_x q(z,\bar{\omega}))\\
&\qquad\quad + u^T(\nabla_x^2(\lambda^T q)(z,\omega) - \nabla_x^2(\lambda^T q)(z,\bar{\omega}))\big)(x-\bar{x})\\
&\quad \le e_1(\omega) + \big(\|\nabla_x q(z,\omega) - \nabla_x q(z,\bar{\omega})\| + \|\nabla_x^2 q(z,\omega) - \nabla_x^2 q(z,\bar{\omega})\|\big)\|x-\bar{x}\|
\end{aligned}$$

and (16) follows. The estimate (17) follows analogously.

To show (18) note that there is some point $\tilde{z}$ belonging to the line segment $[\bar{x},x]$ such that

$$\begin{aligned}
\xi^T(q(x,\omega) - q(x,\bar{\omega})) &= \xi^T(q(\bar{x},\omega) - q(\bar{x},\bar{\omega})) + \xi^T(\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega}))(x-\bar{x})\\
&\qquad + \frac{1}{2}(x-\bar{x})^T(\nabla_x^2(\xi^T q)(\tilde{z},\omega) - \nabla_x^2(\xi^T q)(\tilde{z},\bar{\omega}))(x-\bar{x})\\
&\le \|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\| + e_2(\omega)\|x-\bar{x}\|\\
&\qquad + \frac{1}{2}\|\nabla_x^2 q(\tilde{z},\omega) - \nabla_x^2 q(\tilde{z},\bar{\omega})\|\|x-\bar{x}\|^2
\end{aligned}$$

and from this inequality (18) follows. □

For every $\omega \in \Omega$ we define the linearized cone

$$T_\omega^{\text{lin}}(x) := \{u \in \mathbb{R}^n \,|\, \nabla_x q(x;\omega)u \in T(q(x,\omega);P)\}$$

as well as the multifunction $M_\omega : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $M_\omega(x) := q(x,\omega) - P$. Then we have

$$T(x;\mathscr{F}(\omega)) = T_\omega^{\text{lin}}(x), \quad \hat{N}(x;\mathscr{F}(\omega)) \subset \nabla_x q(x,\omega)^T N(q(x,\omega);P)$$

for every $x \in \mathscr{F}(\omega)$ such that M_ω is metrically subregular at $(x,0)$.

It is well known that the property of metric regularity is stable under small Lipschitzian perturbations, see e.g. [17, Section 4.1], [19, Section 4.2.3]. We state here the following result:

Theorem 2. *Assume that $M_{\bar\omega}$ is metrically regular around $(\bar x,0)$. Then there are neighborhoods U of $\bar x$, V of 0, W of $\bar\omega$ and a constant $\kappa>0$ such that*

$$\mathrm{d}(x,M_\omega^{-1}(y))\leq\kappa\mathrm{d}(y,M_\omega(x))=\kappa\mathrm{d}(q(x,\omega)-y,P)\quad\forall(x,y,\omega)\in U\times V\times W. \tag{19}$$

In particular we have $\mathscr{F}(\omega)\neq\emptyset$, $\mathrm{d}(\bar x,\mathscr{F}(\omega))\leq\kappa\|q(\bar x,\omega)-q(\bar x,\bar\omega)\|\leq\kappa e_1(\omega)$ for all $\omega\in W$ and for every $x\in\mathscr{F}(\omega)\cap U$ the multifunction M_ω is metrically regular with modulus κ around $(x,0)$.

Proof. A similar result was stated in [11, Lemma 3.1]. However, the given proof appears to be not correct and therefore we present a different one. Let $\delta>0$, $\kappa'>0$ be chosen such that

$$\mathrm{d}(x,M_{\bar\omega}^{-1}(y))\leq\kappa'\mathrm{d}(y,M_{\bar\omega}(x))\quad\forall(x,y)\in(\bar x+\delta\mathscr{B}_{\mathbb{R}^n})\times\delta\mathscr{B}_{\mathbb{R}^m}.$$

Setting $\varepsilon:=\frac{1}{24(\kappa'+1)}$, we denote by $\hat U_\varepsilon$ and $\hat W_\varepsilon$ the neighborhoods according to Lemma 3, and we choose some radius $r\in(0,\min\{\frac{\delta}{2},1\})$ with $\bar x+r\mathscr{B}_{\mathbb{R}^n}\subset\hat U_\varepsilon$. Then we choose some positive radius $\bar r\leq\frac12 r$ and some neighborhood $W\subset\hat W_\varepsilon$ such that

$$\|q(x,\omega)-q(\bar x,\bar\omega)\|<\tfrac12\varepsilon r\;\forall(x,\omega)\in(\bar x+\bar r\mathscr{B}_{\mathbb{R}^n})\times W\;\text{ and }\;e_1(\omega)<\varepsilon\bar r\;\forall\omega\in W.$$

We now show that the assertion of the theorem holds with $U:=\bar x+\bar r\mathscr{B}_{\mathbb{R}^n}$, $V:=\frac14\varepsilon r\mathscr{B}_{\mathbb{R}^m}$ and $\kappa=2(\kappa'+1)$. Consider arbitrarily fixed elements $(\xi,y,\omega)\in U\times V\times W$ and define the functions

$$g'(x):=q(x,\bar\omega)-q(x,\omega)+y\;\text{ and }\;g(x):=q(x,\bar\omega)-q(x,\omega)+\zeta,$$

where $\zeta\in M_\omega(\xi)$ is chosen such that

$$\|y-\zeta\|=\mathrm{d}(y,M_\omega(\xi))=\mathrm{d}(y,q(\xi,\omega)-P)\leq\|y-(q(\xi,\omega)-q(\bar x,\bar\omega))\|$$

and consequently $\|\zeta\|\leq2\|y\|+\|q(\xi,\omega)-q(\bar x,\bar\omega)\|<\varepsilon r$. Then g and g' are Lipschitz on $\bar x+r\mathscr{B}_{\mathbb{R}^n}$ with constant less or equal than

$$\sup\{\|\nabla_x q(x,\omega)-\nabla_x q(x,\bar\omega)\|\,|\,x\in\bar x+r\mathscr{B}_{\mathbb{R}^n}\}\leq e_1(\omega)+\varepsilon r<2\varepsilon r\leq2\varepsilon<\frac{1}{2(\kappa'+1)},$$

where the first inequality follows from (16). Further we have

$$\sup\{\|g'(x)\|\,|\,x\in\bar x+r\mathscr{B}_{\mathbb{R}^n}\}<e_1(\omega)+\varepsilon r+\|y\|<3\varepsilon r<\frac{r}{8(\kappa'+1)}$$

and

$$\sup\{\|g(x)\|\,|\,x\in\bar x+r\mathscr{B}_{\mathbb{R}^n}\}\leq e_1(\omega)+\varepsilon r+\|\zeta\|<3\varepsilon r=\frac{r}{8(\kappa'+1)}.$$

Since $g(\xi)\in M_{\bar\omega}(\xi)$ we can apply [17, Theorem 4.3] to find some ξ' such that $g'(\xi')\in M_{\bar\omega}(\xi')$ and $\|\xi-\xi'\|\leq2(\kappa'+1)\|g'(\xi)-g(\xi)\|=2(\kappa'+1)\|y-\zeta\|$. It follows that $y\in q(\xi',\omega)-P$ and thus $\xi'\in M_\omega^{-1}(y)$ and

$$\mathrm{d}(\xi,M_\omega^{-1}(y))\leq2(\kappa'+1)\|y-\zeta\|=2(\kappa'+1)\mathrm{d}(y,M_\omega(\xi))$$

showing (19). Taking $\xi=\bar x$, $y=0$ we obtain $\mathrm{d}(\bar x,M_\omega^{-1}(0))=\mathrm{d}(\bar x,\mathscr{F}(\omega))\leq\kappa\mathrm{d}(0,M_\omega(\bar x))\leq\kappa\|q(\bar x,\omega)-q(\bar x,\bar\omega)\|\leq\kappa e_1(\omega)$ and thus $\mathscr{F}(\omega)\neq\emptyset$. To complete the proof note that metric regularity of M_ω around $(x,0)$, where $x\in\mathscr{F}(\omega)\cap U$, is a simple consequence of (19). □

When we do not assume metric regularity of $M_{\bar\omega}$ around $(\bar x,0)$, then we cannot expect in general that the multifunctions M_ω, for ω near $\bar\omega$, are metrically regular around all points $(x,0)$ with $x\in\mathscr{F}(\omega)$ close to $\bar x$. To handle also this situation, we give the following definition.

Definition 4. *Let $l\in\{1,2\}$. We say that property R_l holds, if there are neighborhoods U of $\bar x$, W of $\bar\omega$ and constants $\kappa>0$ and $\gamma>0$ such that for every $\omega\in W$ and every $x\in\mathscr{F}(\omega)\cap U$ with $\|x-\bar x\|>\gamma e_l(\omega)$ the multifunction M_ω is metrically regular with modulus κ around $(x,0)$.*

In particular properties R_1 and R_2 imply that $M_{\bar\omega}$ is metrically regular with some uniform modulus around every point $(x,0)$ with $x\in\mathscr{F}(\bar\omega)\setminus\{\bar x\}$ close to $\bar x$.

We will now show that property R_1 respectively property R_2 holds if $M_{\bar\omega}$ fulfills FOSCMS respectively SOSCMS at $(\bar x,0)$.

Proposition 1. *1. Assume that FOSCMS is fulfilled for $M_{\bar\omega}$ at $\bar x$, i.e., for every direction $0\neq u\in T^{\rm lin}_{\bar\omega}(\bar x)$ we have*

$$\nabla_x q(\bar x,\bar\omega)^T\lambda=0,\ \lambda\in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u)\Longrightarrow\lambda=0.$$

Then property R_1 holds.

2. Assume that SOSCMS is fulfilled for $M_{\bar\omega}$ at $\bar x$, i.e., for every direction $0\neq u\in T^{\rm lin}_{\bar\omega}(\bar x)$ we have

$$\nabla_x q(\bar x,\bar\omega)^T\lambda=0,\ \lambda\in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u),\ u^T\nabla^2_x(\lambda^Tq)(\bar x,\bar\omega)u\geq 0\ \Longrightarrow\lambda=0.$$

Then property R_2 holds.

Proof. To prove the first part assume on the contrary that for every k we can find $(x^k,\omega^k)\in\hat U_{1/k}\times\hat W_{1/k}$ with $x^k\in\mathscr{F}(\omega^k)\cap(\bar x+\frac1k\mathscr{B}_{R^n})$, $\|\nabla_x q(x^k,\omega^k)-\nabla_x q(\bar x,\bar\omega)\|\leq\frac1k$, $\|x^k-\bar x\|>ke_1(\omega^k)$ such that M_{ω^k} is not metrically regular around $(x^k,0)$ with some modulus less or equal than k, where $\hat U_{1/k}$, $\hat W_{1/k}$ are as in Lemma 3. By [22, Example 9.44] there is some $\lambda^k\in N(q(x^k,\omega^k);P)$ with $\|\lambda^k\|=1$ and $\|\nabla_x q(x^k,\omega^k)^T\lambda^k\|\leq\frac1k$. According to the definition of the limiting normal cone we can find for each k elements $q^k\in(q(x^k,\omega^k)+\frac{\|x^k-\bar x\|}{k}\mathscr{B}_{\mathbb{R}^n})\cap P$ and $\xi^k\in\hat N(q^k,P)\cap(\lambda^k+\frac{\|x^k-\bar x\|}{k}\mathscr{B}_{\mathbb{R}^n})$. By passing to a subsequence if necessary, we can assume that $u^k:=(x^k-\bar x)/\|x^k-\bar x\|\to u$ and $\lim_{k\to\infty}\lambda^k=\lim_{k\to\infty}\xi^k=\lambda\neq 0$. Because of Lemma 3 we have $\|q(x^k,\omega^k)-q(x^k,\bar\omega)\|/\|x^k-\bar x\|\leq e_1(\omega^k)/\|x^k-\bar x\|+1/k\leq 2/k$ and therefore

$$\begin{aligned}\lim_{k\to\infty}\frac{q^k-q(\bar x,\bar\omega)}{\|x^k-\bar x\|}&=\lim_{k\to\infty}\frac{q(x^k,\omega^k)-q(\bar x,\bar\omega)}{\|x^k-\bar x\|}\\&=\lim_{k\to\infty}\left(\frac{q(x^k,\omega^k)-q(x^k,\bar\omega)}{\|x^k-\bar x\|}+\frac{q(x^k,\bar\omega)-q(\bar x,\bar\omega)}{\|x^k-\bar x\|}\right)\\&=\nabla_x q(\bar x,\bar\omega)u\end{aligned}$$

showing $0\neq u\in T^{\rm lin}_{\bar\omega}(\bar x)$ and $\lambda\in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u)$. Since we also have

$$0=\lim_{k\to\infty}\nabla_x q(x^k,\omega^k)^T\lambda^k=\nabla_x q(\bar x,\bar\omega)^T\lambda$$

we obtain a contradiction to the assumption. Hence the first part is proved.

To prove the second part, assume on the contrary that for every k we can find $(x^k,\omega^k)\in \hat U_{1/k}\times \hat W_{1/k}$ with $x^k\in\mathscr F(\omega^k)\cap(\bar x+\frac1k\mathscr B_{R^n})$, $\|\nabla_x q(x^k,\omega^k)-\nabla_x q(\bar x,\bar\omega)\|\le\frac1k$, $\|x^k-\bar x\|>ke_2(\omega^k)$ such that M_{ω^k} is not metrically regular around $(x^k,0)$ with some modulus less or equal than k. Then $\|q(\bar x,\omega^k)-q(\bar x,\bar\omega\|^{\frac12}\le e_2(\omega^k)\le 1/k^2\le 1$ and $e_2(\omega^k)\ge e_1(\omega^k)$ follows. Hence we can proceed similarly as in the first part of the proof to find the sequences (λ^k), (ξ^k) and q^k together with the limits $0\ne u\in T^{\rm lin}_{\bar\omega}(\bar x)$ and $0\ne\lambda\in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u)$ satisfying $\nabla_x q(\bar x,\bar\omega)^T\lambda=0$ with the only difference that we now require $q^k\in(q(x^k,\omega^k)+\frac{\|x^k-\bar x\|^2}{k}\mathscr B_{\mathbb R^n})\cap P$. By passing to a subsequence if necessary we can assume that there are index sets $\mathscr P\subset\{1,\dots,p\}$ and $\mathscr A_i\subset\{1,\dots,m_i\}$, $i\in\mathscr P$ such that $\mathscr P(q^k)=\mathscr P$ and $\mathscr A_i(q^k)=\mathscr A_i$, $i\in\mathscr P$ holds for all k. Further there are numbers $\mu^k_{ij}\ge0$, $j\in\mathscr A_i$, $i\in\mathscr P$ such that

$$\xi^k-\sum_{j\in\mathscr A_i}\mu^k_{ij}a_{ij}=0,\ i\in\mathscr P,$$

and $\sum_{j\in\mathscr A_i}\mu^k_{ij}\le\beta_i\|\xi^k\|$ for some constant β_i, $i\in\mathscr P$. By passing to a subsequence once more we can assume that the sequences μ^k_{ij} converge to $\mu_{ij}\ge0$ for each $j\in\mathscr A_i$, $i\in\mathscr P$ and it follows that $\lambda-\sum_{j\in\mathscr A_i}\mu_{ij}a_{ij}=0,\ i\in\mathscr P$. Since $\mathscr P\subset\mathscr P(q(\bar x))$ and $\mathscr A_i\subset\mathscr A_i(\bar x)$, $i\in\mathscr P$ we obtain for each $i\in\mathscr P$

$$\lambda^Tq^k=\sum_{j\in\mathscr A_i}\mu_{ij}a_{ij}^Tq^k=\sum_{j\in\mathscr A_i}\mu_{ij}b_{ij}=\sum_{j\in\mathscr A_i}\mu_{ij}a_{ij}^Tq(\bar x,\bar\omega)=\lambda^Tq(\bar x,\bar\omega)\ \forall k.$$

Using Lemma 3 we have

$$\|q(x^k,\omega^k)-q(x^k,\bar\omega)\|\le e_2(\omega^k)^2+e_2(\omega^k)\|x^k-\bar x\|+\frac1{2k}\|x^k-\bar x\|^2\le(\frac1{k^2}+\frac1k+\frac1{2k})\|x^k-\bar x\|^2,$$

and so, together with $\|q^k-q(x^k,\omega^k)\|\le\frac1k\|x^k-\bar x\|^2$ and $\nabla_x q(\bar x,\bar\omega)^T\lambda=0$, we obtain

$$\begin{aligned}0&=\lim_{k\to\infty}\frac{\lambda^T(q^k-q(\bar x,\bar\omega))}{\|x^k-\bar x\|^2}=\lim_{k\to\infty}\frac{\lambda^T(q^k-q(x^k,\omega^k)+q(x^k,\omega^k)-q(x^k,\bar\omega)+q(x^k,\bar\omega)-q(\bar x,\bar\omega))}{\|x^k-\bar x\|^2}\\&=\lim_{k\to\infty}\frac{\lambda^T(q(x^k,\bar\omega)-q(\bar x,\bar\omega))}{\|x^k-\bar x\|^2}=\lim_{k\to\infty}\frac{\lambda^T(\nabla_x q(\bar x,\bar\omega)(x^k-\bar x)+\frac12(x^k-\bar x)^T\nabla^2_x q(\bar x,\bar\omega)(x^k-\bar x))}{\|x^k-\bar x\|^2}\\&=\frac12u^T\nabla^2_x(\lambda^Tq)(\bar x,\bar\omega)u,\end{aligned}$$

contradicting the assumption and the proposition is proved. □

Proposition 2. *Let $\bar x\in\mathscr F(\bar\omega)$.*

1. *Assume that there is some $0\ne u\in T^{\rm lin}_{\bar\omega}(\bar x)$ such that*

$$\nabla_x q(\bar x,\bar\omega)^T\lambda=0,\ \lambda\in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u)\implies\lambda=0.$$

Then there is a constant $\kappa>0$ and a neighborhood W of $\bar\omega$ such that

$$\mathrm d(\bar x,\mathscr F(\omega))\le\kappa\|q(\bar x,\omega)-q(\bar x,\bar\omega)\|\le\kappa e_1(\omega)\ \forall\omega\in W.$$

2. *Assume that there is some* $0 \neq u \in T^{\rm lin}_{\bar\omega}(\bar x)$ *such that*

$$\nabla_x q(\bar x,\bar\omega)^T\lambda = 0,\ \lambda \in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u),\ u^T\nabla_x^2(\lambda^T q)(\bar x,\bar\omega)u \geq 0 \implies \lambda = 0.$$

Then there is a constant $\kappa > 0$ *and a neighborhood* W *of* $\bar\omega$ *such that*

$$\mathrm{d}(\bar x,\mathscr{F}(\omega)) \leq \kappa e_2(\omega)\ \forall \omega \in W.$$

Proof. We can assume without loss of generality that $\|u\| = 1$. In both cases the assumption ensures that $M_{\bar\omega}$ is metrically subregular at $(\bar x, 0)$ in direction u. Now we make some preliminary considerations, before proving the assertions of the proposition. For every k we can find a neighborhood W^k of $\bar\omega$ and a radius $\rho^k > 0$ such that $(\bar x + \rho^k\mathscr{B}_{\mathbb{R}^n}) \times W^k \subset \hat U_{1/k} \times \hat W_{1/k}$,

$$\begin{aligned}
&\sup\{\|\nabla_x^2 q(x,\omega) - \nabla_x^2 q(x,\bar\omega)\| \,|\, x \in \bar x + \rho^k\mathscr{B}_{\mathbb{R}^n}, \omega \in W^k\} < \frac{1}{9k^3},\\
&\sup\{\|\nabla_x q(x,\omega) - \nabla_x q(\bar x,\bar\omega)\| \,|\, x \in \bar x + \rho^k\mathscr{B}_{\mathbb{R}^n}, \omega \in W^k\} < \frac{1}{k},\\
&\sup\{e_2(\omega) \,|\, \omega \in W^k\} \leq \min\{\frac{1}{16k^3}, \frac{\rho^k}{16k^2}\}.
\end{aligned}$$

Now consider sequences $(t^k) \downarrow 0$ and (ω^k) such that $t^k < \rho^k/2$ and $\omega^k \in W^k$ hold for all k. Since $\nabla_x q(\bar x,\bar\omega)u \in T(q(\bar x,\bar\omega);P)$ and P is the union of finitely many convex polyhedral sets, we have $q(\bar x,\bar\omega) + t^k\nabla_x q(\bar x,\bar\omega)u \in P$ for all k sufficiently large. Thus there is some constant L such that $\mathrm{d}(q(\bar x + t^k u,\bar\omega),P) \leq L(t^k)^2$ and, by metric subregularity of $M_{\bar\omega}$ in direction u, there is some $\kappa' > 0$ such that for every k there is some point $\bar x^k \in \mathscr{F}(\bar\omega) \cap (\bar x + t^k u + \kappa' L(t^k)^2\mathscr{B}_{\mathbb{R}^n})$. For all k sufficiently large we also have $t^k L\kappa' < \frac{1}{2}$, implying

$$\frac{t^k}{2} < \|\bar x^k - \bar x\| < \frac{3t^k}{2} < \rho^k,$$

and we obtain

$$\begin{aligned}
\mathrm{d}(q(\bar x^k,\omega^k),P) &\leq \mathrm{d}(q(\bar x^k,\bar\omega),P) + \|q(\bar x^k,\omega^k) - q(\bar x^k,\bar\omega)\|\\
&\leq 0 + \|q(\bar x,\omega^k) - q(\bar x,\bar\omega))\| + \|\bar x^k - \bar x\|\|\nabla_x q(\bar x,\omega^k) - \nabla_x q(\bar x,\bar\omega)\|\\
&\qquad + \frac{\|\bar x^k - \bar x\|^2}{2}\sup\{\|\nabla_x^2 q(x,\omega^k) - \nabla_x^2 q(x,\bar\omega)\| \,|\, \|x - \bar x\| \leq \|\bar x^k - \bar x\|\}\\
&\leq \|q(\bar x,\omega^k) - q(\bar x,\bar\omega))\| + \frac{3}{2}t^k\|\nabla_x q(\bar x,\omega^k) - \nabla_x q(\bar x,\bar\omega)\| + \frac{(t^k)^2}{8k^3} =: \varepsilon^k,
\end{aligned}$$

By using Ekeland's variational principle we can find for every k some point $x^k \in \bar x^k + k\varepsilon^k\mathscr{B}_{\mathbb{R}^n}$ such that $\mathrm{d}(q(x^k,\omega^k),P) \leq \mathrm{d}(q(\bar x^k,\omega^k),P)$ and

$$\mathrm{d}(q(x^k,\omega^k),P) \leq \mathrm{d}(q(x,\omega^k),P) + \frac{1}{k}\|x - x^k\|\ \forall x \in \mathbb{R}^n$$

Since P is closed, there is for every k some $q^k \in P$ with $\mathrm{d}(q(x^k,\omega^k),P) = \|q(x^k,\omega^k) - q^k\|$, and we conclude

$$\|q(x^k,\omega^k) - q^k\| \le \|q(x,\omega^k) - y\| + \frac{1}{k}\|x - x^k\| \ \forall (x,y) \in \mathbb{R}^n \times P,$$

i.e. (x^k,q^k) is a solution of the optimization problem

$$\min_{x,y} \|q(x,\omega^k) - y\| + \frac{1}{k}\|x - x^k\| \quad \text{subject to} \quad y \in P.$$

By applying first-order optimality conditions it follows that $\xi^k := (q(x^k,\omega^k) - q^k)/\|q(x^k,\omega^k) - q^k\| \in \hat{N}(q^k;P)$ and $\|\nabla_x q(x^k,\omega^k)^T\xi^k\| \le \frac{1}{k}$ provided that $q(x^k,\omega^k) - q^k \ne 0$.

Now let us prove the first assertion by contraposition. We assume on the contrary that for every k there is some $\omega^k \in W^k$ with $\mathrm{d}(\bar{x},\mathscr{F}(\omega^k)) > 12k^2\|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\|$. Note that $\|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\| = 0$ or $e_2(\omega^k) = 0$ implies $\bar{x} \in \mathscr{F}(\omega^k)$ and thus $\|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\| > 0$. By taking $t^k := 4k^2\|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\| \le 4k^2 e_1(\omega^k) \le 4k^2 e_2(\omega^k) \le \min\{\frac{1}{4k}, \frac{\rho^k}{4}\}$ we obtain

$$\begin{aligned}
\varepsilon^k &\le \|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\|(1 + 6k^2\|\nabla_x q(\bar{x},\omega^k) - \nabla_x q(\bar{x},\bar{\omega})\| + \frac{t^k}{2k}) \\
&\le \|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\|(1 + \frac{3}{8k} + \frac{t^k}{2k}) \le 2\|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\| = \frac{1}{2k^2}t^k < \frac{\|\bar{x}^k - \bar{x}\|}{k^2}
\end{aligned}$$

and therefore $\|x^k - \bar{x}^k\| \le k\varepsilon^k < \frac{\|\bar{x}^k - \bar{x}\|}{k}$, showing

$$\lim_{k\to\infty} \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = \lim_{k\to\infty} \frac{\bar{x}^k - \bar{x}}{\|\bar{x}^k - \bar{x}\|} = u, \tag{20}$$

$$\begin{aligned}
(1 - \frac{1}{k})\|\bar{x}^k - \bar{x}\| &< \|x^k - \bar{x}\| < (1 + \frac{1}{k})\|\bar{x}^k - \bar{x}\| \\
&< \frac{3}{2}(1 + \frac{1}{k})t^k \le 12k^2\|q(\bar{x},\omega^k) - q(\bar{x},\bar{\omega})\| < \mathrm{d}(\bar{x},\mathscr{F}(\omega^k)),
\end{aligned} \tag{21}$$

and

$$\|q(x^k,\omega^k) - q^k\| \le \varepsilon^k < \frac{\|\bar{x}^k - \bar{x}\|}{k^2} \le \frac{\|x^k - \bar{x}\|}{k} \ \forall k \ge 2. \tag{22}$$

From (21) we can conclude $q(x^k,\omega^k) - q^k \ne 0$, since otherwise $\mathrm{d}(\bar{x},\mathscr{F}(\omega^k)) \le \|\bar{x} - x^k\|$ would hold. Hence ξ^k is well defined and we can proceed as in the proof of Proposition 1 to obtain the contradiction, that every limit point λ of the sequence (ξ^k) fulfills $\|\lambda\| = 1$, $\lambda \in N(q(\bar{x});P;\nabla_x q(\bar{x},\bar{\omega})u)$ and $\nabla_x q(\bar{x},\bar{\omega})^T\lambda = 0$.

We proof the second part similarly. Assuming on the contrary that for every k there is some $\omega^k \in W^k$ with $e_2(\omega^k) > 0$ and $\mathrm{d}(\bar{x},\mathscr{F}(\omega^k))/e_2(\omega^k) > 24k^2$ and setting $t^k := 8k^2 e_2(\omega^k) \le \min\{\frac{1}{2k}, \frac{\rho^k}{2}\}$ we obtain $\|\bar{x}^k - \bar{x}\| < \frac{3}{2}t^k \le \frac{3}{4}$ and, together with $\frac{1}{2}t^k < \|\bar{x}^k - \bar{x}\|$,

$$\varepsilon^k \le e_2(\omega^k)^2(1 + 12k^2 + 8k) = (t^k)^2\frac{1 + 12k^2 + 8k}{64k^4} < \frac{\|\bar{x}^k - \bar{x}\|^2}{k^2} \le \frac{\|\bar{x}^k - \bar{x}\|}{k^2} \ \forall k \ge 2.$$

As before we obtain $\|x^k - \bar{x}^k\| \le k\varepsilon^k < \frac{\|\bar{x}^k - \bar{x}\|}{k}$ and thus (20),

$$(1-\frac{1}{k})\|\bar{x}^k - \bar{x}\| < \|x^k - \bar{x}\| < (1+\frac{1}{k})\|\bar{x}^k - \bar{x}\| < \frac{3}{2}(1+\frac{1}{k})t^k \le 24k^2 e_2(\omega^k) < \mathrm{d}(\bar{x}, \mathscr{F}(\omega^k))$$

and (22) hold. This implies again that (ξ^k) is well defined and using the same arguments as in the proof of the second part of Proposition 1 we obtain that every limit point λ of the sequence (ξ^k) fulfills $\|\lambda\| = 1$, $\lambda \in N(q(\bar{x}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$, $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0$ and $u^T \nabla_x^2 (\lambda^T q)(\bar{x}, \bar{\omega})u = 0$, a contradiction. □

4 Quantitative stability of solution mappings

In what follows we denote by $X : \Omega \rightrightarrows \mathbb{R}^n$ the mapping which assigns to every ω the set of local minimizers for the problem $P(\omega)$ in (1) and by $S : \Omega \rightrightarrows \mathbb{R}^n$ and $S_M : \Omega \rightrightarrows \mathbb{R}^n$, respectively, the mappings which assign to every $\omega \in \Omega$ the sets of B-stationary points and M-stationary points, respectively, for the problem $P(\omega)$, i.e.

$$S(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in \nabla_x f(x, \omega) + \hat{N}(x; \mathscr{F}(\omega))\},$$

$$S_M(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in \nabla_x f(x, \omega) + \nabla_x q(x, \omega)^T N(q(x, \omega); P)\}.$$

Then we have $X(\omega) \subset S(\omega)$ $\forall \omega \in \Omega$.

Further we consider the solution mapping $\hat{S} : \Omega \rightrightarrows \mathbb{R}^n$ of the generalized equation (3),

$$\hat{S}(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in F(x, \omega) + \hat{N}(x; \mathscr{F}(\omega))\},$$

and the related mapping

$$\hat{S}_M(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in F(x, \omega) + \nabla_x q(x, \omega)^T N(q(x, \omega); P)\}.$$

The aim of this section is to give sufficient conditions such that estimates of the form

$$(S(\omega) \cup S_M(\omega)) \cap U \subset \bar{x} + L\tau_l(\omega)\mathscr{B}_{\mathbb{R}^n} \ \forall \omega \in W$$

respectively

$$(\hat{S}(\omega) \cup \hat{S}_M(\omega)) \cap U \subset \bar{x} + L\hat{\tau}_l(\omega)\mathscr{B}_{\mathbb{R}^n} \ \forall \omega \in W$$

hold, where $L > 0$ and U and W are neighborhoods of $\bar{x}$ and $\bar{\omega}$. We only state here a result for the solution mappings $\hat{S}(\omega) \cup \hat{S}_M(\omega)$ for $GE(\omega)$, the corresponding statement for $P(\omega)$ follows immediately by taking $F(x, \omega) := \nabla_x f(x, \omega)$.

Proposition 3. *Let* $\bar{x} \in \mathscr{F}(\bar{\omega})$.

1. *If property R_1 holds and there does not exist a triple $(u,\lambda,\mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ satisfying*

$$0 \neq u \in T^{\rm lin}_{\bar\omega}(\bar x), \tag{23}$$
$$\lambda \in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u) \tag{24}$$
$$\mu \in T(\lambda;N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u)) \tag{25}$$
$$F(\bar x,\bar\omega)+\nabla_x q(\bar x,\bar\omega)^T\lambda = 0 \tag{26}$$
$$\nabla_x F(\bar x,\bar\omega)u+\nabla_x^2(\lambda^T q)(\bar x,\bar\omega)u+\nabla_x q(\bar x,\bar\omega)^T\mu = 0, \tag{27}$$

then there are neighborhoods U of $\bar x$, W of $\bar\omega$ and a constant $L>0$ such that

$$(\hat S(\omega)\cup\hat S_M(\omega))\cap U \subset \bar x + L\hat\tau_1(\omega)\mathscr{B}_{\mathbb{R}^n}\ \forall \omega\in W.$$

2. *If property R_2 holds and there does not exist a quadruple $(u,\lambda,\mu,v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ such that (u,λ,μ) fulfills* (23)-(27) *and*

$$\nabla_x q(\bar x,\bar\omega)v + u^T\nabla_x^2 q(\bar x,\bar\omega)u \in T(q(\bar x,\bar\omega);\bigcup_{i\in\mathscr{I}(u)} P_i), \tag{28}$$
$$u^T\nabla_x^2(\lambda^T q)(\bar x,\bar\omega)u = F(\bar x,\bar\omega)^T v,, \tag{29}$$

where $\mathscr{I}(u) := \{i\,|\,\nabla_x q(\bar x,\bar\omega)u \in T(q(\bar x,\bar\omega);P_i)\}$, then there are neighborhoods U of $\bar x$, W of $\bar\omega$ and a constant $L>0$ such that

$$(\hat S(\omega)\cup\hat S_M(\omega))\cap U \subset \bar x + L\hat\tau_2(\omega)\mathscr{B}_{\mathbb{R}^n}\ \forall \omega\in W.$$

Proof. We prove the first part by contraposition. Let W denote the neighborhood according to the definition of property R_1. Assume on the contrary that for every k we can find $(x^k,\omega^k) \in (\hat U_{1/k}\cap(\bar x+\frac{1}{k}\mathscr{B}_{\mathbb{R}^n}))\times(\hat W_{1/k}\cap W)$ with $(x^k)\in\hat S_M(\omega^k)\cup\hat S(\omega^k)$ and $\|x^k-\bar x\| > k\hat\tau_1(\omega^k)$. Because of $e_1(\omega^k)\le\hat\tau_1(\omega^k)$ it follows that $e_1(\omega^k)/\|x^k-\bar x\| \le \hat\tau_1(\omega^k)/\|x^k-\bar x\| < \frac{1}{k}$ and due to property R_1 we have that M_{ω^k} is metrically regular with modulus κ around $(x^k,0)$ for all k sufficiently large. Hence, $x^k\in\hat S_M(\omega^k)$ and there exists a vector $\lambda^k \in N(q(x^k,\omega^k);P)$ with $F(x^k,\omega^k)+\nabla_x q(x^k,\omega^k)^T\lambda^k = 0$. From [22, Example 9.44] we conclude $\|\lambda^k\|\le\kappa\|F(x^k,\omega^k)\|$ showing that (λ^k) is bounded. By the definition of the limiting normal cone we can find for each k elements $q^k\in(q(x^k,\omega^k)+\frac{\|x^k-\bar x\|^2}{k}\mathscr{B}_{\mathbb{R}^n})\cap P$ and $\xi^k\in\hat N(q^k,P)\cap(\lambda^k+\frac{\|x^k-\bar x\|}{k}\mathscr{B}_{\mathbb{R}^n})$. By passing to a subsequence if necessary we can assume that $u^k := (x^k-\bar x)/\|x^k-\bar x\|\to u$ and $\lambda^k\to\lambda$. Because of Lemma 3 we have $\|q(x^k,\omega^k)-q(x^k,\bar\omega)\|/\|x^k-\bar x\| \le \hat\tau_1(\omega^k)/\|x^k-\bar x\|+1/k\to 0$ and therefore

$$\lim_{k\to\infty}\frac{q^k-q(\bar x,\bar\omega)}{\|x^k-\bar x\|} = \lim_{k\to\infty}\frac{q(x^k,\omega^k)-q(\bar x,\bar\omega)}{\|x^k-\bar x\|} = \lim_{k\to\infty}\frac{q(x^k,\bar\omega)-q(\bar x,\bar\omega)}{\|x^k-\bar x\|} = \nabla_x q(\bar x,\bar\omega)u$$

showing $0\neq u\in T^{\rm lin}_{\bar\omega}(\bar x)$ and $\lambda\in N(q(\bar x,\bar\omega);P;\nabla_x q(\bar x,\bar\omega)u)$. Using Lemma 3 again we similarly obtain

$$\lim_{k\to\infty}\frac{\nabla_x q(x^k,\omega^k)-\nabla_x q(\bar x,\bar\omega)}{\|x^k-\bar x\|} = \nabla_x^2 q(\bar x,\bar\omega)u,\quad \lim_{k\to\infty}\frac{F(x^k,\omega^k)-F(\bar x,\bar\omega)}{\|x^k-\bar x\|} = \nabla_x F(\bar x,\bar\omega)u. \tag{30}$$

Further

$$0 = \lim_{k\to\infty} (F(x^k,\omega^k) + \nabla_x q(x^k,\omega^k)^T \lambda^k) = F(\bar{x},\bar{\omega}) + \nabla_x q(\bar{x},\bar{\omega})^T \lambda. \tag{31}$$

By passing to a subsequence once more we can assume that there are index sets $\mathscr{P} \subset \{1,\dots,p\}$ and $\mathscr{A}_i \subset \{1,\dots,m_i\}$, $i \in \mathscr{P}$ such that $\mathscr{P}(q^k) = \mathscr{P}$ and $\mathscr{A}_i(q^k) = \mathscr{A}_i$, $i \in \mathscr{P}$ holds for all k. Further there are numbers $\zeta_{ij}^k \geq 0$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ such that

$$\xi^k - \sum_{j\in\mathscr{A}_i} \zeta_{ij}^k a_{ij} = 0,\ i \in \mathscr{P}$$

and $\sum_{j\in\mathscr{A}_i} \zeta_{ij}^k \leq \beta_i \|\xi^k\|$, $i \in \mathscr{P}$ for some constants β_i. By passing to a subsequence once more we can assume that the sequences ζ_{ij}^k converge to $\zeta_{ij} \geq 0$ for each $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ and it follows that $\lambda - \sum_{j\in\mathscr{A}_i} \zeta_{ij} a_{ij} = 0$, $i \in \mathscr{P}$. Further $F(\bar{x},\bar{\omega}) + \nabla_x q(\bar{x},\bar{\omega})^T \lambda = 0$ and thus, by Hoffman's lemma, there is some real $\bar{\gamma} > 0$ such that for each k we can find $\bar{\xi}^k \in \mathbb{R}^m$ and nonnegative numbers $\bar{\zeta}_{ij}^k$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ satisfying

$$\begin{aligned}
&\|\bar{\xi}^k - \xi^k\| + \sum_{i\in\mathscr{P}} \sum_{j\in\mathscr{A}_i} |\zeta_{ij}^k - \bar{\zeta}_{ij}^k| \leq \bar{\gamma} \|F(\bar{x},\bar{\omega}) + \nabla_x q(\bar{x},\bar{\omega})^T \xi^k\| \\
&\bar{\xi}^k - \sum_{j\in\mathscr{A}_i} \bar{\zeta}_{ij}^k a_{ij} = 0,\ i \in \mathscr{P} \\
&F(\bar{x},\bar{\omega}) + \nabla_x q(\bar{x},\bar{\omega})^T \bar{\xi}^k = 0.
\end{aligned}$$

Taking into account (31) and $\|\nabla_x q(\bar{x},\bar{\omega})^T(\lambda^k - \xi^k)\| \leq 1/k$ we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{\|F(\bar{x},\bar{\omega}) + \nabla_x q(\bar{x},\bar{\omega})^T \xi^k\|}{\|x^k - \bar{x}\|} \\
&= \limsup_{k\to\infty} \frac{\|F(\bar{x},\bar{\omega}) + \nabla_x q(\bar{x},\bar{\omega})^T \xi^k - F(x^k,\omega^k) - \nabla_x q(x^k,\omega^k)^T \lambda^k\|}{\|x^k - \bar{x}\|} \\
&\leq \limsup_{k\to\infty} \frac{\|F(\bar{x},\bar{\omega}) - F(x^k,\omega^k) + (\nabla_x q(\bar{x},\bar{\omega}) - \nabla_x q(x^k,\omega^k))^T \lambda^k\| + \|\nabla_x q(\bar{x},\bar{\omega})^T(\lambda^k - \xi^k)\|}{\|x^k - \bar{x}\|} \\
&= \|\nabla_x F(\bar{x},\bar{\omega}) u + \nabla_x^2(\lambda^T q)(\bar{x},\bar{\omega}) u\| < \infty,
\end{aligned}$$

where the last equation follows from (30). Hence for each $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ the sequence $(\zeta_{ij}^k - \bar{\zeta}_{ij}^k)/\|x^k - \bar{x}\|$ is bounded and we can assume that it converges to some ν_{ij}, where we have eventually passed to a subsequence. Now consider the set $\mathscr{I} := \{(i,j) \,|\, i \in \mathscr{P}, j \in \mathscr{A}_i, \zeta_{ij} = 0\}$. If $\nu_{ij} \geq 0\ \forall (i,j) \in \mathscr{I}$ we set $\bar{\nu}_{ij} = \nu_{ij}$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$. Otherwise we choose an index $\bar{k}$ such that $(\zeta_{ij}^{\bar{k}} - \bar{\zeta}_{ij}^{\bar{k}})/\|x^{\bar{k}} - \bar{x}\| < \nu_{ij}/2$ hold for all $(i,j) \in \mathscr{I}$ with $\nu_{ij} < 0$ and set $\bar{\nu}_{ij} = \nu_{ij} + 2(\bar{\zeta}_{ij}^{\bar{k}} - \zeta_{ij})/\|x^{\bar{k}} - \bar{x}\|$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$. Then we have for every $(i,j) \in \mathscr{I}$

$$\bar{\nu}_{ij} = \nu_{ij} + 2\bar{\zeta}_{ij}^{\bar{k}}/\|x^{\bar{k}} - \bar{x}\| \geq \nu_{ij} + 2(\bar{\zeta}_{ij}^{\bar{k}} - \zeta_{ij}^{\bar{k}})/\|x^{\bar{k}} - \bar{x}\| > 0$$

and hence in any case there is some $\bar{t} > 0$ such that $\zeta_{ij} + t\bar{\nu}_{ij} \geq 0$ holds for all $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ and $t \in [0,\bar{t}]$. Setting $\mu := \sum_{j\in\mathscr{A}_i} \bar{\nu}_{ij} a_{ij}$ for an arbitrarily chosen $i \in \mathscr{P}$ we either have

$\mu = \lim_{k\to\infty} \frac{\xi^k - \bar{\xi}^k}{\|x^k - \bar{x}\|}$ or $\mu = \lim_{k\to\infty} \frac{\xi^k - \bar{\xi}^k}{\|x^k - \bar{x}\|} + 2\frac{\bar{\xi}^{\bar{k}} - \lambda}{\|x^{\bar{k}} - \bar{x}\|}$ and therefore $\mu = \sum_{j\in\mathscr{A}_i} \bar{v}_{ij} a_{ij}$ holds for all $i \in \mathscr{P}$. Hence $\lambda + t\mu \in \hat{N}(q^k); P)$ $\forall k$ and $\lambda + t\mu \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$ and $\mu \in T(\lambda; N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u))$ follows. Taking into account $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = \nabla_x q(\bar{x}, \bar{\omega})^T \bar{\xi}^{\bar{k}} = -F(\bar{x}, \bar{\omega})$ we obtain

$$\begin{aligned}
0 &= \lim_{k\to\infty} \frac{F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \lambda^k}{\|x^k - \bar{x}\|} = \lim_{k\to\infty} \frac{F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \xi^k}{\|x^k - \bar{x}\|} \\
&= \lim_{k\to\infty} \left(\frac{F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \bar{\xi}^k}{\|x^k - \bar{x}\|} + \nabla_x q(x^k, \omega^k)^T \frac{\xi^k - \bar{\xi}^k}{\|x^k - \bar{x}\|} \right) \\
&= \nabla_x q(\bar{x}, \bar{\omega})^T \mu + \lim_{k\to\infty} \frac{F(x^k, \omega^k) - F(\bar{x}, \bar{\omega}) + (\nabla_x q(x^k, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega}))^T \bar{\xi}^k}{\|x^k - \bar{x}\|} \\
&= \nabla_x q(\bar{x}, \bar{\omega})^T \mu + \nabla_x F(\bar{x}, \bar{\omega})u + \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega})u.
\end{aligned}$$

Thus the triple (u, λ, μ) fulfills conditions (23)-(27), a contradiction, and the first part is proved.

We also prove the second assertion by contraposition. Let W now denote the neighborhood according to the definition of property R_2. Assume on the contrary that for every k we can find $(x^k, \omega^k) \in (\hat{U}_{1/k} \cap (\bar{x} + \frac{1}{k}\mathscr{B}_{\mathbb{R}^n})) \times (\hat{W}_{1/k} \cap W)$ with $(x^k) \in \hat{S}_M(\omega^k) \cup \hat{S}(\omega^k)$ and $\|x^k - \bar{x}\| > k\hat{\tau}_2(\omega^k)$. Then $\hat{\tau}_2(\omega^k) \le 1/k^2$ and consequently $\hat{\tau}_1(\omega^k)/\|x^k - \bar{x}\| \le \hat{\tau}_2(\omega^k)/\|x^k - \bar{x}\| \le 1/k$ for all k and we can proceed as in the first part to find (u, λ, μ). Thus, in order to prove the second assertion it remains to show that there is some v such that (28)-(29) holds. Since $\mathscr{P} \subset \mathscr{P}(q(\bar{x}, \bar{\omega}))$ and $\mathscr{A}_i \subset \mathscr{A}_i(q(\bar{x}, \bar{\omega}))$, $i \in \mathscr{P}$, we have $\lambda^T(q^k - q(\bar{x}, \bar{\omega})) = 0$ for all k. Now fix any $\bar{i} \in \mathscr{P}$ and consider an arbitrarily fixed vector $\xi \in \mathbb{R}^m$ such that $F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \xi = 0$ and there exist nonnegative numbers $\tau_{\bar{i}j} \ge 0$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))$ with $\xi = \sum_{j\in\mathscr{A}_{\bar{i}}(q(\bar{x},\bar{\omega}))} \tau_{\bar{i}j} a_{\bar{i}j}$. Since $q^k \in P_{\bar{i}}$ we have $a_{\bar{i}j}^T q^k \le b_{\bar{i}j} = a_{\bar{i}j}^T q(\bar{x}, \bar{\omega})$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))$ and therefore $\xi^T(q^k - q(\bar{x}, \bar{\omega})) \le 0$. Hence

$$\begin{aligned}
0 &\ge (\xi - \lambda)^T (q^k - q(\bar{x}, \bar{\omega})) \\
&= (\xi - \lambda)^T \big(q^k - q(x^k, \omega^k) + q(x^k, \omega^k) - q(x^k, \bar{\omega}) \\
&\qquad + \nabla_x q(\bar{x}, \bar{\omega})^T (x^k - \bar{x}) + \tfrac{1}{2}(x^k - \bar{x})^T \nabla_x^2 q(\bar{x}, \bar{\omega})(x^k - \bar{x})\big) + r^k,
\end{aligned}$$

where $r^k := (\xi - \lambda)^T \left(q(x^k, \bar{\omega}) - q(\bar{x}, \bar{\omega}) - \nabla_x q(\bar{x}, \bar{\omega})(x^k - \bar{x}) - \frac{1}{2}(x^k - \bar{x})^T \nabla_x^2 q(\bar{x}, \bar{\omega})(x^k - \bar{x})\right)$. By Lemma 3 we have $(q(x^k, \omega^k) - q(x^k, \bar{\omega}))/\|x^k - \bar{x}\|^2 \to 0$. Since $(\xi - \lambda)^T \nabla_x q(\bar{x}, \bar{\omega}) = 0$, $(q^k - q(x^k, \omega^k))/\|x^k - \bar{x}\|^2 \to 0$ and $r^k/\|x^k - \bar{x}\|^2 \to 0$, by dividing by $\|x^k - \bar{x}\|^2$ and taking the limit $k \to \infty$, we obtain

$$0 \ge \frac{1}{2} u^T \nabla_x^2((\xi - \lambda)^T q)(\bar{x}, \bar{\omega})u.$$

Setting $\zeta_{\bar{i}j} = 0$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega})) \setminus \mathscr{A}_{\bar{i}}$ we obtain that $\zeta_{\bar{i}j}$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))$ is a solution of the linear optimization problem

$$\begin{aligned}
\min \sum_{j\in\mathscr{A}_{\bar{i}}(q(\bar{x},\bar{\omega}))} & -\tau_{\bar{i}j} a_{\bar{i}j}^T (u^T \nabla_x^2 q(\bar{x}, \bar{\omega})u) \\
\text{subject to} \sum_{j\in\mathscr{A}_{\bar{i}}(q(\bar{x},\bar{\omega}))} \tau_{\bar{i}j} \nabla_x q(\bar{x}, \bar{\omega})^T a_{\bar{i}j} &= -F(\bar{x}, \bar{\omega}) \\
\tau_{\bar{i}j} &\ge 0, \ j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))
\end{aligned}$$

Then, by duality theory of linear optimization, the dual program also has a solution $v \in \mathbb{R}^n$ and

$$\begin{array}{rcl} -F(\bar{x},\bar{\omega})^T v &=& -u^T \nabla_x^2 (\lambda^T q)(\bar{x},\bar{\omega}) u \\ a_{\bar{i}j}^T \nabla_x q(\bar{x},\bar{\omega}) v &\leq& -a_{\bar{i}j}^T (u^T \nabla_x^2 q(\bar{x},\bar{\omega}) u),\ j \in \mathscr{A}_{\bar{i}}(q(\bar{x},\bar{\omega})) \end{array}$$

Hence $\nabla_x q(\bar{x},\bar{\omega})v + u^T \nabla_x^2 q(\bar{x},\bar{\omega})u \in T(q(\bar{x},\bar{\omega}),P_{\bar{i}})$ and since $\bar{i} \in \mathscr{P} \subset \mathscr{I}(u)$, the quadruple (u,λ,μ,v) fulfills (23)-(26), (28),(29), a contradiction, and the second part is also proved. □

5 Quantitative stability of optimal solutions

In this section we consider the problem (1). Recall that a point $\bar{x} \in \mathscr{F}(\bar{\omega})$ is an *essential local minimizer of second order* for the problem $P(\bar{\omega})$, if there is a neighborhood U of $\bar{x}$ and a constant $\eta > 0$ such that

$$\max\{f(x,\bar{\omega}) - f(\bar{x},\bar{\omega}),\ \mathrm{d}(q(x,\bar{\omega}),P)\} \geq \eta \|x-\bar{x}\|^2\ \forall x \in U. \tag{32}$$

This implies that the *quadratic growth condition* for the problem $P(\bar{\omega})$ holds at $\bar{x}$, i.e., $f(x,\bar{\omega}) - f(\bar{x},\bar{\omega}) \geq \eta \|x-\bar{x}\|^2\ \forall x \in U \cap \mathscr{F}(\bar{\omega})$. The opposite direction is true if $M_{\bar{\omega}}$ is metrically subregular at $(\bar{x},\bar{\omega})$. To see this one could use similar arguments as in [6, Section 3] by noting that convexity of P is not needed and the assumption of metric regularity used in [6] can be replaced by assuming metric subregularity.

Note that the proof of the following proposition does not use the polyhedral form of P, it suffices to suppose that P is a nonempty closed set.

Proposition 4. *Let $\bar{x}$ be an essential local minimizer of second order for the problem $P(\bar{\omega})$. Then there are constants $\gamma_1,\gamma_2 > 0$ and a neighborhood W of $\bar{\omega}$ such that for all $\omega \in W$ with $\mathrm{d}(\bar{x},\mathscr{F}(\omega)) < \gamma_1$ one has*

$$\mathrm{d}(\bar{x},X(\omega)) \leq \gamma_2 \left(\mathrm{d}(\bar{x},\mathscr{F}(\omega))^{\frac{1}{2}} + \tau_2(\omega)\right)$$

Proof. Let $\eta > 0$ and $\rho > 0$ be such that (32) hold for all $x \in \bar{x} + \rho\mathscr{B}_{\mathbb{R}^n}$ and define $L := \sup\{\|\nabla_x f(x,\bar{\omega})\| \,|\, x \in \bar{x}+\rho\mathscr{B}_{\mathbb{R}^n}\} + 1$, $\gamma_2 := \max\{\frac{1+\sqrt{2\eta+1}}{\eta}, 2\sqrt{L/\eta}\}$. Then we choose $0 < \bar{\rho} \leq \rho$ and $W \subset \hat{W}_{\eta/2}$ such that $\bar{x} + \bar{\rho}\mathscr{B}_{\mathbb{R}^n} \subset \hat{U}_{\eta/2}$,

$$\|\nabla_x^2 f(x,\omega) - \nabla_x^2 f(\bar{x},\bar{\omega})\| \leq \frac{\eta}{4}\ \forall (x,\omega) \in (\bar{x}+\bar{\rho}\mathbb{R}_{\mathscr{B}^n}) \times W,$$

and finally, that $\tau_2(\omega) < \min\{L/2, \bar{\rho}/(2\gamma_2)\}\ \forall \omega \in W$. Further we set $\gamma_1 := \min\{2L/\eta, \bar{\rho}^2/(4\gamma_2^2)\} \leq \min\{2L/\eta, \bar{\rho}^2/(16L/\eta)\} \leq \bar{\rho}$. Now let $\omega \in W$ with $\mathrm{d}(\bar{x},\mathscr{F}(\omega)) < \gamma_1$ be arbitrarily fixed and choose $y \in \mathscr{F}(\omega)$ with $\|y-\bar{x}\| = \mathrm{d}(\bar{x},\mathscr{F}(\omega))$. For $x \in \bar{x}+\bar{\rho}\mathscr{B}_{\mathbb{R}^n}$ we obtain

$$\begin{aligned} &|f(x,\omega) - f(x,\bar{\omega}) - (f(y,\omega) - f(y,\bar{\omega})) - (\nabla_x f(\bar{x},\omega) - \nabla_x f(\bar{x},\bar{\omega}))(x-y)| \\ &\leq \frac{\eta}{4}(\|x-\bar{x}\|^2 + \|y-\bar{x}\|^2) \end{aligned}$$

and, together with Lemma 3

$$
\begin{aligned}
&|f(x,\omega)-f(x,\bar\omega)-(f(y,\omega)-f(\bar x,\bar\omega))|+\|q(x,\omega)-q(x,\bar\omega)\|\\
&\le e_2(\omega)^2+\tau_2(\omega)(\|x-\bar x\|+\|y-\bar x\|)+\frac{\eta}{2}\|x-\bar x\|^2+\frac{\eta}{4}\|y-\bar x\|^2+|f(y,\bar\omega)-f(\bar x,\bar\omega)|\\
&\le \tau_2(\omega)^2+\tau_2(\omega)\|x-\bar x\|+\frac{\eta}{2}\|x-\bar x\|^2+(\tau_2(\omega)+\frac{\eta}{4}\|y-\bar x\|+L)\|y-\bar x\|\\
&\le \tau_2(\omega)^2+\tau_2(\omega)\|x-\bar x\|+\frac{\eta}{2}\|x-\bar x\|^2+2L\|y-\bar x\|,
\end{aligned}
$$

implying

$$
\begin{aligned}
\alpha(x,y,\omega) &:= \max\{f(x,\omega)-f(y,\omega),\mathrm{d}(q(x,\omega),P)\}\\
&\ge \max\{f(x,\bar\omega)-f(\bar x,\bar\omega),\mathrm{d}(q(x,\bar\omega),P)\}\\
&\qquad -|f(x,\omega)-f(x,\bar\omega)-(f(y,\omega)-f(\bar x,\bar\omega))|-\|q(x,\omega)-q(x,\bar\omega)\|\\
&\ge \frac{\eta}{2}\|x-\bar x\|^2-\tau_2(\omega)^2-\tau_2(\omega)\|x-\bar x\|-2L\|y-\bar x\|.
\end{aligned}
$$

Now assume $\|x-\bar x\|>\gamma_2(\|y-\bar x\|^{\frac12}+\tau_2(\omega))$. Then

$$
\|x-\bar x\|>2\sqrt{\frac{L}{\eta}}\|y-\bar x\|^{\frac12}+\frac{1+\sqrt{2\eta+1}}{\eta}\tau_2(\omega)
$$

and therefore

$$
\begin{aligned}
\frac{\eta}{2}\left(\|x-\bar x\|-\frac{\tau_2(\omega)}{\eta}\right)^2 &> \frac{\eta}{2}\left(2\sqrt{\frac{L}{\eta}}(\|y-\bar x\|^{\frac12}+\frac{\sqrt{2\eta+1}}{\eta}\tau_2(\omega)\right)^2\\
&\ge 2L\|y-\bar x\|+\frac{2\eta+1}{2\eta}\tau_2(\omega)^2,
\end{aligned}
$$

showing $\alpha(x,y,\omega)>0$. Hence we conclude that for every $x\in\mathscr{F}(\omega)$ with $\gamma_2(\|y-\bar x\|^{\frac12}+\tau_2(\omega))<\|x-\bar x\|\le\bar\rho$ we have $f(x,\omega)>f(y,\omega)$, showing, together with $\gamma_2(\|y-\bar x\|^{\frac12}+\tau_2(\omega))<\bar\rho$, that there is a global solution $\bar x_\omega$ of the problem

$$
\min\ f(x,\omega)\ \text{subject to}\ q(x,\omega)\in P,\ x\in\bar x+\bar\rho B_{\mathbb{R}^n}
$$

with $\bar x_\omega\in X(\omega)$ and $\|\bar x_\omega-\bar x\|\le\gamma_2(\|y-\bar x\|^{\frac12}+\tau_2(\omega))<\bar\rho$. □

Given $x\in\mathscr{F}(\omega)$, we denote by $\mathscr{C}_\omega(x):=\{u\in T_\omega^{\mathrm{lin}}(x)\,|\,\nabla_x f(x,\omega)u\le 0\}$, the cone of critical directions at x. Further we introduce the Lagrangian $\mathscr{L}:\mathbb{R}^n\times\mathbb{R}^m\times\Omega\to\mathbb{R}$,

$$
\mathscr{L}(x,\lambda,\omega):=f(x,\omega)+\lambda^Tq(x,\omega) \tag{33}
$$

and for every $\omega\in\Omega$, every $x\in\mathscr{F}(\omega)$ and every $u\in\mathscr{C}_\omega(x)$ we define the set of multipliers

$$
\Lambda_\omega^1(x;u):=\{\lambda\in N(q(x,\omega);P;\nabla_x q(x,\omega)u)\,|\,\nabla_x\mathscr{L}(x,\lambda,\omega)=0\}
$$

Definition 5. *Let $\bar{x} \in \mathscr{F}(\bar{\omega})$. We say that the* refined strong second-order sufficient condition *(RSSOSC) holds at $\bar{x}$ for the problem $P(\bar{\omega})$, if for every nonzero critical direction $0 \neq u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ one has*

$$u^T \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega}) u > 0 \; \forall \lambda \in \Lambda_{\bar{\omega}}^1(\bar{x}; u).$$

Note that RSSOSC is sufficient for $\bar{x}$ being an essential local minimizer of second order only under some additional first-order optimality condition, e.g., if $\bar{x}$ is an extended M-stationary point in the sense of [10], i.e. $\Lambda_{\bar{\omega}}^1(\bar{x}; u) \neq \emptyset \; \forall 0 \neq u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$.

Recall that the multifunction M_ω is defined by $M_\omega(x) = q(x, \omega) - P$. In what follows, the first- and second order sufficient conditions for metric subregularity FOSCMS and SOSCMS are used as in Proposition 1.

Theorem 3. *Let $\bar{x} \in \mathscr{F}(\bar{\omega})$, and assume that Assumption 1 is fulfilled.*

1. *If FOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ and RSSOSC holds at $\bar{x}$ for $P(\bar{\omega})$ then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that*

$$(S_M(\omega) \cup S(\omega)) \cap U \subset \bar{x} + L\tau_1(\omega)\mathscr{B}_{\mathbb{R}^n} \; \forall \omega \in W. \tag{34}$$

   In addition, if $\bar{x}$ is an essential local minimizer for $P(\bar{\omega})$ and either $T_{\bar{\omega}}^{\text{lin}}(\bar{x}) \neq \{0\}$ or $M_{\bar{\omega}}$ is metrically regular around $(\bar{x}, 0)$, then $X(\omega) \cap U \neq \emptyset \; \forall \omega \in W$.

2. *If SOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ and $\bar{x}$ is an essential local minimizer of second order for $P(\bar{\omega})$, then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that*

$$(S_M(\omega) \cup S(\omega)) \cap U \subset \bar{x} + L\tau_2(\omega)\mathscr{B}_{\mathbb{R}^n} \; \forall \omega \in W. \tag{35}$$

   In addition, if either $T_{\bar{\omega}}^{\text{lin}}(\bar{x}) \neq \{0\}$ or $M_{\bar{\omega}}$ is metrically regular around $(\bar{x}, 0)$, then $X(\omega) \cap U \neq \emptyset \; \forall \omega \in W$.

Proof. We show the first part by contraposition. If the assertion does not hold, by virtue of Proposition 3 with $F = \nabla_x f$ together with Proposition 1 there is some triple (u, λ, μ) satisfying $\lambda \in \Lambda_{\bar{\omega}}^1(\bar{x}; u)$, (23), (25) and $0 = \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega}) u + \nabla_x q(\bar{x}; \bar{\omega})^T \mu$. By (25) there are sequences $(\alpha^k) \downarrow 0$ and $(\mu^k) \to \mu$ with $\lambda + \alpha^k \mu^k \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega}) u)$ and by using [10, Lemma 2.1] we obtain $\lambda^T \nabla_x q(\bar{x}, \bar{\omega}) u = (\lambda + \alpha^k \mu^k)^T \nabla_x q(\bar{x}, \bar{\omega}) u = 0 \; \forall k$ showing $\lambda^T \nabla_x q(\bar{x}, \bar{\omega}) u = \mu^T \nabla_x q(\bar{x}, \bar{\omega}) u = 0$. Hence $u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ because of $0 = \nabla_x \mathscr{L}(\bar{x}, \lambda, \bar{\omega}) u = \nabla_x f(\bar{x}, \bar{\omega}) u$. It follows $0 = u^T \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega}) u$ contradicting RSSOSC.

We also prove the second part by contraposition. Assuming on the contrary that the assertion does not hold, by applying Proposition 3 with $F = \nabla_x f$ together with Proposition 1 we find (u, λ, μ, v) satisfying $\lambda \in \Lambda_{\bar{\omega}}^1(\bar{x}; u)$, (23), (25) and $0 = \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega}) u + \nabla_x q(\bar{x}; \bar{\omega})^T \mu$, (28) and $u^T \nabla_x^2 (\lambda^T q)(\bar{x}, \bar{\omega}) u = \nabla_x f(\bar{x}, \bar{\omega})^T v$. Proceeding as before we obtain $u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ and $0 = u^T \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega}) u = u^T \nabla_x^2 f(\bar{x}, \bar{\omega}) u + \nabla_x f(\bar{x}, \bar{\omega})^T v \in T(\nabla_x f(\bar{x}, \bar{\omega}) u; \mathbb{R}_-)$. Further we have

$T(q(\bar{x},\bar{\omega});P_i) \subset T(\nabla_x q(\bar{x},\bar{\omega})u; T(q(\bar{x},\bar{\omega});P_i))$ for each $i \in \mathscr{I}(u)$ and thus

$$\begin{aligned}
\nabla_x q(\bar{x},\bar{\omega})v + u^T \nabla_x^2 q(\bar{x},\bar{\omega})u &\in T(q(\bar{x},\bar{\omega}); \bigcup_{i\in\mathscr{I}(u)} P_i) = \bigcup_{i\in\mathscr{I}(u)} T(q(\bar{x},\bar{\omega});P_i) \\
&\subset \bigcup_{i\in\mathscr{I}(u)} T(\nabla_x q(\bar{x},\bar{\omega})u; T(q(\bar{x},\bar{\omega});P_i)) \\
&= T(\nabla_x q(\bar{x},\bar{\omega})u; T(q(\bar{x},\bar{\omega}); \bigcup_{i\in\mathscr{I}(u)} P_i)) \\
&= T(\nabla_x q(\bar{x},\bar{\omega})u; T(q(\bar{x},\bar{\omega});P)).
\end{aligned}$$

Hence we can conclude from [10, Lemma 3.16] that $\bar{x}$ is not an essential local minimizer of second order for $P(\bar{\omega})$ contradicting our assumption.

Finally, in both cases the assertion that $X(\omega) \cap U \neq \emptyset \ \forall \omega \in W$ follows from Proposition 2 respectively Theorem 2 and Proposition 4. □

Remark 2. *The statements* (34) *and* (35)*, respectively, remain valid if we replace FOSCMS and SOSCMS by the formally weaker assumption that properties R_1 and R_2, respectively, hold. In order that the statement $X(\omega) \cap U \neq \emptyset$ holds in case that $\bar{x}$ is an essential local minimizer, we must ensure that $\mathscr{F}(\omega) \neq \emptyset$ for ω close to $\bar{\omega}$, e.g. by the assumption that there is a direction $0 \neq u \in T_{\bar{\omega}}^{\text{lin}}(\bar{x})$ fulfilling the assumptions of Proposition 2 or by the assumption of metric regularity of $M_{\bar{\omega}}$ around $(\bar{x},0)$.*

Remark 3. *If $p = 1$, i.e. if P is a polyhedron, then it follows from [7, Proposition 3.9] that the conditions FOSCMS for $M_{\bar{\omega}}$ and $T_{\bar{\omega}}^{\text{lin}}(\bar{x}) \neq \{0\}$ imply metric regularity of $M_{\bar{\omega}}$ around $(\bar{x},0)$.*

6 Application to MPECs

In this section we consider the special case $MPEC(\omega)$ as given in Example 1. In what follows we denote by $\bar{x}$ a point feasible for the problem $MPEC(\bar{\omega})$. Recall that the set P is given by $P = \mathbb{R}_-^{m_I} \times \{0\}^{m_E} \times Q_{EC}^{m_C}$. Hence, by [22, Proposition 6.41] and by Lemma 1, for every $(\tilde{g},\tilde{h},\tilde{a}) \in P \subset \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times (\mathbb{R}^2)^{m_C}$ we have

$$\begin{aligned}
T((\tilde{g},\tilde{h},\tilde{a});P) &= T(\tilde{g};\mathbb{R}_-^{m_I}) \times \{0\}^{m_E} \times \prod_{i=1}^{m_C} T(\tilde{a}_i;Q_{EC}), \\
N((\tilde{g},\tilde{h},\tilde{a});P) &= N(\tilde{g};\mathbb{R}_-^{m_I}) \times \mathbb{R}^{m_E} \times \prod_{i=1}^{m_C} N(\tilde{a}_i;Q_{EC}), \\
\hat{N}((\tilde{g},\tilde{h},\tilde{a});P) &= \hat{N}(\tilde{g};\mathbb{R}_-^{m_I}) \times \mathbb{R}^{m_E} \times \prod_{i=1}^{m_C} \hat{N}(\tilde{a}_i;Q_{EC}).
\end{aligned}$$

Further, for every direction $(u,v,w) \in T((\tilde{g},\tilde{h},\tilde{a});P)$ we have

$$N((\tilde{g},\tilde{h},\tilde{a});P;(u,v,w)) = N(\tilde{g};\mathbb{R}_-^{m_I};u) \times \mathbb{R}^{m_E} \times \prod_{i=1}^{m_C} N(\tilde{a}_i;Q_{EC};w_i).$$

For the inequality constraints we obviously have

$$\begin{array}{rcl} T(\tilde g;\mathbb{R}^{m_I}_-) &=& \{u\in\mathbb{R}^{m_I}\,|\,u_i\le 0\ \forall i:\tilde g_i=0\},\\ N(\tilde g;\mathbb{R}^{m_I}_-) &=& \hat N(\tilde g;\mathbb{R}^{m_I}_-)=\{\lambda\in\mathbb{R}^{m_I}_+\,|\,\lambda_i\tilde g_i=0,\ i=1,\ldots,m_I\},\\ N(\tilde g;\mathbb{R}^{m_I}_-;u) &=& \{\lambda\in\hat N(\tilde g;\mathbb{R}^{m_I}_-)\,|\,\lambda_i u_i=0,\ i=1,\ldots,m_I\}. \end{array}$$

By straightforward calculation we can obtain the formulas for the regular normal cone, the limiting normal cone and the contingent cone of the set Q_{EC} as follows: For all $a=(a_1,a_2)\in Q_{\mathrm{EC}}$ we have

$$\hat N(a;Q_{\mathrm{EC}})=\left\{(\xi_1,\xi_2)\,|\ \begin{array}{ll}\xi_2=0 & \text{if } 0=a_1>a_2\\ \xi_1\ge 0,\xi_2\ge 0 & \text{if } a_1=a_2=0\\ \xi_1=0 & \text{if } a_1<a_2=0\end{array}\right\},$$

$$N(a;Q_{\mathrm{EC}})=\begin{cases}\hat N(a;Q_{\mathrm{EC}}) & \text{if } a\ne(0,0)\\ \{(\xi_1,\xi_2)\,|\,\text{either } \xi_1>0,\ \xi_2>0 \text{ or } \xi_1\xi_2=0\} & \text{if } a=(0,0),\end{cases}$$

$$T(a;Q_{\mathrm{EC}})=\left\{(u_1,u_2)\,|\ \begin{array}{ll}u_1=0 & \text{if } 0=a_1>a_2\\ u_1\le 0,u_2\le 0,u_1u_2=0 & \text{if } a_1=a_2=0\\ u_2=0 & \text{if } a_1<a_2=0\end{array}\right\}$$

and for all $u=(u_1,u_2)\in T(a;Q_{\mathrm{EC}})$ we have

$$T(u;T(a;Q_{\mathrm{EC}}))=\begin{cases}T(a;Q_{\mathrm{EC}}) & \text{if } a\ne(0,0)\\ T(u;Q_{\mathrm{EC}})) & \text{if } a=(0,0),\end{cases}$$

$$\hat N(u;T(a;Q_{\mathrm{EC}}))=\begin{cases}\hat N(a;Q_{\mathrm{EC}}) & \text{if } a\ne(0,0)\\ \hat N(u;Q_{\mathrm{EC}}) & \text{if } a=(0,0),\end{cases}$$

$$N(a;Q_{\mathrm{EC}};u)=\begin{cases}N(a;Q_{\mathrm{EC}}) & \text{if } a\ne(0,0)\\ N(u;Q_{\mathrm{EC}}) & \text{if } a=(0,0).\end{cases}$$

Denoting

$$\begin{array}{l}\bar I_g:=\{i\in\{1,\ldots,m_I\}\,|\,g_i(\bar x,\bar\omega)=0\},\\ \bar I^{+0}:=\{i\in\{1,\ldots,m_C\}\,|\,G_i(\bar x,\bar\omega)>0=H_i(\bar x,\bar\omega)\},\\ \bar I^{0+}:=\{i\in\{1,\ldots,m_C\}\,|\,G_i(\bar x,\bar\omega)=0<H_i(\bar x,\bar\omega)\},\\ \bar I^{00}:=\{i\in\{1,\ldots,m_C\}\,|\,G_i(\bar x,\bar\omega)=0=H_i(\bar x,\bar\omega)\},\end{array}$$

the cone $T^{\mathrm{lin}}_{\bar\omega}(\bar x)$ is given by

$$T^{\mathrm{lin}}_{\bar\omega}(\bar x)=\left\{u\in\mathbb{R}^n\,|\ \begin{array}{l}\nabla_x g_i(\bar x,\bar\omega)u\le 0,\ i\in\bar I_g,\\ \nabla_x h_i(\bar x,\bar\omega)u=0,\ i=1,\ldots,m_E,\\ \nabla_x G_i(\bar x,\bar\omega)u=0,\ i\in\bar I^{0+},\\ \nabla_x H_i(\bar x,\bar\omega)u=0,\ i\in\bar I^{+0},\\ -(\nabla_x G_i(\bar x,\bar\omega)u,\nabla_x H_i(\bar x,\bar\omega)u)\in Q_{\mathrm{EC}},\ i\in\bar I^{00}\end{array}\right\}.$$

Given $u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ we define

$$\begin{aligned}
&I_g(u) := \{i \in \bar{I}_g \,|\, \nabla_x g_i(\bar{x},\bar{\omega})u = 0\}\\
&I^{+0}(u) := \{i \in \bar{I}^{00} \,|\, \nabla_x G_i(\bar{x},\bar{\omega})u > 0 = \nabla_x H_i(\bar{x},\bar{\omega})u\},\\
&I^{0+}(u) := \{i \in \bar{I}^{00} \,|\, \nabla_x G_i(\bar{x},\bar{\omega})u = 0 < \nabla_x H_i(\bar{x},\bar{\omega})u\},\\
&I^{00}(u) := \{i \in \bar{I}^{00} \,|\, \nabla_x G_i(\bar{x},\bar{\omega})u = 0 = \nabla_x H_i(\bar{x},\bar{\omega})u\}.
\end{aligned}$$

For given $u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ the set $N(q(\bar{x},\bar{\omega});P;\nabla_x q(\bar{x},\bar{\omega})u)$ consists of all $\lambda = (\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ satisfying

$$\lambda_i^g \geq 0,\ i \in I_g(u), \quad \lambda_i^g = 0,\ i \notin I_g(u), \tag{36}$$

$$\lambda_i^H = 0,\ i \in \bar{I}^{0+} \cup I^{0+}(u), \quad \lambda_i^G = 0,\ i \in \bar{I}^{+0} \cup I^{+0}(u), \tag{37}$$

$$\text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0,\ i \in I^{00}(u) \tag{38}$$

Then we have the following characterizations of metric subregularity: FOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ if and only if for every direction $0 \neq u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ the only multiplier $\lambda = (\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ satisfying (36)-(38) and

$$\sum_{i=1}^{m_I} \lambda_i^g \nabla_x g_i(\bar{x},\bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla_x h_i(\bar{x},\bar{\omega}) - \sum_{i=1}^{m_C} (\lambda_i^G \nabla_x G_i(\bar{x},\bar{\omega}) + \lambda_i^H \nabla_x H_i(\bar{x},\bar{\omega})) = 0 \tag{39}$$

is the trivial multiplier $\lambda = 0$.

Similarly, SOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ if and only if for every direction $0 \neq u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ the only multiplier $\lambda = (\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ satisfying (36)-(39) and

$$u^T \left(\sum_{i=1}^{m_I} \lambda_i^g \nabla_x^2 g_i(\bar{x},\bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla_x^2 h_i(\bar{x},\bar{\omega}) - \sum_{i=1}^{m_C} (\lambda_i^G \nabla_x^2 G_i(\bar{x},\bar{\omega}) + \lambda_i^H \nabla_x^2 H_i(\bar{x},\bar{\omega})) \right) u \geq 0 \tag{40}$$

is $\lambda = 0$.

Further, $M_{\bar{\omega}}$ is metrically regular around $(\bar{x},0)$ if and only if in case $u = 0$ the only multiplier λ fulfilling (36)-(39) is $\lambda = 0$. Note that $I_g(0) = \bar{I}_g$, $I^{+0}(0) = I^{0+}(0) = \emptyset$ and $I^{00}(0) = \bar{I}^{00}$.

We now translate Proposition 3 into terms of $MPEC(\omega)$. To do this it is convenient to consider the (extended) Lagrangian

$$\mathscr{L}^{\lambda_0}(x,\lambda,\omega) := \lambda_0 f(x,\omega) + \sum_{i=1}^{m_I} \lambda_i^g g_i(\bar{x},\omega) + \sum_{i=1}^{m_E} \lambda_i^h h_i(\bar{x},\omega) - \sum_{i=1}^{m_C} (\lambda_i^G G_i(\bar{x},\omega) + \lambda_i^H H_i(\bar{x},\omega))$$

where $\lambda_0 \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\lambda = (\lambda^g,\lambda^h,\lambda^G,\lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ and $\omega \in \Omega$. The Lagrangian $\mathscr{L}^1$ corresponds to the Lagrangian $\mathscr{L}$ as defined in (33).

Corollary 3. *Let $\bar{x}$ be feasible for $MPEC(\bar{\omega})$.*

1. *If property R_1 holds and there does not exist $0 \neq u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ and $\mu = (\mu^g, \mu^h, \mu^G, \mu^H)$ satisfying* (36)-(38)*,*

$$\mu_i^g \geq 0,\ i \in I_g(u) : \lambda_i^g = 0, \quad \mu_i^g = 0,\ i \notin I_g(u), \tag{41}$$

$$\mu_i^H = 0,\ i \in \bar{I}^{0+} \cup I^{0+}(u), \quad \mu_i^G = 0,\ i \in \bar{I}^{+0} \cup I^{+0}(u), \tag{42}$$

$$\text{either } \mu_i^G > 0, \mu_i^H > 0 \text{ or } \mu_i^G \mu_i^H = 0,\ i \in I^{00}(u) : \lambda_i^G = \lambda_i^H = 0 \tag{43}$$

$$\mu_i^G \geq 0,\ i \in I^{00}(u) : \lambda_i^G = 0, \lambda_i^H > 0, \quad \mu_i^G = 0,\ i \in I^{00}(u) : \lambda_i^G = 0, \lambda_i^H < 0, \tag{44}$$

$$\mu_i^H \geq 0,\ i \in I^{00}(u) : \lambda_i^H = 0, \lambda_i^G > 0, \quad \mu_i^H = 0,\ i \in I^{00}(u) : \lambda_i^H = 0, \lambda_i^G < 0, \tag{45}$$

$$\nabla_x \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega}) = 0, \tag{46}$$

$$\nabla_x^2 \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega}) u + \nabla_x \mathscr{L}^0(\bar{x}, \mu, \bar{\omega}) = 0, \tag{47}$$

then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that for the stationary solution mappings $S(\omega)$ and $S_M(\omega)$ of $MPEC(\omega)$ one has

$$(S(\omega) \cup S_M(\omega)) \cap U \subset \bar{x} + L\tau_1(\omega)\ \forall \omega \in W.$$

2. *If property R_2 holds and there does not exist $0 \neq u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$, $\mu = (\mu^g, \mu^h, \mu^G, \mu^H)$ and $v \in \mathbb{R}^n$ satisfying* (36)-(38)*,* (41)-(47) *and*

$$\nabla_x g_i(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 g_i(\bar{x}, \bar{\omega}) u \leq 0,\ i \in \bar{I}_g, \tag{48}$$

$$\nabla_x h_i(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 h_i(\bar{x}, \bar{\omega}) u = 0,\ i = 1, \ldots, m_E, \tag{49}$$

$$\nabla_x G_i(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 G_i(\bar{x}, \bar{\omega}) u = 0,\ i \in \bar{I}^{0+} \cup I^{0+}(u), \tag{50}$$

$$\nabla_x H_i(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 H_i(\bar{x}, \bar{\omega}) u = 0,\ i \in \bar{I}^{+0} \cup I^{+0}(u), \tag{51}$$

$$-(\nabla_x G_i(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 G_i(\bar{x}, \bar{\omega}) u, \nabla_x H_i(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 H_i(\bar{x}, \bar{\omega}) u) \in T((0,0), Q_{EC}),\ i \in I^{00}(u), \tag{52}$$

$$u^T \nabla_x^2 \mathscr{L}^0(\bar{x}, \lambda, \bar{\omega}) u = \nabla_x f(\bar{x}, \bar{\omega}) v \tag{53}$$

then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that for the stationary solution mappings $S(\omega)$ and $S_M(\omega)$ of $MPEC(\omega)$ one has

$$(S(\omega) \cup S_M(\omega)) \cap U \subset \bar{x} + L\tau_2(\omega)\ \forall \omega \in W.$$

In the following table we specify for each condition of Proposition 3 its corresponding counterpart in Corollary 3:

Prop.3(1.)	Cor.3(1.)	Prop.3(2.)	Cor.3(2.)
(24)	(36)-(38)	(28)	(48)-(52)
(25)	(41)-(45)	(29)	(53)
(26)	(46)		
(27)	(47)		

Finally let us mention that the definition of M-stationary solutions for $MPEC(\omega)$ is the same as coined by Scholtes [24].

We demonstrate now our results in the following example:

Example 3. *Consider the following MPEC depending on the parameter $\omega \in \Omega := \mathbb{R}_+$*

$$\begin{array}{rrcl} & MPEC(\omega) \qquad -2x_1 + x_2 & & \\ \textit{subject to} & g_1(x,\omega) := x_1 - x_2 - \omega & \leq & 0, \\ & g_2(x,\omega) := \omega\phi(x_1) - x_2 & \leq & 0, \\ & -(G_1(x,\omega,H_1(x,\omega)) := -(x_1,x_2) & \in & Q_{EC}, \end{array}$$

where $\phi(x) := x^6(1-\cos\frac{1}{x})$ for $x \neq 0$ and $\phi(0) := 0$, with $\bar{x} = (0,0)$ and $\bar{\omega} = 0$. Then we have $\bar{I}_g = \{1,2\}$, $\bar{I}^{0+} = \bar{I}^{+0} = \emptyset$, $\bar{I}^{00} = \{1\}$,

$$T_{\bar{\omega}}^{\text{lin}}(\bar{x}) = \{(u_1,u_2)\,|\,u_1 - u_2 \leq 0,\ -u_2 \leq 0,\ -(u_1,u_2) \in Q_{EC}\} = \{(0,u_2)\,|\,u_2 \geq 0\}$$

and we obtain, that $I_g(u) = I^{+0}(u) = I^{00}(u) = \emptyset$, $I^{0+}(u) = \{1\}$ for every $0 \neq u = (0,u_2) \in T_{\bar{\omega}}^{\text{lin}}(\bar{x})$ and the set of multipliers $\lambda = (\lambda_1^g, \lambda_2^g, \lambda^G, \lambda^H)$ fulfilling (36)-(38) *is given by the relations*

$$\lambda_1^g = 0,\ \lambda_2^g = 0,\ \lambda^H = 0. \tag{54}$$

Since (39) *reads as*

$$\lambda_1^g(1,-1) + \lambda_2^g(0,-1) - \lambda^G(1,0) - \lambda^H(0,1) = (0,0)$$

we conclude that FOSCMS is fulfilled and therefore property R_1 holds. But for $u = (0,0)$ we see, that $\lambda = (0,1,0,-1)$ satisfies (36)-(39) *and therefore metric regularity around $(\bar{x},0)$ does not hold.*

It is easy to see that $\mathscr{C}_{\bar{\omega}}(\bar{x}) = \{0\}$. Hence RSSOSC is fulfilled, $\bar{x}$ is an extended M-stationary point in the sense of [10] and therefore $\bar{x}$ is an essential local minimizer of second order by [10, Theorem 3.21]. Hence we can apply Theorem 3(1.) to obtain

$$(S(\omega) \cup S_M(\omega)) \cap U \subset L\omega\mathscr{B}_{\mathbb{R}^2}\ \forall \omega \in W$$

for some neighborhoods U of $\bar{x}$ and W of $\bar{\omega}$ and some constant $L > 0$. Further we have $X(\omega) \cap U \neq \emptyset\ \forall \omega \in W$, since $(0,1) \in T_{\bar{\omega}}^{\text{lin}}(\bar{x})$.

Now let us compute the sets $S(\omega)$, $S_M(\omega)$ and $X(\omega)$. For every $\omega > 0$ the feasible set $\mathscr{F}(\omega)$ consists of the union of the nonnegative x_2 axis, $\{(0,x_2)\,|\,x_2 \geq 0\}$, and the set $\tilde{X}(\omega) := \{(x_1,0)\,|\,0 < x_1 \leq \omega, \phi(x_1) = 0\} = \{(\frac{1}{2k\pi},0)\,|\,k \in \mathbb{N}, 2k\pi\omega \geq 1\}$ consisting of a sequence of isolated points with limit $\bar{x}$. It is easily checked that $(0,0)$ is M-stationary for every $\omega > 0$, but not a local minimizer. Further, every element of $\tilde{X}(\omega)$ is a local minimizer and consequently B-stationary, but all these local minimizers are not M-stationary because the constraints are degenerated, the only possible exception being the point $(\omega,0)$ if $\frac{1}{2\pi\omega} \in \mathbb{N}$. Summarizing all we have

$$S(\omega) = X(\omega) = \tilde{X}(\omega), S_M(\omega) = \begin{cases} \{(0,0)\} & \text{if } \frac{1}{2\pi\omega} \notin \mathbb{N} \\ \{(0,0),(\omega,0)\} & \text{else} \end{cases}$$

for $\omega > 0$ and $S(0) = S_M(0) = X(0) = \{0,0\}$. It is quite surprising that we could prove existence and upper Lipschitz continuity of solutions in the absence of metric regularity of the constraints, a situation which is not possible in case of NLP (see,e.g. [17, Lemma 8.31]).

For the sake of completeness we also formulate explicitly the sufficient conditions for upper Lipschitz continuity of Corollary 3, although we know by the proof of Theorem 3 that they are implied by RSSOSC. The conditions of Corollary 3 are, that there are no elements $u = (0, u_2)$ *with* $u_2 > 0$ *and* $\lambda = (\lambda_1^g, \lambda_2^g, \lambda^G, \lambda^H)$, $\mu = (\mu_1^g, \mu_2^g, \mu^G, \mu^H)$ *fulfilling* (54), $\mu_1^g = \mu_2^g = \mu^H = 0$,

$$(-2,1) + \lambda_1^g(1,-1) + \lambda_2^g(0,-1) - \lambda^G(1,0) - \lambda^H(0,1) = (0,0) \tag{55}$$

and

$$(0,0) + \mu_1^g(1,-1) + \mu_2^g(0,-1) - \mu^G(1,0) - \mu^H(0,1) = (0,0).$$

It is easy to see that this holds true, because (54) *and* (55) *are inconsistent.*

Corollary 3 recovers well-known results for standard nonlinear programs in the case of C^2 data. Consider the parametric nonlinear optimization problem

$$NLP(\omega) \qquad \min f(x,\omega) \quad \text{subject to} \quad g(x,\omega) \le 0,\ h(x,\omega) = 0, \tag{56}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, and $q = (g,h) : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}$ satisfy the Basic Assumption 1 and, in addition, that $f(\bar{x},\cdot)$, $\nabla_x f(\bar{x},\cdot)$, $q(\bar{x},\cdot)$ and $\nabla_x q(\bar{x},\cdot)$ are Lipschitzian near $\bar{\omega}$.

We use the same notation as above in this section, the results and representations reduce to forms which omit the complementarity data G_i and H_i. In particular, the cone $T_{\bar{\omega}}^{\text{lin}}(\bar{x})$ is given by

$$T_{\bar{\omega}}^{\text{lin}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \nabla_x g_i(\bar{x},\bar{\omega})u \le 0,\ i \in \bar{I}_g, \\ \nabla_x h_i(\bar{x},\bar{\omega})u = 0,\ i = 1,\dots,m_E \end{array} \right\},$$

with $\bar{I}_g = \{i \in \{1,\dots,m_I\} \,|\, g_i(\bar{x},\bar{\omega}) = 0\}$, and the extended Lagrangian defined above reduces to

$$\mathscr{L}^{\lambda_0}(x,\lambda^g,\lambda^h,\omega) := \lambda_0 f(x,\omega) + \sum_{i=1}^{m_I} \lambda_i^g g_i(x,\omega) + \sum_{i=1}^{m_E} \lambda_i^h h_i(x,\omega).$$

Then a point $x \in \mathscr{F}(\omega)$ feasible for $NLP(\omega)$ is M-stationary if and only if it is stationary in the Karush-Kuhn-Tucker (KKT) sense, i.e. the set of multipliers associated with x,

$$\Lambda(x,\omega) := \{(\lambda^g,\lambda^h) \in \mathbb{R}_+^{m_I} \times \mathbb{R}^{m_E} \,|\, \nabla_x \mathscr{L}^1(x,\lambda^g,\lambda^h,\omega) = 0, {\lambda^g}^T g(\bar{x},\bar{\omega}) = 0\}$$

is not empty. Further recall that under some constraint qualification (CQ), e.g. the Mangasarian-Fromovitz CQ (MFCQ), also the concepts of B-stationarity and KKT-stationarity coincide. Since MFCQ persists under small perturbations in our setting, hence $S(\omega) \cap U$ and $S_M(\omega) \cap U$ coincide for some neighborhood U of $\bar{x} \in S(\bar{\omega})$ and all ω close to $\bar{\omega}$, provided that $\bar{x}$ satisfies MFCQ.

We show how to prove two classical results by means of Corollary 3 or Theorem 3, respectively.

Corollary 4. *(Klatte, Kummer [17, Thm. 8.24]) Given a stationary solution $\bar{x} \in S(\bar{\omega})$ of $NLP(\bar{\omega})$ in the KKT sense, i.e., $\Lambda(\bar{x},\bar{\omega}) \neq \emptyset$, we suppose that MFCQ is satisfied at $(\bar{x},\bar{\omega})$. Then S is locally upper Lipschitz at $(\bar{\omega},\bar{x})$ if for every $\lambda = (\lambda^g,\lambda^h) \in \Lambda(\bar{x},\bar{\omega})$, the system*

$$\begin{array}{ll} u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x}), \quad (\alpha,\beta) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}, & \\ \nabla^2_x \mathscr{L}^1(\bar{x},\lambda^g,\lambda^h,\bar{\omega})u + \nabla_x \mathscr{L}^0(\bar{x},\alpha,\beta,\bar{\omega}) = 0, & \\ \nabla_x g_i(\bar{x},\bar{\omega})\, u = 0, & \text{if } i \in \bar{I}_g : \ \lambda^g_i > 0, \\ \alpha_i \geq 0, \ \alpha_i \nabla_x g_i(\bar{x},\bar{\omega})\, u = 0, & \text{if } i \in \bar{I}_g : \ \lambda^g_i = 0, \\ \alpha_i = 0, & \text{if } i \notin \bar{I}_g. \end{array} \tag{57}$$

has no solution (u,α,β) with $u \neq 0$.

Proof. Take any (u,λ,μ) which satisfies $u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$, (36), (41), (46) and (47). It is sufficient to show that (u,λ,μ) also satisfies both $\lambda \in \Lambda(\bar{x},\bar{\omega})$ and (57) when putting $\alpha = \mu^g$ and $\beta = \mu^h$. Indeed, then the assumption of the corollary says that $u = 0$. Since MFCQ at $(\bar{x},\bar{\omega})$ implies property R_1, Corollary 3 thus immediately gives in the special case $NLP(\omega)$

$$S(\omega) \cap U \subset \bar{x} + L\tau_1(\omega) \ \forall \omega \in W.$$

By our assumptions for f and $q = (g,h)$, we have for all ω sufficiently close to $\bar{\omega}$,

$$\begin{aligned} \tau_1(\omega) &= \|\nabla_x f(\bar{x},\omega) - \nabla_x f(\bar{x},\bar{\omega})\| + \|\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega})\| + \|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\| \\ &\leq \ const.\, \|\omega - \bar{\omega}\| \end{aligned}$$

and so we are done.

It remains to show that (u,λ,μ) satisfies both $\lambda \in \Lambda(\bar{x},\bar{\omega})$ and (57) when putting $\alpha = \mu^g$ and $\beta = \mu^h$. First, we observe that (36) says that $\lambda^g_i = 0$ if $i \notin \bar{I}_g$ or if $i \in \bar{I}_g \setminus I_g(u)$, and $\lambda^g_i \geq 0$ if $i \in I_g(u)$. Together with (46), this gives $\lambda \in \Lambda(\bar{x},\bar{\omega})$. Second, $u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ and (47) directly appear in the first two lines of (57). Moreover, (41) says (i) $\alpha_i = \mu^g_i = 0$ if $i \notin \bar{I}_g$ or if $i \in \bar{I}_g :$ $\nabla_x g_i(\bar{x},\bar{\omega})\, u < 0$, and (ii) $\alpha_i = \mu^g_i \geq 0$ if $i \in \bar{I}_g : \ \lambda^g_i = 0$, which implies that (u,α) fulfills the last two lines of (57). Finally, assuming that for some $i \in \bar{I}_g : \ \lambda^g_i > 0$ one has $\nabla_x g_i(\bar{x},\bar{\omega})\, u \neq 0$, i.e., $i \notin I_g(u)$, this contradicts (36). Therefore, also the third line of (57) is satisfied. □

Remark 4. *The opposite direction of Corollary 4 is true if the parametric problem (56) includes canonical perturbations, i.e. if f and $q = (g,h)$ are defined by $f(x,\omega) = \tilde{f}(t,x) - \langle a,x\rangle$ and $(g,h)(x,\omega) = (\tilde{g},\tilde{h})(t,x) - b$ for varying $\omega = (t,a,b) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{m_I+m_L}$, see [17, Thm. 8.24].*

The following Hölder stability result is a generalization of Proposition 4.41 in Bonnans and Shapiro [4], where global minimizing sets of $NLP(\omega)$ are considered instead of the sets of stationary solutions or local minimizers as here. Recall that $\bar{x} \in \mathscr{F}(\bar{\omega})$ satisfies the quadratic growth condition for $NLP(\bar{\omega})$ if $f(x,\bar{\omega}) - f(\bar{x},\bar{\omega}) \geq \eta \|x - \bar{x}\|^2 \ \forall x \in V \cap \mathscr{F}(\bar{\omega})$ holds for some $\eta > 0$ and some neighborhood V of $\bar{x}$.

Corollary 5. *(upper Hölder continuity of local minimizers)*
Suppose that $\bar{x} \in \mathscr{F}(\bar{\omega})$ satisfies the quadratic growth condition for NLP($\bar{\omega}$), MFCQ is satisfied at $\bar{x}$, and the functions f, g, and h are twice continuously differentiable near $(\bar{x}, \bar{\omega})$. Then X and S are locally nonempty-valued and upper Hölder of order $\frac{1}{2}$ at $(\bar{\omega}, \bar{x})$, i.e., there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that for the solution mapping $X(\omega)$ of $P(\omega)$ one has

$$\emptyset \neq X(\omega) \cap U \subset S(\omega) \cap U \subset \{\bar{x}\} + L\|\omega - \bar{\omega}\|^{\frac{1}{2}} \quad \forall \omega \in W.$$

Proof. The corollary is an immediate consequence of the second part of Theorem 3 by taking into account that MFCQ is equivalent to metric regularity of the multifunction $M_{\bar{\omega}}$ around $(\bar{x}, 0)$ implying SOSCMS by virtue of Theorem 1, and that under MFCQ the quadratic growth condition is equivalent with the property that $\bar{x}$ is an essential local minimizer of second-order. □

Results of this type are classical, but concern global or so-called complete local minimizing (CLM) sets, see e.g. [2, 3, 4, 16]. Corollary 5 extends this by providing even existence and upper Hölder continuity for stationary solution sets and the sets of *all* local minimizers, while part 2 of Theorem 3 allows us in addition to weaken the assumption MFCQ by assuming SOSCMS (which implies R_2) instead. Note that upper Hölder stability of the global minimizing or CLM set mapping even holds if $\bar{x}$ is not locally isolated (see e.g. [4, 16]). Recently, Kummer [18] presented an alternative characterization of upper Hölder stability of KKT- stationary solutions via convergence properties of suitable iteration procedures.

Remark 5. *For nonlinear programs, a standard second order sufficient optimality condition at $\bar{x} \in \mathscr{F}(\bar{\omega})$ (see for example [4, Prop.5.48]) is equivalent with the property* (32) *(i.e., $\bar{x}$ is an essential local minimizer of second order for the problem NLP($\bar{\omega}$)). This is true without any constraint qualification, see the remarks following Thm. 5.11 in [6].*

Further note that in the case (56), SOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ if and only if for every direction $0 \neq u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$, the only multiplier $\lambda = (\lambda^g, \lambda^h) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}$ satisfying

$$\lambda_i^g \geq 0,\ i \in I_g(u),\ \lambda_i^g = 0,\ i \notin I_g(u),\ u^T \left(\sum_{i=1}^{m_I} \lambda_i^g \nabla_x^2 g_i(\bar{x}, \bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla_x^2 h_i(\bar{x}, \bar{\omega}) \right) u \geq 0$$

is $\lambda = 0$, where as above $I_g(u) = \{i \in \bar{I}_g \mid \nabla_x g_i(\bar{x}, \bar{\omega}) u = 0\}$.

Example 4. *This example shows, as mentioned above, that part 2 of Theorem 3 gives a stronger result than the (more classical) Corollary 5. Consider the parameter dependent program*

$$P(\omega) \quad \min x_1^2 - x_2^2 \quad s.t. \ -x_2 + \omega \leq 0,\ x_2 - x_1^2 \leq 0,$$

with $\Omega = \mathbb{R}$, $\bar{\omega} = 0$ and $\bar{x} = (0,0)$. $M_{\bar{\omega}}$ is not metrically regular near $(\bar{x}, 0)$. One easily sees that SOSCMS holds for $M_{\bar{\omega}}$ at $\bar{x}$: one has to check only the directions $u = (\pm 1, 0) \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$. Obviously, $\bar{x}$ is an an essential local minimizer of second order for the problem $P(\bar{\omega})$, and one has

$$S(\omega) = S_M(\omega) = \begin{cases} \{(0,0), (0,\omega)\} & \text{if } \omega < 0, \\ \{(\pm\sqrt{\omega}, \omega)\} & \text{if } 0 \leq \omega \leq \frac{1}{2}, \end{cases}$$

$$X(\omega) = \begin{cases} \{(0,\omega)\} & \text{if } \omega < 0, \\ \{(\pm\sqrt{\omega},\omega)\} & \text{if } 0 \le \omega \le \frac{1}{2}. \end{cases}$$

Indeed, the assumptions and the statement of part 2 of Theorem 3 are fulfilled.

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References

[1] W. ACHTZIGER, C. KANZOW, *Mathematical programs with vanishing constraints: Optimality conditions and constraint qualifications*, Math. Program., 114 (2008), pp. 69-99.

[2] W. ALT, *Lipschitzian perturbations of infinite optimization problems*, in A.V. Fiacco, editor, *Mathematical Programming with Data Perturbations*, pages 7–21. M. Dekker, New York, 1983.

[3] A. V. ARUTYUNOV, A. F. IZMAILOV, *Sensitivity analysis for cone-constrained optimization problems under the relaxed constraint qualifications*, Math. Oper. Res., 30 (2005), pp. 333–353.

[4] J. F. BONNANS, A. SHAPIRO, *Perturbation analysis of optimization problems*, Springer, New York, 2000.

[5] A.L. DONTCHEV, R.T. ROCKAFELLAR, *Implicit Functions and Solution Mappings*, Springer, New York, 2009.

[6] H. GFRERER, *Second-order optimality conditions for scalar and vector optimization problems in Banach spaces*, SIAM J. Control Optim., 45 (2006), pp. 972–997.

[7] H. GFRERER, *First order and second order characterizations of metric subregularity and calmness of constraint set mappings*, SIAM J. Optim., 21 (2011), pp. 1439–1474.

[8] H. GFRERER, *On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs*, Set-Valued Var. Anal., 21 (2013), pp. 151–176.

[9] H. GFRERER, *On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs*, SIAM J. Optim., 23 (2013), pp. 632–665.

[10] H. GFRERER, *Optimality conditions for disjunctive programs based on generalized differentiation with application to mathematical programs with equilibrium constraints*, SIAM J. Optim., to appear.

[11] L. GUO,G. H. LIN, J. J. YE, *Stability analysis for parametric mathematical programs with geometric constraints and its applications*, SIAM J. Optim., 22 (2012), pp. 1151–1176.

[12] T. HOHEISEL, C. KANZOW, *Stationary conditions for mathematical programs with vanishing constraints using weak constraint qualifications*, J. Math. Anal. Appl. 337 (2008), pp. 292–310.

[13] A. F. IZMAILOV, *Mathematical programs with complementarity constraints: Regularity, optimality conditions and sensitivity*, Comput. Math. Math. Phys., 44 (2004), pp. 1145–1164.

[14] A. F. IZMAILOV, M. V. SOLODOV, *Mathematical programs with vanishing constraints: Optimality conditions, sensitivity, and a relaxation method*, J. Optim. Theory Appl., 142 (2009), pp. 501–532.

[15] H. TH. JONGEN, V. SHIKHMAN, S. STEFFENSEN, *Characterization of strong stability for C-stationary points in MPCC*, Math. Program., 132 (2012), pp. 295–308.

[16] D. KLATTE, *On quantitative stability for non-isolated minima*, Control and Cybernetics, 23 (1994), pp.$\tilde{1}$83–200.

[17] D. KLATTE, B. KUMMER, *Nonsmooth equations in optimization. Regularity, calculus, methods and applications*, Nonconvex Optimization and its Applications 60, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[18] B. KUMMER, *Inclusions in general spaces: Hölder stability, Solution schemes and Ekeland's principle* J. Math. Anal. Appl. 358 (2009), pp. 327–344.

[19] B. S. MORDUKHOVICH, *Variational analysis and generalized differentiation, I: Basic theory*, Springer, Berlin, Heidelberg, 2006.

[20] B. S. MORDUKHOVICH, *Variational analysis and generalized differentiation, II: Applications*, Springer, Berlin, Heidelberg, 2006.

[21] S. M. ROBINSON, *Some continuity properties of polyhedral multifunctions*, Math. Program. Studies, 14 (1981), pp. 206–214.

[22] R. T. ROCKAFELLAR, R. J-B. WETS, *Variational analysis*, Springer, Berlin, 1998.

[23] H. SCHEEL, S. SCHOLTES, *Mathematical programs with complementarity constraints: Stationarity, optimality, and sensitivity*, Math. Oper. Res., 25 (2000), pp. 1–22.

[24] S. SCHOLTES, *Convergence properties of a regularization scheme for mathematical programs with complementarity constraints*, SIAM J. Optim., 11 (2001), pp. 918–936.

[25] V. SHIKHMAN, *Topological Aspects of Nonsmooth Optimization*, Springer, New York, 2012.

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