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On Computation of Generalized Derivatives of the Normal-Cone Mapping and their Applications

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On computation of generalized derivatives of the normal-cone mapping and their applications

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Abstract

The paper concerns the computation of the graphical derivative and the regular (Fréchet) coderivative of the normal-cone mapping related to C^2 inequality constraints under very weak qualification conditions. This enables us to provide the graphical derivative and the regular coderivative of the solution map to a class of parameterized generalized equations with the constraint set of the investigated type. On the basis of these results we obtain finally a characterization of the isolated calmness property of the mentioned solution map and derive strong stationarity conditions for an MPEC with control constraints.

Key words. Parameterized generalized equation - Graphical derivatives - Regular coderivative - Mathematical program with equilibrium constraints.
AMS subject classification. 49J53, 90C31, 90C46.

1 Introduction

Various generalized derivatives introduced in modern variational analysis represent an efficient tool in stability analysis of multifunctions ([30, 14, 19, 5]). This concerns in particular the so-called solution maps associated with parameter-dependent variational inequalities or generalized equations. Their stability properties have been thoroughly analyzed already in the seventies, above all in the papers by Robinson. A (particular) attention has been paid to the case of polyhedral constraint sets, independent of the parameter; see, for instance, [25, 26, 28, 4]. An overview of available results in this situation can be found in

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[5, Chapter 2E]. Concerning non-polyhedral constraint sets, one can find a huge number of works related to constraint sets with nonlinear programming structure, possibly even parameter-dependent. They deal with various types of stability, including the strong regularity of Robinson ([27]) and various other types of "regular" behavior, see [14] and the references therein.

Another group of results concerns the so-called conic constraints, cf. [2]. They are formulated either in the general framework or for special important classes (SDP cones, Lorentz cones etc.).

In the first part of the present paper our main attention is paid to the graphical derivative and the regular coderivative of the normal-cone mapping

$$y \mapsto \widehat{N}_\Gamma(y), \tag{1}$$

where

$$\Gamma = \{y \in \mathbb{R}^m | q_i(y) \le 0, i = 1, 2, \ldots, l\} \tag{2}$$

with twice continuously differentiable functions $q_i : \mathbb{R}^m \to \mathbb{R}$. In (1), $\widehat{N}_\Gamma$ stands for the regular normal cone to Γ defined at the beginning of the next section. The graphical derivative of (1) has been computed in [30, Cor.13.43(a) and Exercise 13.17] in the case when Γ is a fully amenable set (i.e., locally, the pre-image of a polyhedral set under a constraint qualification). In the case of Γ given by (2) and $\bar{y}^* \in \widehat{N}_\Gamma(\bar{y})$ this formula attains the form

$$D\widehat{N}_\Gamma(\bar{y}, \bar{y}^*)(v) = \{\nabla^2(\lambda^T q)(\bar{y})v | \lambda \in \bar{\Lambda}(v)\} + N_{K(\bar{y},\bar{y}^*)}(v), v \in \mathbb{R}^m, \tag{3}$$

where

$$\bar{\Lambda}(v) = \mathop{\arg\max}_{\substack{\nabla q(\bar{y})^T\lambda = \bar{y}^* \\ \lambda \in N_{\mathbb{R}^l_-}(q(\bar{y}))}} \langle v, \nabla^2(\lambda^T q)(\bar{y})v\rangle$$

and

$$K(\bar{y}, \bar{y}^*) := T_\Gamma(\bar{y}) \cap \{\bar{y}^*\}^\perp$$

is the critical cone to Γ at $\bar{y}$ with respect to $\bar{y}^*$. Formula (3) can be viewed as a starting points of two lines of research. The first one is directed to the relaxation of the assumed full amenability of Γ in order to be able to deal with the problems of second-order or semidefinite cone programming [22]. In this paper, on the other hand, we will concentrate on Γ given by (2) and prove the validity of (1) under a substantially weaker condition than the Mangarian-Fromovitz constraint qualification (MFCQ) required in this case in [30] and all other works dealing with this subject.

Concerning the regular coderivative of $\widehat{N}_\Gamma$ for Γ given by (2), it has been studied in [8] and [9] under MFCQ and the Constant Rank constraint qualification (CRCQ) which are also substantially more restrictive than the conditions imposed in this paper.

In the next part of the paper we consider the *solution map* S which assigns to each value of the *parameter* $x \in \mathbb{R}^n$ the corresponding set of solutions to the (parameterized) generalized equation (GE).

$$0 \in F(x, y) + \widehat{N}_\Gamma(y), \tag{4}$$

and a modified solution map $\tilde{S}$ taking into account also possible parameter constraints. In (4), $y \in \mathbb{R}^m$ is the *decision* variable and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a single-valued, continuously differentiable mapping. On the basis of the graphical derivative of $\widehat{N}_\Gamma$ it is not difficult to compute the graphical derivative of S (or its outer estimate). On the contrary, the computation of the regular coderivative of S (respectively $\tilde{S}$) requires apart from the regular coderivative of $\widehat{N}_\Gamma$ also the fulfillment of another qualification condition which is typically rather restrictive. In order to relax it we have invoked the idea of *nondegeneracy* which we have extended from the original convex framework (see [2, p.315]) to a nonconvex one. This technique has enabled us to derive a new calculus rule for the regular normal cone to a "set with constraint structure" ([30, Theorem 6.14]) and, eventually, to compute the regular coderivative of S (respectively $\tilde{S}$).

The last part of the paper is devoted to applications. On the basis of the graphical derivative of S we state there a new criterion for the isolated calmness of S (at a given point from the graph of S), which is a valuable stability property and may be used e.g. in postoptimal analysis. Further, on the basis of the regular coderivative of S we have derived sharp necessary optimality conditions for an optimization problem, where (4) arises among the constraints. Such problems are termed mathematical programs with equilibrium constraints (MPECs) and represent a typical application area for new techniques of variational analysis.

The structure of the paper is as follows. In the first from two preparatory sections (Section 2) we provide a background from variational analysis. Apart from standard notions and properties we introduce there in Definition 2 a new stability property for multifunctions which plays a crucial role in further development. In Section 3 we fix the notation and state some simple auxiliary results. The main results are collected in Sections 4 and 5. These sections deal with the two mentioned generalized derivatives of $\widehat{N}_\Gamma$ and with the regular coderivatives of S and $\tilde{S}$, respectively. The concluding Section 6 is devoted to applications.

Our notation is basically standard: conv Ω and ri Ω denote the convex hull and the relative interior of the set Ω, respectively, gphΦ stands for the graph of the map Φ and span$\{a, b\}$ signifies the linear subspace generated by vectors a, b. Furthermore, $\xrightarrow{\Omega}$ denotes convergence within the set Ω, for a cone K its negative polar is denoted by K°, $\|\cdot\|$ stands for the Euclidean norm and $\ker A$ means the kernel of the matrix A. Finally, $|I|$ is the cardinality of the index set I, $\Omega^\perp$ denotes the orthogonal complement to Ω, $o : \mathbb{R}^+ \to \mathbb{R}$ denotes a function with the property that $o(\lambda)/\lambda \to 0$ when $\lambda \downarrow 0$ and $\mathrm{d}(\cdot, \Omega)$ signifies the (Euclidean) distance function to Ω.

2 Background from variational analysis

In this section we briefly review some generalized differential constructions employed in the paper, confining ourselves only to the settings that appear below. The reader can find more details and extended frameworks in the monographs [19, 30] and in the papers we refer to.

Let us start with geometric objects. Given a set $\Omega \subset \mathbb{R}^d$ and a point $\bar{z} \in \Omega$, define the

(Bouligand-Severi) *tangent/contingent cone* to Ω at $\bar{z}$ by

$$T_\Omega(\bar{z}) := \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega - \bar{z}}{t} = \Big\{u \in \mathbb{R}^d \Big|\ \exists\, t_k \downarrow 0,\ u_k \to u \ \text{ with } \ \bar{z} + t_k u_k \in \Omega\ \forall\ k\Big\}. \tag{5}$$

The (Fréchet) *regular normal cone* to Ω at $\bar{z} \in \Omega$ can be equivalently defined by

$$\widehat{N}_\Omega(\bar{z}) := \Big\{v^* \in \mathbb{R}^d \Big|\ \limsup_{z \xrightarrow{\Omega} \bar{z}} \frac{\langle v^*, z - \bar{z}\rangle}{\|z - \bar{z}\|} \le 0\Big\} = (T_\Omega(\bar{z}))^\circ. \tag{6}$$

The (Clarke) *regular tangent cone* to Ω at $\bar{z} \in \Omega$ can be defined by

$$\hat{T}_\Omega(\bar{z}) := \operatorname*{Lim\,inf}_{z \xrightarrow{\Omega} \bar{z}} T_\Omega(z).$$

The above notation Lim sup and Lim inf stands for the outer and the inner set limits in the sense of Painlevé–Kuratowski, see e.g. [30, Chapter 4]. Note that the Clarke tangent cone and the regular normal cone reduce to the classical tangent cone and normal cone of convex analysis, respectively, when the set Ω is convex.

Considering next set-valued (in particular, single-valued) mappings $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, we define for them the corresponding derivative and coderivative constructions generated by the tangent cone (5) and the normal cone (6), respectively. Given $(\bar{z}, \bar{w}) \in \operatorname{gph}\Psi$, the *graphical derivative* $D\Psi(\bar{z}, \bar{w}) : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ of Ψ at $(\bar{z}, \bar{w})$ is

$$D\Psi(\bar{z}, \bar{w})(u) := \big\{v \in \mathbb{R}^s \big|\ (u, v) \in T_{\operatorname{gph}\Psi}(\bar{z}, \bar{w})\big\}, \quad u \in \mathbb{R}^d. \tag{7}$$

From the dual prospectives we define the *regular coderivative* $\widehat{D}^*\Psi(\bar{z}, \bar{w}) : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ of Ψ at $(\bar{z}, \bar{w}) \in \operatorname{gph}\Psi$ generated by the regular normal cone (6) as

$$\widehat{D}^*\Psi(\bar{z}, \bar{w})(v^*) := \big\{u^* \in \mathbb{R}^d \big|\ (u^*, -v^*) \in \widehat{N}_{\operatorname{gph}\Psi}(\bar{z}, \bar{w})\big\}, \quad v^* \in \mathbb{R}^s. \tag{8}$$

If Ψ is single-valued at $\bar{z}$, we drop $\bar{w}$ in the notation of (7)–(8). In the case of smooth single-valued mappings, for all $u \in \mathbb{R}^d$ and $v^* \in \mathbb{R}^s$ we have the representation

$$D\Psi(\bar{z})(u) = \big\{\nabla\Psi(\bar{z})u\big\} \ \text{ and } \ \widehat{D}^*\Psi(\bar{z})(v^*) = \big\{\nabla\Psi(\bar{z})^T v^*\big\}.$$

In variational analysis an important role is played by various stability notions for multifunctions. In the sequel we will be extensively using the following two of them.

Definition 1. *Let $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ be a multifunction, let $(\bar{u}, \bar{v}) \in \operatorname{gph}\Psi$ and let $\kappa > 0$.*

1. *Ψ is called* metrically regular with modulus κ *near $(\bar{u}, \bar{v})$ if there are neighborhoods U of $\bar{u}$ and V of $\bar{v}$ such that*

$$\mathrm{d}(u, \Psi^{-1}(v)) \le \kappa \mathrm{d}(v, \Psi(u))\ \forall (u, v) \in U \times V. \tag{9}$$

2. Ψ *is called* metrically subregular with modulus κ *at* $(\bar{u}, \bar{v})$ *if there is a neighborhood* U *of* $\bar{u}$ *such that*
$$\mathrm{d}(u, \Psi^{-1}(\bar{v})) \leq \kappa \mathrm{d}(\bar{v}, \Psi(u)) \ \forall u \in U. \tag{10}$$

It is well known that metric regularity of the multifunction Ψ near $(\bar{u}, \bar{v})$ is equivalent to the Aubin property of the inverse multifunction Ψ^{-1} and metric subregularity of Ψ at $(\bar{u}, \bar{v})$ is equivalent with the property of *calmness* of its inverse.

For general multifunctions the property of metric regularity is characterized by the so-called Mordukhovich criterion, see e.g. [18], [19, Theorem 4.18].

In the sequel we are dealing mostly with the perturbation mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^l$ associated with (2), which is defined by

$$M(y) := q(y) - \mathbb{R}^l_-.$$

In this case one can ensure the metric regularity or the metric subregularity via the following statements. The first one follows immediately from [30, Exercise 9.44].

Proposition 1. *Given* $\bar{y} \in \Gamma$, M *is metrically regular near* $(\bar{y}, 0)$ *if and only if*

$$\ker(\nabla q(\bar{y})^T) \cap \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y})) = \{0\}. \tag{11}$$

The infimum of the moduli κ *for which the metric regularity property holds is equal to*

$$\max_{\substack{\lambda \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y})),\\ \|\lambda\|=1}} \frac{1}{\|\nabla q(\bar{y})^T \lambda\|}. \tag{12}$$

It is well known that condition (11) is equivalent to the classical Mangasarian-Fromovitz constraint qualification (MFCQ) at $\bar{y}$.

For the next statement we need the notion of the *linearized tangent cone* at some point $y \in \Gamma$ defined by
$$T^{\mathrm{lin}}_{\Gamma}(y) := \{v \in \mathbb{R}^m \,|\, \nabla q_i(y) v \leq 0, \ i \in \mathcal{I}(y)\},$$
where $\mathcal{I}(y) := \{i \in \{1, \ldots, l\} \,|\, q_i(y) = 0\}$ denotes the index set of constraints, active at y.

Proposition 2 (Second order sufficient condition for metric subregularity [7, Theorem 6.1])**.** *Let* $\bar{y} \in \Gamma$ *and assume that for every* $0 \neq u \in T^{\mathrm{lin}}_{\Gamma}(\bar{y})$ *one has*

$$\lambda \in \ker(\nabla q(\bar{y})^T) \cap \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y})), \ u^T \nabla^2 (\lambda^T q)(\bar{y}) u \geq 0 \implies \lambda = 0.$$

Then M *is metrically subregular at* $(\bar{y}, 0)$.

We will refer to this condition by using the acronym SOSCMS. In the literature one can find also other sufficient conditions for metric subregularity, see e.g. [10, 13, 31, 32].

In some situations M is not metrically regular near $\bar{y} \in \Gamma$ but enjoys a weaker property defined below.

Definition 2. *Let $\bar{y} \in \Gamma$. We say that M is* locally uniformly metrically regular except for $\bar{y}$, *if there is some neighborhood V of $\bar{y}$ and some constant $\kappa > 0$ such that for every $y \in M^{-1}(0) \cap V$, $y \neq \bar{y}$ the multifunction M is metrically regular near $(y, 0)$ with modulus κ.*

Since metric regularity is an open property in the sense that it holds in a neighborhood of the point in question, we easily conclude that metric regularity near $(\bar{y}, 0)$ implies locally uniformly metric regularity except for $\bar{y}$.

Proposition 3. *Let $\bar{y} \in \Gamma$. Under SOSCMS the mapping M is locally uniformly metrically regular except for $\bar{y}$.*

Proof. The proof follows from [6, Proposition 1] and the observation following Definition 4 therein. □

3 Notation and auxiliary results

Given elements $y \in \Gamma$ and $y^* \in \widehat{N}_\Gamma(y)$ we define by

$$\Lambda(y, y^*) := \{\lambda \in \widehat{N}_{\mathbb{R}^l_-}(q(y)) \mid \nabla q(y)^T \lambda = y^*\},$$

the set of *Lagrange multipliers* associated with (y, y^*). Moreover

$$K(y, y^*) := T_\Gamma(y) \cap (y^*)^\perp$$

stands for the *critical cone* to Γ at y with respect to y^*.

For a given reference pair $(\bar{y}, \bar{y}^*)$, $\bar{y} \in \Gamma$, $\bar{y}^* \in \widehat{N}_\Gamma(\bar{y})$, fixed throughout this paper, we shortly set $\bar{\mathcal{I}} := \mathcal{I}(\bar{y})$, $\bar{\Lambda} := \Lambda(\bar{y}, \bar{y}^*)$ and $\bar{K} := K(\bar{y}, \bar{y}^*)$. Furthermore we employ the set

$$\bar{\mathcal{N}} := \{v \in \mathbb{R}^m \mid \nabla q_i(\bar{y})v = 0,\ i \in \bar{\mathcal{I}}\}$$

(nullspace of gradients of constraints active at $\bar{y}$). Given a multiplier $\lambda \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))$ we introduce the index sets

$$I^+(\lambda) := \{i \in \{1, \ldots, l\} \mid \lambda_i > 0\},\ \bar{I}^0(\lambda) := \bar{\mathcal{I}} \setminus I^+(\lambda),$$

the sets of strongly and weakly active constraints. Apart from them we will be working with

$$\bar{I}^+ = \bigcup_{\lambda \in \bar{\Lambda}} I^+(\lambda), \quad \bar{I}^0 := \bar{\mathcal{I}} \setminus \bar{I}^+.$$

With a direction $v \in T_\Gamma^{\text{lin}}(\bar{y})$ let us now associate the *directional multiplier set*

$$\bar{\Lambda}(v) := \arg\max_{\lambda \in \bar{\Lambda}} v^T \nabla^2 (\lambda^T q)(\bar{y}) v.$$

Directly from the definition of metric subregularity one can infer that under metric subregularity of M at $(y, 0)$ one has

$$T_\Gamma(y) = T_\Gamma^{\text{lin}}(y).$$

It follows that under this condition the regular normal cone to Γ at y amounts to

$$\widehat{N}_\Gamma(y) = \nabla q(y)^T \widehat{N}_{\mathbb{R}^l_-}(q(y))$$

and consequently $\Lambda(y, y^*) \neq \emptyset$. In the rest of this section the metric subregularity of M at $(\bar{y}, 0)$ will be assumed.

Lemma 1. *Let $v \in \bar{K}$, $\lambda \in \bar{\Lambda}$. Then*

$$\widehat{N}_{\bar{K}}(v) = \{\nabla q(\bar{y})^T \mu \,|\, \mu^T \nabla q(\bar{y}) v = 0, \mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda)\}$$

Proof. Note that $\widehat{N}_{\bar{K}}(v) = \bar{K}^\circ \cap \{v\}^\perp$ and $\bar{K}^\circ = T_\Gamma^{\text{lin}}(\bar{y})^\circ + \mathbb{R}\bar{y}^*$. Hence, for every $v^* \in \widehat{N}_{\bar{K}}(v)$ there is some $\tilde{\mu} \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))$ and some $\alpha \in \mathbb{R}$ with $v^* = \nabla q(\bar{y})^T \tilde{\mu} + \alpha \bar{y}^* = \nabla q(\bar{y})^T (\tilde{\mu} + \alpha\lambda)$. Setting $\mu = \tilde{\mu} + \alpha\lambda$ we have $\mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda)$, $v^* = \nabla q(\bar{y})^T \mu$ and $\mu^T \nabla q(\bar{y}) v = v^{*T} v = 0$.

Conversely, let $\mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda)$ with $\mu^T \nabla q(\bar{y})^T v = 0$ be arbitrarily fixed. Then there is some $t > 0$ such that $\lambda + t\mu \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))$ and therefore for all $w \in \bar{K} \subset T_\Gamma^{\text{lin}}(\bar{y})$ we have

$$0 \geq (\lambda + t\mu)^T \nabla q(\bar{y}) w = \bar{y}^{*T} w + t\mu^T \nabla q(\bar{y}) w = t(\nabla q(\bar{y})^T \mu)^T w$$

showing $\nabla q(\bar{y})^T \mu \in \bar{K}^\circ$. Together with $(\nabla q(\bar{y})^T \mu)^T v = 0$ we conclude $(\nabla q(\bar{y})^T \mu) \in \widehat{N}_{\bar{K}}(v)$. $\square$

Lemma 2. *There is some $\tilde{\lambda} \in \bar{\Lambda}$ such that $I^+(\tilde{\lambda}) = \bar{I}^+$. Further we have*

$$\bar{K} = \left\{ v \in \mathbb{R}^m \,\middle|\, \begin{array}{l} \nabla q_i(\bar{y}) v = 0, \;\; i \in \bar{I}^+ \\ \nabla q_i(\bar{y}) v \leq 0, \;\; i \in \bar{I}^0 \end{array} \right\}$$

and there is some $v \in \bar{K}$ satisfying

$$\nabla q_i(\bar{y}) v = 0, \;\; i \in \bar{I}^+, \quad \nabla q_i(\bar{y}) v < 0, \;\; i \in \bar{I}^0. \tag{13}$$

Proof. Since $\bar{I}^+$ is a finite index set, there are $\lambda^i \in \bar{\Lambda}$, $i = 1, \ldots, N$ such that $\bar{I}^+ = \bigcup_{i=1}^N I^+(\lambda^i)$. Setting $\tilde{\lambda} := \frac{1}{N} \sum_{i=1}^N \lambda^i$ it easily follows that $\tilde{\lambda} \in \bar{\Lambda}$ and $I^+(\tilde{\lambda}) = \bar{I}^+$. The second assertion follows from the equivalences

$$\begin{aligned} v \in \bar{K} \quad &\Leftrightarrow \quad \left(v \in T_\Gamma^{\text{lin}}(\bar{y}) \wedge 0 = \bar{y}^{*T} v = (\nabla q(\bar{y})^T \tilde{\lambda})^T v = \tilde{\lambda}^T \nabla q(\bar{y}) v = \sum_{i \in I^+(\tilde{\lambda})} \tilde{\lambda}_i \nabla q_i(\bar{y}) v \right) \\ &\Leftrightarrow \quad \left(v \in T_\Gamma^{\text{lin}}(\bar{y}) \wedge \nabla q_i(\bar{y}) v = 0, i \in I^+(\tilde{\lambda}) = \bar{I}^+ \right) \end{aligned}$$

We prove the last statement by contraposition. Assuming that the system

$$\nabla q_i(\bar{y})v = 0,\ i \in \bar{I}^+, \quad \nabla q_i(\bar{y})v \le -1,\ i \in \bar{I}^0 \tag{14}$$

does not have a solution, by the Farkas lemma there is some $\mu \in \mathbb{R}^l$ such that $\nabla q(\bar{y})^T \mu = 0$, $\mu_i = 0,\ i \notin \bar{\mathcal{I}}$, $\mu_i \ge 0, i \in \bar{I}^0$ and $\sum_{i\in\bar{I}^0} \mu_i > 0$. It follows that $\tilde{\lambda} + t\mu \in \bar{\Lambda}$ for some $t > 0$ and from $\sum_{i\in\bar{I}^0} \mu_i > 0$ we conclude that there must be some index $i \in \bar{I}^0$ with $\tilde{\lambda}_i + t\mu_i > 0$ implying $i \in \bar{I}^+$, a contradiction. Hence the system (14) has a solution and this completes the proof. □

Consider for every $v \in \bar{K}$ the linear optimization problem

$$LP(v) \qquad \min -v^T\nabla^2(\lambda^T q)(\bar{y})v \quad \text{subject to} \quad \lambda \in \bar{\Lambda}$$

together with its dual program

$$DP(v) \qquad \max \bar{y}^{*T} z \quad \text{subject to} \quad \nabla q_i(\bar{y})z \le -v^T\nabla^2 q_i(\bar{y})v,\ i \in \bar{\mathcal{I}}.$$

Then, by definition, $\bar{\Lambda}(v)$ is the solution set of $LP(v)$ and, by duality theory of linear programming, $\bar{\Lambda}(v) \neq \emptyset$ if and only if $DP(v)$ is solvable. Further, given $\lambda \in \bar{\Lambda}$ and z feasible for $DP(v)$, we have $\lambda \in \Lambda(v)$ and z solves $DP(v)$ if and only if

$$\lambda_i \left(\nabla q_i(\bar{y})z + v^T\nabla^2 q_i(\bar{y})v\right) = 0, i \in \bar{\mathcal{I}}. \tag{15}$$

Lemma 3. *For every $\bar{v} \in \bar{K}$ there is a neighborhood U of $\bar{v}$ such that*

$$\bar{\Lambda}(v) \subset \bar{\Lambda}(\bar{v})\ \forall v \in U \cap \bar{K}$$

Proof. If $\bar{\Lambda}(\bar{v}) = \emptyset$, we conclude from [1, Theorems 5.4.1,5.4.2] that $\bar{\Lambda}(v) = \emptyset$ for all v belonging to some neighborhood U of $\bar{v}$. Now assume $\bar{\Lambda}(\bar{v}) \neq \emptyset$. Again by [1, Theorems 5.4.1,5.4.2] for every $\epsilon > 0$ there is some neighborhood U_ϵ such that $\mathrm{d}(\lambda, \bar{\Lambda}(\bar{v})) < \epsilon$ for every $v \in U_\epsilon$ and every $\lambda \in \bar{\Lambda}(v)$. Further we know that the solution set of a linear optimization problem is a face of the feasible set. Combining both properties we see that for all v near $\bar{v}$ the solution set $\bar{\Lambda}(v)$ is a face of $\bar{\Lambda}(\bar{v})$, provided it is not empty. □

Definition 3. *Let $\bar{\Lambda} \neq \emptyset$. An index set $\mathcal{B} \subset \bar{\mathcal{I}}$ is called a basis, if $|\mathcal{B}| = \mathrm{rank}\,(\nabla q_i(\bar{y}))_{i\in\bar{\mathcal{I}}}$ and the family $(\nabla q_i(\bar{y}))_{i\in\mathcal{B}}$ is linearly independent. A basis $\mathcal{B}$ is called feasible, if the unique solution $\lambda^{\mathcal{B}} \in \mathbb{R}^l$ of the system*

$$\sum_{i\in\mathcal{B}} \lambda_i^{\mathcal{B}} \nabla q_i(\bar{y}) = \bar{y}^*, \lambda_i^{\mathcal{B}} = 0, i \notin \mathcal{B}$$

fulfills $\lambda_i^{\mathcal{B}} \ge 0,\ i \in \mathcal{B}$. Then $\lambda^{\mathcal{B}}$ is called a feasible basis solution. The collection of all feasible bases is denoted by $\mathcal{B}(\bar{y}, \bar{y}^*)$ and we denote by $\mathcal{E} = \{\lambda^{\mathcal{B}} \,|\, \mathcal{B} \in \mathcal{B}(\bar{y}, \bar{y}^*)\}$ the collection of all feasible basis solutions.

It is well known that the feasible basis solutions are exactly the extreme points of the polyhedron $\bar{\Lambda}$. Moreover, the polyhedron $\bar{\Lambda}$ can be represented as the sum of the convex hull of its extreme points and its recession cone $\mathcal{R} := \{\lambda \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y})) \mid \nabla q(\bar{y})^T\lambda = 0\}$, i.e. $\bar{\Lambda} = \operatorname{conv}\mathcal{E} + \mathcal{R}$. From the theory of linear programming it is well known that $\bar{\Lambda}(v) \neq \emptyset$ if and only if

$$v^T\nabla^2(\lambda^T q)(\bar{y})v \leq 0 \ \forall \lambda \in \mathcal{R}.$$

In this case the set $\bar{\Lambda}(v) \cap \mathcal{E}$ is not empty and contains exactly the extreme points of $\bar{\Lambda}(v)$. In what follows we denote by $\Lambda^{\mathcal{E}}(v)$ the compact convex polyhedron $\Lambda^{\mathcal{E}}(v) := \bar{\Lambda}(v) \cap \operatorname{conv}\mathcal{E}$.

4 Graphical derivative and regular coderivative of $\widehat{N}_\Gamma$

We denote by $\mathcal{T}(\bar{y}, \bar{y}^*)$ the set

$$\mathcal{T}(\bar{y}, \bar{y}^*) := \{(v, v^*) \mid \exists \lambda \in \bar{\Lambda}(v) : v^* \in \nabla^2(\lambda^T q)(\bar{y})v + \widehat{N}_{\bar{K}}(v)\}$$

Theorem 1. *Assume that M is metrically subregular at $(\bar{y}, 0)$. Then*

$$\mathcal{T}(\bar{y}, \bar{y}^*) \subset T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) \tag{16}$$

and equality holds if in addition M is locally uniformly metrically regular except for $\bar{y}$.

Proof. To show (16) let $(v, v^*) \in \mathcal{T}(\bar{y}, \bar{y}^*)$ be arbitrarily fixed. In order to show $(v, v^*) \in T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$ we must prove the existence of sequences $(t_k) \downarrow 0$ and $(y_k) \to \bar{y}$ such that

$$\lim_{k\to\infty} \frac{\bar{y} + t_k v - y_k}{t_k} = 0, \ \lim_{k\to\infty} \frac{\mathrm{d}(\bar{y}^* + t_k v^*, \widehat{N}_\Gamma(y_k))}{t_k} = 0.$$

Let $\lambda \in \bar{\Lambda}(v)$ such that $v^* \in \nabla^2(\lambda^T q)(\bar{y})v + \widehat{N}_{\bar{K}}(v)$ and choose $\alpha > 0$ so small such that $\alpha\|\nabla^2(\lambda^T q)(\bar{y})\| < \frac{1}{2}$. It follows that $2I + \nabla^2((\alpha\lambda)^T q)(\bar{y})$ is positive definite and hence, by applying the standard second order sufficient conditions of nonlinear programming (see, e.g., [2, Proposition 5.48]), we can conclude that there is some positive radius $\rho > 0$ such that $\bar{y}$ is the unique global solution of the problem

$$\min_y \|\bar{y} + \alpha\bar{y}^* - y\|^2 \text{ subject to } q(y) \leq 0, \ \|y - \bar{y}\| \leq \rho.$$

By duality theory of linear programming, there is some vector z solving $DP(v)$ with

$$\nabla q_i(\bar{y})z + v^T\nabla^2 q_i(\bar{y})v \leq 0, \ \lambda_i(\nabla q_i(\bar{y})z + v^T\nabla^2 q_i(\bar{y})v) = 0, \ i \in \bar{\mathcal{I}}.$$

Since $\lambda_i = 0$, $i \notin \bar{\mathcal{I}}$, we obtain

$$\lambda^T\nabla q(\bar{y})z + v^T\nabla^2(\lambda^T q)(\bar{y})v = 0. \tag{17}$$

By Lemma 1 there is some $\mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda) \cap (\nabla q(\bar{y})v)^\perp$ with $v^* = \nabla^2(\lambda^T q)(\bar{y})v + \nabla q(\bar{y})^T \mu$ and thus

$$v^{*T} v = v^T \nabla^2(\lambda^T q)(\bar{y})v + \mu^T \nabla q(\bar{y})v = v^T \nabla^2(\lambda^T q)(\bar{y})v. \tag{18}$$

The condition $\mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda)$ ensures the existence of some $\bar{t} > 0$ such that $\lambda + t\mu \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))$ for all $t \in [0, \bar{t}]$ and since $q_i(\bar{y}) < 0$, $i \notin \bar{\mathcal{I}}$, we can also assume that for every $t \in [0, \bar{t}]$ we have $q_i(\bar{y} + tv + \frac{1}{2}t^2 z) < 0$, $i \notin \bar{\mathcal{I}}$. We now consider for each $t \in [0, \bar{t}]$ a global solution y_t of the optimization problem

$$\min \|\bar{y} + tv + \frac{1}{2}t^2 z + \alpha(\bar{y}^* + tv^*) - y\|^2 \text{ subject to } y \in \Gamma,\ \|y - \bar{y}\| \le \rho.$$

In order to show the inclusion $(v, v^*) \in T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$ it suffices to show

$$\lim_{t \downarrow 0} \frac{\bar{y} + tv - y_t}{t} = 0 \tag{19}$$

because then for all $t > 0$ sufficiently small we have $\|y_t - \bar{y}\| < \rho$ and therefore the standard optimality condition at y_t reads as

$$\alpha(\bar{y}^* + tv^*) + t(\frac{\bar{y} + tv - y_t}{t} + \frac{1}{2}tz) \in \widehat{N}_\Gamma(y_t).$$

This, because $\widehat{N}_\Gamma(y_t)$ is a cone, implies that

$$\lim_{t \downarrow 0} \frac{\mathrm{d}(\bar{y}^* + tv^*, \widehat{N}_\Gamma(y_t))}{t} = 0.$$

Our choice of α and ρ guarantees that $y_0 = \bar{y}$ and, by using [2, Proposition 4.4], we conclude $\lim_{t \downarrow 0} y_t = \bar{y}$. Since

$$q_i(\bar{y} + tv + \frac{1}{2}t^2 z) = q_i(\bar{y}) + t\nabla q_i(\bar{y})v + \frac{1}{2}t^2(\nabla q_i(\bar{y})z + v^T \nabla^2 q_i(\bar{y})v) + o(t^2) \le o(t^2),\ i \in \bar{\mathcal{I}},$$

$q_i(\bar{y} + tv + \frac{1}{2}t^2 z) < 0$, $i \notin \bar{\mathcal{I}}$, and M is assumed to be metrically subregular at $(\bar{y}, 0)$, we can find for every $t \in [0, \bar{t}]$ some point $\tilde{y}_t \in \Gamma$ with $\|\bar{y} + tv + \frac{1}{2}t^2 z - \tilde{y}_t\| = o(t^2)$. Hence

$$\|\bar{y} + tv + \frac{1}{2}t^2 z + \alpha(\bar{y}^* + tv^*) - y_t\|^2 \le \|\bar{y} + tv + \frac{1}{2}t^2 z + \alpha(\bar{y}^* + tv^*) - \tilde{y}_t\|^2$$

for all $t \ge 0$ sufficiently small, implying

$$\|\bar{y} + tv + \frac{1}{2}t^2 z - y_t\|^2 + 2\alpha(\bar{y}^* + tv^*)^T(\bar{y} + tv + \frac{1}{2}t^2 z - y_t) \le o(t^2).$$

From $\lambda + t\mu \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))$, $q(y_t) - q(\bar{y}) \in T_{\mathbb{R}^l_-}(q(\bar{y}))$ and $v^* = \nabla^2(\lambda^T q)(\bar{y})v + \nabla q(\bar{y})^T\mu$ we obtain

$$\begin{aligned}
0 &\geq (\lambda + t\mu)^T(q(y_t) - q(\bar{y})) \\
&= (\lambda + t\mu)^T\nabla q(\bar{y})(y_t - \bar{y}) + \frac{1}{2}(y_t - \bar{y})^T\nabla^2(\lambda^T q)(\bar{y})(y_t - \bar{y}) + o(\|y_t - \bar{y}\|^2) \\
&= (\bar{y}^* + tv^*)^T(y_t - \bar{y}) - tv^T\nabla^2(\lambda^T q)(\bar{y})(y_t - \bar{y}) + \frac{1}{2}(y_t - \bar{y})^T\nabla^2(\lambda^T q)(\bar{y})(y_t - \bar{y}) + o(\|y_t - \bar{y}\|^2)
\end{aligned}$$

and, consequently, by taking into account ${\bar{y}^*}^T v = 0$ and relations (17), (18),

$$\begin{aligned}
&(\bar{y}^* + tv^*)^T(\bar{y} + tv + \frac{1}{2}t^2 z - y_t) \\
&\quad= (\bar{y}^* + tv^*)^T(\bar{y} - y_t) + t{\bar{y}^*}^T v + \frac{1}{2}t^2{\bar{y}^*}^T z + t^2{v^*}^T v + o(t^2) \\
&\quad\geq tv^T\nabla^2(\lambda^T q)(\bar{y})(\bar{y} - y_t) + \frac{1}{2}(y_t - \bar{y})^T\nabla^2(\lambda^T q)(\bar{y})(y_t - \bar{y}) + t{\bar{y}^*}^T v + \\
&\qquad\quad \frac{1}{2}t^2\lambda^T\nabla q(\bar{y})z + t^2 v^T\nabla^2(\lambda^T q)(\bar{y})v + o(t^2) + o(\|y_t - \bar{y}\|^2) \\
&\quad= tv^T\nabla^2(\lambda^T q)(\bar{y})(\bar{y} - y_t) + \frac{1}{2}(y_t - \bar{y})^T\nabla^2(\lambda^T q)(\bar{y})(y_t - \bar{y}) + \\
&\qquad\quad \frac{1}{2}t^2 v^T\nabla^2(\lambda^T q)(\bar{y})v + o(t^2) + o(\|y_t - \bar{y}\|^2) \\
&\quad= \frac{1}{2}(\bar{y} + tv - y_t)^T\nabla^2(\lambda^T q)(\bar{y})(\bar{y} + tv - y_t) + o(t^2) + o(\|y_t - \bar{y}\|^2).
\end{aligned}$$

Since $\alpha\|\nabla^2(\lambda^T q)(\bar{y})\| < \frac{1}{2}$, it follows that

$$\begin{aligned}
&\|\bar{y} + tv + \frac{1}{2}t^2 z - y_t\|^2 - \frac{1}{2}\|\bar{y} + tv - y_t\|^2 \\
&\quad\leq \|\bar{y} + tv + \frac{1}{2}t^2 z - y_t\|^2 + 2\alpha(\bar{y}^* + tv^*)^T(\bar{y} + tv + \frac{1}{2}t^2 z - y_t) + o(t^2) + o(\|y_t - \bar{y}\|^2) \\
&\quad\leq o(t^2) + o(\|y_t - \bar{y}\|^2)
\end{aligned}$$

and

$$\frac{1}{2}\|\bar{y} + tv - y_t\|^2 \leq o(t^2) + o(\|y_t - \bar{y}\|^2). \tag{20}$$

Hence there is some $\tilde{t} > 0$ such that $\frac{1}{2}\|\bar{y} + tv - y_t\|^2 \leq \frac{1}{4}(t^2 + \|y_t - \bar{y}\|^2)$ $\forall t \in [0, \tilde{t}]$. After rearranging we obtain $\frac{1}{4}\|\bar{y} - y_t\|^2 + tv^T(\bar{y} - y_t) + \frac{1}{2}t^2\|v\|^2 \leq \frac{1}{4}t^2$ showing

$$\frac{1}{4}\|\bar{y} + 2tv - y_t\|^2 = \frac{1}{4}\|\bar{y} - y_t\|^2 + tv^T(\bar{y} - y_t) + t^2\|v\|^2 \leq \frac{1}{4}t^2(1 + 2\|v\|^2)$$

and $\|\bar{y} - y_t\| \leq t(2\|v\| + \sqrt{1 + 2\|v\|^2})$. From (20) we conclude that (19) holds and therefore $(v, v^*) \in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$ follows.

Next we show that the reverse inclusion $\mathcal{T}(\bar{y},\bar{y}^*)\supset T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ is valid under the additional assumption that M is locally uniformly metrically regular except for $\bar{y}$. Let $(v,v^*)\in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ and consider sequences $(t_k)\downarrow 0$, $(v_k)\to v$ and $(v_k^*)\to v^*$ such that $y_k^*:=\bar{y}^*+t_kv_k^*\in\widehat{N}_\Gamma(y_k)$, where $y_k:=\bar{y}+t_kv_k$. By passing to a subsequence if necessary we can assume that there is some index set $\tilde{\mathcal{I}}\subset\bar{\mathcal{I}}$ such that $\mathcal{I}(y_k)=\tilde{\mathcal{I}}$ holds for all k. For every $i\in\bar{\mathcal{I}}$ we have

$$q_i(y_k)=q_i(\bar{y})+t_k\nabla q_i(\bar{y})v_k+o(t_k)=t_k\nabla q_i(\bar{y})v_k+o(t_k)\begin{cases}=0 & \text{if } i\in\tilde{\mathcal{I}},\\ \leq 0 & \text{if } i\in\bar{\mathcal{I}}\setminus\tilde{\mathcal{I}}.\end{cases}$$

Dividing by t_k and passing to the limit we obtain

$$\nabla q_i(\bar{y})v\begin{cases}=0 & \text{if } i\in\tilde{\mathcal{I}},\\ \leq 0 & \text{if } i\in\bar{\mathcal{I}}\setminus\tilde{\mathcal{I}}.\end{cases}\tag{21}$$

Next consider for each $y^*\in\mathbb{R}^m$ the set

$$\Psi_{\tilde{\mathcal{I}}}(y^*):=\{\lambda\in\mathbb{R}^l\mid\nabla q(\bar{y})^T\lambda=y^*,\lambda_i\geq 0,\ i\in\tilde{\mathcal{I}},\lambda_i=0,i\notin\tilde{\mathcal{I}}\}.\tag{22}$$

By Hoffman's Lemma there is some constant β such that for every $\lambda\in\mathbb{R}^l$ and every $y^*\in\mathbb{R}^m$ with $\Psi_{\tilde{\mathcal{I}}}(y^*)\neq\emptyset$ one has

$$\mathrm{d}(\lambda,\Psi_{\tilde{\mathcal{I}}}(y^*))\leq\beta(\|\nabla q(\bar{y})^T\lambda-y^*\|+\sum_{i\in\tilde{\mathcal{I}}}\min\{\lambda_i,0\}+\sum_{i\notin\tilde{\mathcal{I}}}|\lambda_i|).$$

If $y_k\neq\bar{y}$ then, as a consequence of the assumption that M is locally uniformly metrically regular except for $\bar{y}$, there is some multiplier $\lambda^k\in\widehat{N}_{\mathbb{R}^l_-}(q(y_k))$ with $y_k^*=\nabla q(y_k)^T\lambda^k$ and using Proposition 1, we have $\|\lambda^k\|\leq\kappa\|y_k^*\|$. On the other hand, if $y_k=\bar{y}$, since M is assumed to be metrically subregular at $(\bar{y},0)$, there is also some multiplier $\lambda^k\in\widehat{N}_{\mathbb{R}^l_-}(q(y_k))$ with $y_k^*=\nabla q(y_k)^T\lambda^k$ and by using Hoffman's Lemma, we can choose λ^k such that $\|\lambda^k\|=\mathrm{d}(0,\Psi_{\tilde{\mathcal{I}}}(y_k^*))\leq\beta\|y_k^*\|$. Hence we can assume that the sequence (λ^k) is uniformly bounded by some constant c_1. Since

$$\nabla q(\bar{y})^T\lambda^k-\bar{y}^*=t_kv_k^*+(\nabla q(\bar{y})-\nabla q(y_k))^T\lambda^k$$

and $\|\nabla q(\bar{y})-\nabla q(y_k)\|\leq c_2\|y_k-\bar{y}\|=c_2t_k\|v_k\|$ for some constant $c_2\geq 0$, we can find for each k some $\tilde{\lambda}^k\in\Psi_{\tilde{\mathcal{I}}}(\bar{y}^*)\subset\bar{\Lambda}$ with $\|\tilde{\lambda}^k-\lambda^k\|\leq\beta t_k(\|v_k^*\|+c_1c_2\|v_k\|)$. Taking $\mu^k:=(\lambda_k-\tilde{\lambda}_k)/t_k$ we have that (μ^k) is uniformly bounded. By passing to subsequences if necessary we can assume that both sequences $(\tilde{\lambda}^k)$ and (μ^k) are convergent to some $\tilde{\lambda}\in\Psi_{\tilde{\mathcal{I}}}(\bar{y}^*)\subset\bar{\Lambda}$ and some μ. Since $\lambda_i^k=\tilde{\lambda}_i^k=0$, $i\notin\tilde{\mathcal{I}}$, we have ${\mu^k}^T\nabla q(\bar{y})v=0\ \forall k$ implying $\mu\in(\nabla q(\bar{y})v)^\perp$.

Taking into account ${\tilde{\lambda}^k}{}^Tq(y_k)=0\ \forall k$, we obtain

$$0=\lim_{k\to\infty}\frac{{\tilde{\lambda}^k}{}^Tq(y_k)}{t_k}=\lim_{k\to\infty}{\tilde{\lambda}^k}{}^T\nabla q(\bar{y})v_k=\bar{y}^{*T}v$$

which, together with (21), shows $v \in \bar{K}$.

Further we have for all $\lambda \in \bar{\Lambda}$

$$\begin{aligned} 0 &\leq (\tilde{\lambda}^k - \lambda)^T q(y_k) = (\tilde{\lambda}^k - \lambda)^T (q(\bar{y}) + t_k \nabla q(\bar{y}) v_k + \frac{1}{2} t_k^2 v_k^T \nabla^2 q(\bar{y}) v^k + o(t_k^2)) \\ &= (\tilde{\lambda}^k - \lambda)^T (\frac{1}{2} t_k^2 v_k^T \nabla^2 q(\bar{y}) v^k + o(t_k^2)). \end{aligned}$$

Dividing by t_k^2 and passing to the limit we obtain $(\tilde{\lambda} - \lambda)^T v^T \nabla^2 q(\bar{y}) v \geq 0 \ \forall \lambda \in \bar{\Lambda}$ and hence $\tilde{\lambda} \in \bar{\Lambda}(v)$ follows.

Since

$$y_k^* = \nabla q(\bar{y})^T \tilde{\lambda}^k + t_k v_k^* = \nabla q(y_k)^T \lambda^k,$$

we obtain

$$\begin{aligned} v^* &= \lim_{k \to \infty} v_k^* = \lim_{k \to \infty} \frac{\nabla q(y_k)^T \lambda^k - \nabla q(\bar{y})^T \tilde{\lambda}^k}{t_k} \\ &= \lim_{k \to \infty} \frac{(\nabla q(y_k) - \nabla q(\bar{y}))^T \tilde{\lambda}^k + \nabla q(y_k)^T (\lambda^k - \tilde{\lambda}^k)}{t_k} \\ &= \nabla^2 (\tilde{\lambda}^T q)(\bar{y}) v + \nabla q(\bar{y})^T \mu. \end{aligned}$$

If $\mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\tilde{\lambda})$, by using Lemma 1, the assertion is proved. Otherwise the set $J := \{i \in \tilde{\mathcal{I}} \,|\, \tilde{\lambda}_i = 0, \mu_i < 0\}$ is not empty. Let us choose some index $\bar{k}$ such that $(\lambda_i^{\bar{k}} - \tilde{\lambda}_i^{\bar{k}})/t_{\bar{k}} \leq \mu_i/2$ $\forall i \in J$ and set $\tilde{\mu} := \mu + 2(\tilde{\lambda}^{\bar{k}} - \tilde{\lambda})/t_{\bar{k}}$. Then for all i with $\tilde{\lambda}_i = 0$ we have $\tilde{\mu}_i \geq \mu_i$ and for all $i \in J$ we have

$$\tilde{\mu}_i = \mu_i + 2(\tilde{\lambda}_i^{\bar{k}} - \tilde{\lambda}_i)/t_{\bar{k}} \geq \mu_i + 2(\tilde{\lambda}_i^{\bar{k}} - \lambda_i^{\bar{k}})/t_{\bar{k}} \geq 0$$

and therefore $\tilde{\mu} \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\tilde{\lambda})$. Observing that $\nabla q(\bar{y})^T \tilde{\mu} = \nabla q(\bar{y})^T \mu$ because of $\tilde{\lambda}, \tilde{\lambda}^{\bar{k}} \in \bar{\Lambda}$ and taking into account Lemma 1 completes the proof. □

Under the assumptions ensuring equality in (16) one has thus the formula

$$D\widehat{N}_\Gamma(\bar{y}, \bar{y}^*)(v) = \{\nabla^2(\lambda^T q)(\bar{y}) v \,|\, \lambda \in \bar{\Lambda}(v)\} + \widehat{N}_{\bar{K}}(v), \ v \in \mathbb{R}^m.$$

In this way we have recovered formula (3) under substantially weaker assumptions.

Let us turn our attention to the regular coderivative of $\widehat{N}_\Gamma$.

Proposition 4. *Assume that $\bar{\Lambda} \neq \emptyset$. Then*

$$\mathcal{T}(\bar{y}, \bar{y}^*)^\circ = \{(w^*, w) \,|\, w \in \bar{K}, {w^*}^T v + w^T \nabla^2 (\lambda^T q)(\bar{y}) v \leq 0, \ v \in \bar{K}, \lambda \in \bar{\Lambda}(v)\}.$$

Proof. By definition of the polar cone, we have $(w^*, w) \in \mathcal{T}(\bar{y}, \bar{y}^*)^\circ$ if and only if ${w^*}^T v + w^T v^* \leq 0 \ \forall (v, v^*) \in \mathcal{T}(\bar{y}, \bar{y}^*)$, i.e.

$${w^*}^T v + w^T (\nabla^2 (\lambda^T q)(\bar{y}) v + \xi^*) \leq 0, \ v \in \bar{K}, \ \lambda \in \bar{\Lambda}(v), \ \xi^* \in \widehat{N}_{\bar{K}}(v)$$

Taking $v = 0$, $\lambda \in \bar\Lambda(0) = \bar\Lambda \neq \emptyset$, we obtain $w^T v^* \leq 0 \ \forall v^* \in \widehat{N}_{\bar K}(0) = \bar K^\circ$ showing $w \in \bar K$. Since $\widehat{N}_{\bar K}(v) = \{\xi^* \in \bar K^\circ \,|\, {\xi^*}^T v = 0\} \subset \bar K^\circ$ we have $w^T \xi^* \leq 0 \ \forall \xi^* \in \widehat{N}_{\bar K}(v)$ and, because of $0 \in \widehat{N}_{\bar K}(v)$, we see that $(w^*, w) \in \mathcal{T}(\bar y, \bar y^*)^\circ$ if and only if $w \in \bar K$ and

$$ {w^*}^T v + w^T \nabla^2(\lambda^T q)(\bar y) v \leq 0,\ v \in \bar K, \lambda \in \bar\Lambda(v). $$

□

By using Theorem 1 we obtain that $\widehat{N}_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar y, \bar y^*) \subset \mathcal{T}(\bar y, \bar y^*)^\circ$ if M is metrically subregular at $(\bar y, 0)$ and this inclusion holds with equality if in addition M is locally uniformly metrically regular except for $\bar y$. However, the representation of $\mathcal{T}(\bar y, \bar y^*)^\circ$ by Proposition 4 is not very useful in practise because of the simultaneous appearance of v and $\lambda \in \bar\Lambda(v)$.

We now define for each $v \in \bar{\mathcal{N}}$ the sets

$$ \bar{\mathcal{W}}(v) := \{w \in \bar K \,|\, w^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar y) v = 0, \forall \lambda^1, \lambda^2 \in \bar\Lambda(v)\}, $$

$$ \tilde\Lambda^{\mathcal{E}}(v) := \begin{cases} \bar\Lambda^{\mathcal{E}}(v) & \text{if } v \neq 0, \\ \mathrm{conv}\,(\bigcup\limits_{0 \neq u \in \bar K} \Lambda^{\mathcal{E}}(u)) & \text{if } v = 0,\ \bar K \neq \{0\} \end{cases} $$

and for each $w \in \bar K$

$$ \bar L(v; w) := \begin{cases} \{-\nabla^2(\lambda^T q)(\bar y) w \,|\, \lambda \in \tilde\Lambda^{\mathcal{E}}(v)\} + \bar K^\circ & \text{if } \bar K \neq \{0\} \\ \mathbb{R}^m & \text{if } \bar K = \{0\}. \end{cases} $$

Note that $\tilde\Lambda^{\mathcal{E}}(0)$ is a convex compact polyhedron, since there are only finitely many subsets of the finite set $\mathcal{E}$.

Proposition 5. *Assume that $\bar\Lambda(v) \neq \emptyset \ \forall v \in \bar K$. Then*

$$ \mathcal{T}(\bar y, \bar y^*)^\circ \subset \{(w^*, w) \,|\, w \in \bigcap_{v \in \bar{\mathcal{N}}} \bar{\mathcal{W}}(v), w^* \in \bigcap_{v \in \bar{\mathcal{N}}} \bar L(v; w)\} \tag{23} $$

and equality holds, if either for any $0 \neq v_1, v_2 \in \bar K$ it holds $\bar\Lambda^{\mathcal{E}}(v_1) = \bar\Lambda^{\mathcal{E}}(v_2)$ or $\bar I^+ = \bar{\mathcal{I}}$.

Proof. If $\bar K = \{0\}$, by Proposition 4 we have $\mathcal{T}(\bar y, \bar y^*)^\circ = \mathbb{R}^m \times \{0\}$ and (23) holds with equality. Now assume that $\bar K \neq 0$, consider $(w^*, w) \in \mathcal{T}(\bar y, \bar y^*)^\circ$ and fix $v \in \bar{\mathcal{N}} \subset \bar K$. Then also $-v \in \bar{\mathcal{N}}$, $\bar\Lambda(v) = \bar\Lambda(-v)$ and by Proposition 4 we obtain ${w^*}^T(\pm v) + w^T \nabla^2(\lambda^T q)(\bar y)(\pm v) \leq 0$, $\lambda \in \bar\Lambda(v)$ and therefore ${w^*}^T v + w^T \nabla^2(\lambda^T q)(\bar y) v = 0$, $\lambda \in \bar\Lambda(v)$, implying $w \in \bar{\mathcal{W}}(v)$. By Lemma 3 together with the assumption of the Proposition there is some compact convex neighborhood U of v such that $\emptyset \neq \bar\Lambda(u) \subset \bar\Lambda(v) \ \forall u \in U \cap \bar K$ and therefore also $\emptyset \neq \bar\Lambda^{\mathcal{E}}(u) \subset \bar\Lambda^{\mathcal{E}}(v)$. Clearly, if $v = 0$, then we have $\emptyset \neq \bar\Lambda^{\mathcal{E}}(u) \subset \tilde\Lambda^{\mathcal{E}}(v)$ $\forall u \in U \cap K(y, y^*)$ by the definition. Again by Proposition 4, for every $u \in U \cap K(y, y^*)$ there is some $\lambda \in \bar\Lambda^{\mathcal{E}}(u) \subset \tilde\Lambda^{\mathcal{E}}(v)$ such that ${w^*}^T u + w^T \nabla^2(\lambda^T q)(\bar y) u \leq 0$ and therefore

$$ 0 \geq \max_{u \in U \cap \bar K} \min_{\lambda \in \tilde\Lambda^{\mathcal{E}}(v)} {w^*}^T u + w^T \nabla^2(\lambda^T q)(\bar y) u = \min_{\lambda \in \tilde\Lambda^{\mathcal{E}}(v)} \max_{u \in U \cap \bar K} {w^*}^T u + w^T \nabla^2(\lambda^T q)(\bar y) u. $$

Hence there is some $\bar\lambda \in \tilde\Lambda^{\mathcal{E}}(v)$ such that $\max_{u\in U\cap\bar K} {w^*}^T u + w^T\nabla^2(\bar\lambda^T q)(\bar y)u \le 0$. Together with ${w^*}^T v + w^T\nabla^2(\bar\lambda^T q)(\bar y)v = 0$ we conclude $(w^* + \nabla^2(\bar\lambda^T q)(\bar y)w)^T(u-v) \le 0\ \forall u \in U\cap\bar K$ and thus $w^* + \nabla^2(\bar\lambda^T q)(\bar y)w \in \widehat N_{\bar K}(v)$. Taking into account that for $v \in \bar{\mathcal{N}}$ we have $\widehat N_{\bar K}(v) = \bar K^\circ$, we obtain $w^* \in \bar L(v;w)$. Since $v \in \bar{\mathcal{N}}$ was arbitrarily fixed, inclusion (23) follows.

To prove equality we may also assume that $\bar K \ne \{0\}$ and consider first the case when $\hat\Lambda := \bar\Lambda^{\mathcal{E}}(v_1) = \bar\Lambda^{\mathcal{E}}(v_2)$ for all $0 \ne v_1, v_2 \in \bar K$. Then we have $\tilde\Lambda^{\mathcal{E}}(v) = \hat\Lambda\ \forall v \in \bar{\mathcal{N}}$ and the set on the right hand side of the inclusion (23) amounts to

$$\Xi := \bigcup_{\lambda\in\hat\Lambda}\{(w^*,w) \,|\, w \in \bigcap_{v\in\bar{\mathcal{N}}} \bar{\mathcal{W}}(v), w^* \in -\nabla^2(\lambda^T q)(\bar y)w + \bar K^\circ\}.$$

Now assume that there is some element $(w^*,w) \in \Xi\setminus \mathcal{T}(\bar y,\bar y^*)^\circ$, i.e., there is some element $(v,v^*) \in \mathcal{T}(\bar y,\bar y^*)$ with ${w^*}^T v + w^T v^* > 0$. Then there are multipliers $\hat\lambda \in \hat\Lambda$, $\lambda \in \bar\Lambda(v)$ with $w^* \in -\nabla^2(\hat\lambda^T q)(\bar y)w + \bar K^\circ$ and $v^* \in \nabla^2(\lambda^T q)(\bar y)v + \widehat N_{\bar K}(v)$ and hence

$$0 < {w^*}^T v + w^T v^* \le w^T\nabla^2((\lambda-\hat\lambda)^T q)(\bar y)v.$$

Thus $v \ne 0$, $\bar\Lambda^{\mathcal{E}}(v) = \hat\Lambda$ and therefore $v^T\nabla^2((\lambda-\hat\lambda)^T q)(\bar y)v = 0$. For $\alpha > 0$ with $0 < 2w^T\nabla^2((\lambda-\hat\lambda)^T q)(\bar y)v + \alpha w^T\nabla^2((\lambda-\hat\lambda)^T q)(\bar y)w$ it follows that $0 < (v+\alpha w)^T\nabla^2((\lambda-\hat\lambda)^T q)(\bar y)(v+\alpha w)$ and hence $v+\alpha w \ne 0$ and $\hat\lambda \notin \bar\Lambda(v+\alpha w)$ contradicting $\hat\lambda \in \hat\Lambda = \bar\Lambda^{\mathcal{E}}(v+\alpha w) \subset \bar\Lambda(v+\alpha w)$. Hence $\Xi \subset \mathcal{T}(\bar y,\bar y^*)^\circ$ and equality in (23) is established.

To prove equality in the second case, note that $\bar I^+ = \bar{\mathcal{I}}$ implies $\bar K = \bar{\mathcal{N}}$ by Lemma 2. Consider now (w^*,w) belonging to the set on the right hand side of (23) and an arbitrary element $(v,v^*) \in \mathcal{T}(\bar y,\bar y^*)$, i.e., $v \in \bar{\mathcal{N}}$ and $v^* \in \nabla^2(\lambda^T q(\bar y))w + \widehat N_{\bar K}(v)$ for some $\lambda \in \bar\Lambda(v)$. Because of $v \in \bar{\mathcal{N}}$ we have $\widehat N_{\bar K}(v) = \bar K^\circ$ and, by using the representation $w^* \in -\nabla^2(\bar\lambda^T q)(\bar y)w + \bar K^\circ$ with $\bar\lambda \in \tilde\Lambda^{\mathcal{E}}(\bar y)$, we obtain

$${w^*}^T v + w^T v^* \le w^T\nabla^2((\lambda-\bar\lambda)^T q)(\bar y)v = 0$$

due to $w \in \bar{\mathcal{W}}(v)$. Hence $(w^*,w) \in \mathcal{T}(\bar y,\bar y^*)^\circ$ and equality in (23) follows. □

Theorem 2. *Assume that M is metrically subregular at $(\bar y, 0)$. Then*

$$\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*) \subset \{(w^*,w) \,|\, w \in \bigcap_{v\in\bar{\mathcal{N}}} \bar{\mathcal{W}}(v), w^* \in \bigcap_{v\in\bar{\mathcal{N}}} \bar L(v;w)\}. \tag{24}$$

Equality holds if, in addition, M is locally uniformly metrically regular except for $\bar y$ and either for any $0 \ne v_1, v_2 \in \bar K$ it holds $\bar\Lambda^{\mathcal{E}}(v_1) = \bar\Lambda^{\mathcal{E}}(v_2)$ or $\bar I^+ = \bar{\mathcal{I}}$.

Proof. If M is metrically subregular, then by [7, Theorem 6.1 (2.b)] one has that for every $v \in T^{\rm lin}_\Gamma(\bar y)$ we have $v^T\nabla^2(\lambda^T q)(\bar y)v \le 0$ for all λ belonging to the recession cone $\mathcal{R}$ of $\bar\Lambda$. Hence, $\bar\Lambda(v) \ne \emptyset\ \forall v \in T^{\rm lin}_\Gamma(\bar y) \supset \bar K$ and the assertion of the theorem follows from Theorem 1 and Proposition 5. □

It follows from the definition of the regular coderivative that under metric subregularity of M at $(\bar{y}, 0)$ one has

$$\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{y}^*)(w) \begin{cases} \subset \bigcap_{v \in \bar{\mathcal{N}}} \bar{L}(v; -w) & \text{if } w \in \bigcap_{v \in \bar{\mathcal{N}}} -\bar{\mathcal{W}}(v), \\ = \emptyset & \text{else.} \end{cases} \tag{25}$$

Equality holds in this formula provided M is locally uniformly metrically regular except for $\bar{y}$ and either for any $0 \neq v_1, v_2 \in \bar{K}$ it holds $\bar{\Lambda}^{\mathcal{E}}(v_1) = \bar{\Lambda}^{\mathcal{E}}(v_2)$ or $\bar{I}^+ = \bar{\mathcal{I}}$.

Example 1. *The normal cone mapping of the set*

$$\Gamma := \left\{ y \in \mathbb{R}^3 \mid \begin{array}{rl} y_1 + \frac{1}{2}y_2^2 & \leq 0 \\ y_1 + y_2^2 - y_2 y_3 & \leq 0 \end{array} \right\}$$

is given by

$$\widehat{N}_\Gamma(y) = \begin{cases} \{(0,0,0)\} & \textit{if } y_1 < -\max\{\frac{1}{2}y_2^2, y_2^2 - y_2 y_3\}, \\ \{(\lambda_1, \lambda_1 y_2, 0) \mid \lambda_1 \geq 0\} & \textit{if } y_1 = -\frac{1}{2}y_2^2 < -y_2^2 + y_2 y_3, \\ \{(\lambda_2, \lambda_2(2y_2 - y_3), -\lambda_2 y_2) \mid \lambda_2 \geq 0\} & \textit{if } y_1 = -y_2^2 + y_2 y_3 < -\frac{1}{2}y_2^2, \\ \{(\lambda_1 + \lambda_2, (2\lambda_1 + 3\lambda_2)y_3, -2\lambda_2 y_3) \mid \lambda_1, \lambda_2 \geq 0\} & \textit{if } y_1 = -y_2^2 + y_2 y_3 = -\frac{1}{2}y_2^2,\ y_2 \neq 0, \\ \{(\lambda_1 + \lambda_2, -\lambda_2 y_3, 0) \mid \lambda_1, \lambda_2 \geq 0\} & \textit{if } y_1 = y_2 = 0, \\ \emptyset & \textit{else.} \end{cases}$$

Note that MFCQ is fulfilled at every point $y \in \Gamma$. The tangent cone and the Fréchet normal cone to $\operatorname{gph} \widehat{N}_\Gamma$ *at $\bar{y} = (0,0,0)$, $\bar{y}^* = (1,0,0)$ are given by*

$$\begin{aligned} T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) \;=\; & \{((0, v_2, v_3), (v_1^*, v_2, 0)) \mid (0 \leq v_2 \leq 2v_3) \vee (0 \geq v_2 \geq 2v_3)\} \\ & \cup \{((0, v_2, v_3), (v_1^*, 2v_2 - v_3, -v_2)) \mid (v_2 \leq 0 \wedge 2v_3 \geq v_2) \vee (v_2 \geq 0 \wedge 2v_3 \leq v_2)\} \\ & \cup \{((0, 2v_3, v_3), (v_1^*, (2 + \lambda_2)v_3, -2\lambda_2 v_3)) \mid 0 \leq \lambda_2 \leq 1\} \\ & \cup \{((0, 0, v_3), (v_1^*, -\lambda_2 v_3, 0)) \mid 0 \leq \lambda_2 \leq 1\}\} \end{aligned}$$

and

$$\begin{aligned} \widehat{N}_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) \;=\; & \{((w_1^*, -w_2, 0), (0, w_2, w_3))\} \\ & \cap \{((w_1^*, w_3 - 2w_2, w_2), (0, w_2, w_3))\} \\ & \cap \{((w_1^*, w_2^*, w_3^*), (0, 2w_3, w_3)) \mid 2w_2^* + w_3^* + 4w_3 = 0\} \\ & \cap \{((w_1^*, w_2^*, 0), (0, 0, w_3))\} \\ = \;& \{((w_1^*, 0, 0), (0, 0, 0))\}. \end{aligned}$$

Further we have $\bar{K} = \bar{\mathcal{N}} = \{0\} \times \mathbb{R} \times \mathbb{R}$, $\bar{\Lambda} = \{(\lambda_1, \lambda_2) \mid \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1\}$ and for $v \in \bar{K}$ we obtain

$$\bar{\Lambda}(v) = \begin{cases} \{(1,0)\} & \textit{if } v_2^2 > 2v_2 v_3, \\ \bar{\Lambda} & \textit{if } v_2 = 0 \textit{ or } v_2 = 2v_3, \\ \{(0,1)\} & \textit{if } v_2^2 < 2v_2 v_3. \end{cases}$$

For $\tilde v := (0,2,1) \in \bar{\mathcal N}$, $\hat v := (0,0,1) \in \bar{\mathcal N}$ *we have* $\bar\Lambda(\tilde v) = \bar\Lambda(\hat v) = \bar\Lambda$ *and for every* $\lambda^1,\lambda^2 \in \bar\Lambda$ *we obtain*

$$\begin{aligned} w^T\nabla^2((\lambda^1-\lambda^2)^Tq)(\bar y)\tilde v &= w_2\tilde v_2(\lambda^1_1-\lambda^2_1+2(\lambda^1_2-\lambda^2_2)) - (w_2\tilde v_3 + w_3\tilde v_2)(\lambda^1_2-\lambda^2_2)\\ &= (\lambda^1_2-\lambda^2_2)(w_2\tilde v_2 - w_2\tilde v_3 - w_3\tilde v_2) = (\lambda^1_2-\lambda^2_2)(w_2-2w_3)\end{aligned}$$

and, similarly,

$$w^T\nabla^2((\lambda^1-\lambda^2)^Tq)(\bar y)\hat v = -(\lambda^1_2-\lambda^2_2)w_2.$$

Thus $\bar{\mathcal W}(\tilde v) = \{(0,w_2,w_3)\,|\,w_2 = 2w_3\}$, $\bar{\mathcal W}(\hat v) = \{(0,0,w_3)\}$ *and* $\bigcap_{v\in\bar{\mathcal N}}\bar{\mathcal W}(v) = \{(0,0,0)\}$. *Since for every* $v \in \bar{\mathcal N}$ *we have*

$$\bar L(v;0) = \bar K^\circ = \mathbb R\times\{0\}\times\{0\},$$

we obtain

$$\{(w^*,w)\,|\,w\in\bigcap_{v\in\bar{\mathcal N}}\bar{\mathcal W}(v), w^*\in\bigcap_{v\in\bar{\mathcal N}}\bar L(v;w)\} = \bar K^\circ\times\{(0,0,0)\} = \widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*)$$

or $\widehat D^*\widehat N_\Gamma(\bar y,\bar y^*)(0) = \bar K^\circ$, $\widehat D^*\widehat N_\Gamma(\bar y,\bar y^*)(w) = \emptyset$, $w\neq 0$ *and equality in* (24) *holds.* △

Example 2. *The normal cone mapping of the set*

$$\Gamma := \left\{ y\in\mathbb R^3 \,\middle|\, \begin{array}{rl} y_1+\frac12 y_2^2 & \le 0\\ y_1+y_2^2-y_2y_3 & \le 0\\ y_2 & \le 0\end{array}\right\}$$

is given by

$$\widehat N_\Gamma(y) = \begin{cases} \{(0,\lambda_3,0)\,|\,\lambda_3\ge 0\} & \text{if } y_1<0,\ y_2=0,\\ \{(\lambda_1+\lambda_2,-\lambda_2y_3+\lambda_3,0)\,|\,\lambda_1,\lambda_2,\lambda_3\ge 0\} & \text{if } y_1=y_2=0,\\ \{(\lambda_1,\lambda_1y_2,0)\,|\,\lambda_1\ge 0\} & \text{if } 2y_3<y_2<0,\ y_1=-\frac12y_2^2,\\ \{(\lambda_2,\lambda_2(2y_2-y_3),-\lambda_2y_2)\,|\,\lambda_2\ge 0\} & \text{if } 2y_3>y_2,\ y_2<0,\ y_1=-y_2^2+y_2y_3,\\ \{(\lambda_1+\lambda_2,(2\lambda_1+3\lambda_2)y_3,-2\lambda_2y_3)\,|\,\lambda_1,\lambda_2\ge 0\} & \text{if } 2y_3=y_2<0,\ y_1=-\frac12y_2^2,\\ \{(0,0,0)\} & \text{if } y_2<0,\ y_1<-\max\{\frac12y_2^2,y_2^2-y_2y_3\},\\ \emptyset & \text{else.}\end{cases}$$

The tangent cone and the Fréchet normal cone to $\mathrm{gph}\,\widehat N_\Gamma$ *at* $\bar y = (0,0,0)$, $\bar y^* = (1,0,0)$ *are given by*

$$\begin{aligned} T_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*) &= \{((0,0,v_3),(v_1^*,v_2^*,0))\,|\,v_2^*\ge\min\{-v_3,0\}\}\\ &\cup \{((0,v_2,v_3),(v_1^*,v_2,0))\,|\,2v_3\le v_2\le 0\}\\ &\cup \{((0,v_2,v_3),(v_1^*,2v_2-v_3,-v_2))\,|\,2v_3\ge v_2, v_2\le 0\}\\ &\cup \{((0,2v_3,v_3),(v_1^*,(2+\lambda_2)v_3,-2\lambda_2v_3))\,|\,v_3\le 0, 0\le\lambda_2\le 1\}\end{aligned}$$

and

$$\begin{aligned}
\widehat{N}_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*) &= \{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,w_2\le w_3^*\le 0\}\\
&\quad\cap\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,w_2^*+\frac{1}{2}w_3^*+w_2\ge 0, w_3^*\ge 0\}\\
&\quad\cap\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,w_2-w_3^*\ge 0, w_2^*+\frac{3}{2}w_2+\frac{1}{2}w_3^*-w_3\ge 0\}\\
&\quad\cap\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,2w_2^*+w_3^*+2w_2+\min\{w_2-2w_3,0\}\ge 0\}\\
&= \{((w_1^*,w_2^*,0),(0,0,w_3))\,|\,w_2^*\ge\max\{w_3,0\}\}.
\end{aligned}$$

Then $\bar{K}=\{0\}\times\mathbb{R}_-\times\mathbb{R}$, $\bar{\mathcal{N}}=\{0\}\times\{0\}\times\mathbb{R}$, $\bar{\Lambda}=\{(\lambda_1,\lambda_2,0)\,|\,\lambda_1\ge 0,\lambda_2\ge 0,\lambda_1+\lambda_2=1\}$ *and for* $v\in\bar{K}$ *we have*

$$\bar{\Lambda}(v)=\begin{cases}\{(1,0,0)\} & \text{if } 0>v_2>2v_3,\\ \bar{\Lambda} & \text{if } v_2=0 \text{ or } 0>v_2=2v_3,\\ \{(0,1,0)\} & \text{if } v_2<2v_3 \text{ and } v_2<0.\end{cases}$$

For every $0\ne v=(0,0,v_3)\in\bar{\mathcal{N}}$ *we obtain*

$$\begin{aligned}
\bar{\mathcal{W}}(v) &= \{w\in\bar{K}\,|\,w^T\nabla^2((\lambda^1-\lambda^2)^Tq)(\bar{y})v=w_2(\lambda_2^2-\lambda_2^1)v_3=0\forall\lambda^1,\lambda^2\in\bar{\Lambda}(v)=\bar{\Lambda}\}\\
&= \{0\}\times\{0\}\times\mathbb{R},
\end{aligned}$$

$$\tilde{\Lambda}^{\mathcal{E}}(v)=\bar{\Lambda}^{\mathcal{E}}(v)=\bar{\Lambda}$$

and for $w\in\bar{\mathcal{W}}(v)$

$$\bar{L}(v;w)=\{(0,\lambda_2w_3,0)\,|\,0\le\lambda_2\le 1\}+\bar{K}^\circ=\mathbb{R}\times[\min\{w_3,0\},\infty)\times\{0\}.$$

Hence

$$\begin{aligned}
\{(w^*,w)\,|\,w\in\bigcap_{v\in\bar{\mathcal{N}}}\bar{\mathcal{W}}(v), w^*\in\bigcap_{v\in\bar{\mathcal{N}}}\bar{L}(v;w)\} &= \{((w_1^*,w_2^*,0),(0,0,w_3))\,|\,w_2^*\ge\min\{w_3,0\}\}\\
&\ne \{((w_1^*,w_2^*,0),(0,0,w_3))\,|\,w_2^*\ge\max\{w_3,0\}\}\\
&= \widehat{N}_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)
\end{aligned}$$

yielding

$$\widehat{D}^*\widehat{N}_\Gamma(\bar{y},\bar{y}^*)(w)\begin{cases}\subset\{(w_1^*,w_2^*,0)\,|\,w_2^*\ge\min\{-w_3,0\}\} & \text{if } w_1=w_2=0,\\ =\emptyset & \text{else}\end{cases}$$

and equality does not hold. $\triangle$

In case when the set $\bar{\Lambda}(v)$ remains constant for all $0\ne v\in\bar{K}$, the formulas for the contingent cone and the regular normal cone of gph $\widehat{N}_\Gamma$ can be simplified considerably.

Lemma 4. *Assume that $\bar{\Lambda}(v_1) = \bar{\Lambda}(v_2)\ \forall 0 \neq v_1, v_2 \in \bar{K}$. Then for all $v \in \bar{K}$ and all $\lambda^1, \lambda^2 \in \bar{\Lambda}(v)$ we have*

$$\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \in \operatorname{span}\{\nabla q_i(\bar{y})^T \mid i \in \bar{I}^+\}.$$

Proof. By contraposition. Assume on the contrary that there are $v \in \bar{K}$ and $\lambda^1, \lambda^2 \in \bar{\Lambda}(v)$ such that $\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \notin \operatorname{span}\{\nabla q_i(\bar{y})^T \mid i \in \bar{I}^+\}$. This is equivalent with the existence of some w satisfying $\nabla q_i(\bar{y})w = 0$, $i \in \bar{I}^+$ and $w^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{x})v \neq 0$. By Lemma 2 there is some $\tilde{v}$ with

$$\nabla q_i(\bar{y})\tilde{v} \begin{cases} = 0 & i \in \bar{I}^+, \\ < 0 & i \in \bar{I}^0 \end{cases}$$

and hence $w + \alpha\tilde{v} \in \bar{K}$ for all α sufficiently large. Moreover, we can choose α such that, in addition, $\tilde{w}^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \neq 0$, where $\tilde{w} = w + \alpha\tilde{v}$. We can assume without loss of generality that $\tilde{w}^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v < 0$, because otherwise we can interchange λ^1 and λ^2. Then we can choose $\beta > 0$ with $v + \beta\tilde{w} \neq 0$ and $\tilde{w}^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})(2v + \beta\tilde{w}) < 0$. It follows $v + \beta\tilde{w} \in \bar{K}$ and, together with $v^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v = 0$ because of $\lambda^1, \lambda^2 \in \bar{\Lambda}(v)$,

$$(v + \beta\tilde{w})^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})(v + \beta\tilde{w}) = \beta\tilde{w}^T\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})(2v + \beta\tilde{w}) < 0$$

showing $\lambda^1 \notin \bar{\Lambda}(v + \beta\tilde{w})$. Therefore $\bar{\Lambda}(v + \beta\tilde{w}) \neq \bar{\Lambda}(v)$, a contradiction, since we also have $v \neq 0$. □

Note that the condition $\bar{\Lambda}(v_1) = \bar{\Lambda}(v_2)\ \forall\ 0 \neq v_1, v_2 \in \bar{K}$ implies in particular the corresponding property for the sets $\bar{\Lambda}^{\mathcal{E}}$ used in Theorem 2.

Theorem 3. *Assume that M is metrically subregular at $(\bar{y}, 0)$ and locally uniformly metrically regular except for $\bar{y}$. Further assume that $\bar{\Lambda}(v_1) = \bar{\Lambda}(v_2)\ \forall 0 \neq v_1, v_2 \in \bar{K}$ and let $\bar{\lambda}$ be an arbitrary multiplier from $\bar{\Lambda}(v)$ for some $0 \neq v \in \bar{K}$, if $\bar{K} \neq \{0\}$ and $\bar{\lambda} \in \bar{\Lambda}$ otherwise. Then*

$$T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) = \{(v, v^*) \mid v^* \in \nabla^2(\bar{\lambda}^T q)(\bar{y})v + \widehat{N}_{\bar{K}}(v)\} \tag{26}$$

and

$$\widehat{N}_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) = \{(w^*, w) \mid w \in \bar{K}, w^* \in -\nabla^2(\bar{\lambda}^T q)(\bar{y})w + \bar{K}^\circ\}. \tag{27}$$

Consequently,

$$D\widehat{N}_\Gamma(\bar{y}, \bar{y}^*)(v) = \nabla^2(\bar{\lambda}^T q)(\bar{y})v + \widehat{N}_{\bar{K}}(v),\ v \in \mathbb{R}^m,$$

$$\widehat{D}^*\widehat{N}_\Gamma(\bar{y}, \bar{y}^*)(w) = \begin{cases} \nabla^2(\bar{\lambda}^T q)(\bar{y})w + \bar{K}^\circ & \text{if } -w \in \bar{K}, \\ \emptyset & \text{else.} \end{cases}$$

Proof. By Theorem 1 it is clear that the set on the right hand side of (26) is contained in $T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$. To show the reverse inclusion, fix any $(v, v^*) \in T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$. Then there is some $\lambda \in \bar{\Lambda}(v)$ with $v^* \in \nabla^2(\lambda^T q)(\bar{y})v + \widehat{N}_{\bar{K}}(v)$ and by Lemma 4 together with the identity span $\{\nabla q_i(\bar y)^T \mid i \in \bar I^+\} + \widehat N_{\bar K}(v) = \widehat N_{\bar K}(v)$ we obtain $v^* \in \nabla^2(\bar\lambda^T q)(\bar y)v +$span $\{\nabla q_i(\bar y)^T \mid i \in \bar I^+\} + \widehat N_{\bar K}(v) = \nabla^2(\bar\lambda^T q)(\bar y)v + \widehat N_{\bar K}(v)$, showing the desired inclusion.

To show (27), note that by our assumptions equality holds in (24). Further, by Lemma 4 and Lemma 2 we obtain $\bar{\mathcal W}(v) = \bar K\ \forall v \in \bar{\mathcal N}$ and, by using the same arguments as above, $L(v; w) = -\nabla^2(\bar\lambda^T q)(\bar y)w + \bar K^\circ\ \forall w \in \bar K$. Equality (27) follows now from Theorem 2. □

The behavior of the mapping $\bar\Lambda$ required in the above theorem is automatically fulfilled whenever CRCQ holds at $\bar y$, see [9]. The following example shows, however, that the requirements of the theorem can very well be satisfied even without CRCQ.

Example 3. *Let $\Gamma \subset \mathbb{R}^2$ be given by*

$$q(y) = \begin{pmatrix} -y_1^2 + y_2 \\ -y_1^2 - y_2 \\ y_1 \end{pmatrix}.$$

Put $\bar y = 0, \bar y^* = 0$ and let us compute $T_{\operatorname{gph} \widehat N_\Gamma}(0,0)$. It follows that

$$\bar K = T_\Gamma(0) = \{v | v_1 \le 0,\ v_2 = 0\},$$

$$\bar\Lambda = \{\lambda \in \mathbb{R}^3_+ | \lambda_1 = \lambda_2, \lambda_3 = 0\}$$

and

$$\bar\Lambda(v) = \begin{cases} \{\lambda \in \bar\Lambda | \lambda_1 + \lambda_2 = 0\} = \{0\} & \text{if } 0 \ne v \in \bar K, \\ \bar\Lambda & \text{if } v = 0. \end{cases}$$

It is easy to show that the second-order sufficient conditions for metric subregularity are fulfilled and thus M is metrically subregular and even locally uniformly metrically regular except for 0. It follows that we can compute $T_{\operatorname{gph} \widehat N_\Gamma}(0,0)$ according to Theorem 3 and obtain

$$T_{\operatorname{gph} \widehat N_\Gamma}(0,0) = \{(v, v^*) | v^* \in \widehat N_{\bar K}(v)\},$$

and, since $\bar K^\circ = \widehat N_\Gamma(0) = \mathbb{R}_+ \times \mathbb{R}$, one has

$$T_{\operatorname{gph} \widehat N_\Gamma}(0,0) = (\{v \mid v_1 \le 0,\ v_2 = 0\} \times \{v^* \mid v_1^* = 0\}) \cup (\{0,0\} \times \mathbb{R}_+ \times \mathbb{R}).$$

Consequently,

$$\widehat N_{\operatorname{gph} \widehat N_\Gamma}(0,0) = (T_{\operatorname{gph} \widehat N_\Gamma}(0,0))^\circ = \{(v_1^*, v_2^*) \mid v_1^* \ge 0\} \times \{(v_1, 0) \mid v_1 \le 0\} = (\bar K)^\circ \times \bar K$$

verifying (27) with $\bar\lambda = 0$. △

5 Regular coderivative of the solution map

In the preceding section we have computed (an upper estimate of) the regular coderivative of $\widehat{N}_\Gamma$. To compute the regular coderivative of S we need, in addition, a chain rule for regular normal cones without any convexity assumptions. Such a chain rule is provided in the next statement which is important for its own sake and can be used also in completely different situations.

Theorem 4. *Let*

$$\Omega := \{x \in \mathbb{R}^n \,|\, G(x) \in D\}$$

for a closed set $D \subset \mathbb{R}^m$ and a mapping $G : \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable at $\bar{x} \in \Omega$. If the multifunction $x \rightrightarrows G(x) - D$ is metrically subregular at $(\bar{x}, 0)$ and there exists a subspace $L \subset R^m$ such that

$$T_D(G(\bar{x})) + L \subset T_D(G(\bar{x})) \tag{28}$$

and

$$\nabla G(\bar{x})\mathbb{R}^n + L = \mathbb{R}^m, \tag{29}$$

then

$$\widehat{N}_\Omega(\bar{x}) = \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x}).$$

Proof. The inclusion $\widehat{N}_\Omega(\bar{x}) \supset \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x})$ follows immediately from [30, Theorem 6.14]. To show the reverse inclusion, let $x^* \in \widehat{N}_\Omega(\bar{x})$ and consider $h \in \mathcal{S} := \{h \in \mathbb{R}^n \,|\, \nabla G(\bar{x})h \in \operatorname{conv} T_D(G(\bar{x}))\}$. Then $\nabla G(\bar{x})h$ can be written as convex combination of elements of $T_D(G(\bar{x}))$:

$$\nabla G(\bar{x})h = \sum_{i=1}^N \alpha_i t_i,\ t_i \in T_D(G(\bar{x})),\ \alpha_i \geq 0,\ i = 1, \ldots, N;\ \sum_{i=1}^N \alpha_i = 1.$$

By the assumptions of the theorem each of the tangents t_i can be represented as $t_i = \nabla G(\bar{x})h_i + l_i$, where $h_i \in \mathbb{R}^n$ and $l_i \in L$. Since L is a subspace we also have $-l_i \in L$ showing $\nabla G(\bar{x})h_i = t_i - l_i \in T_D(G(\bar{x})) + L \subset T_D(G(\bar{x}))$ and, because $G(\cdot) - D$ is assumed to be metrically subregular at $(\bar{x}, 0)$, we conclude $h_i \in T_\Omega(\bar{x})$ and $\langle x^*, h_i \rangle \leq 0$. Further we have

$$\nabla G(\bar{x})(h - \sum_{i=1}^N \alpha_i h_i) = \sum_{i=1}^N \alpha_i (t_i - \nabla G(\bar{x})h_i) = \sum_{i=1}^N \alpha_i l_i \in L \subset T_D(G(\bar{x})),$$

and, again by metric subregularity of $G(\cdot) - D$, we obtain $h - \sum_{i=1}^N \alpha_i h_i \in T_\Omega(\bar{x})$. Thus $\langle x^*, h - \sum_{i=1}^N \alpha_i h_i \rangle \leq 0$ implying

$$\langle x^*, h \rangle \leq \langle x^*, \sum_{i=1}^N \alpha_i h_i \rangle = \sum_{i=1}^N \alpha_i \langle x^*, h_i \rangle \leq 0.$$

From this it follows that $x^* \in \mathcal{S}^\circ$ and, by [29, Corollary 16.3.2], we can conclude $\widehat{N}_\Omega(\bar{x}) \subset \mathcal{S}^\circ = \nabla G(\bar{x})^T(\operatorname{conv} T_D(G(\bar{x})))^\circ = \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x}))$, provided there exists some u with $\nabla G(\bar{x})u \in \operatorname{ri}\operatorname{conv} T_D(G(\bar{x}))$. To show the existence of such an element u, choose any $t \in \operatorname{ri}\operatorname{conv} T_D(G(\bar{x}))$ and select u and $l \in L$ such that $\nabla G(\bar{x})u + l = t$. Since we also have $\operatorname{conv} T_D(\bar{x}) + L \subset \operatorname{conv} T_D(\bar{x})$, we obtain from [29, Theorem 6.1] that $\nabla G(\bar{x})u = \frac{1}{2}(t - 2l) + \frac{1}{2}t \in \operatorname{ri}\operatorname{conv} T_D(G(\bar{x}))$. Hence $\widehat{N}_\Omega(\bar{x}) \subset \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x})$ holds and this completes the proof. $\square$

If D is convex, then the conditions (28), (29) amount to

$$\nabla G(\bar{x})\mathbb{R}^n + \operatorname{lin} T_D(G(\bar{x})) = \mathbb{R}^m, \tag{30}$$

where lin stands for the lineality space. This condition is known as degeneracy [2, Definition 4.70]. In the nonconvex case, however, the formulation (30) cannot be used because of (28).

In verifying the assumptions of the theorem, a difficult task is sometimes checking of metric subregularity of the multifunction $G(\cdot) - D$. One possibility of doing this would be considering the stronger property of metric regularity, c.f. [30, Example 9.44]: $G(\cdot) - D$ is metrically regular near $(\bar{x}, 0)$ if and only if

$$\nabla G(\bar{x})^T\lambda = 0,\ \lambda \in N_D(G(\bar{x})) \implies \lambda = 0. \tag{31}$$

However, this criterion involves the limiting normal cone $N_D(G(\bar{x}))$ which is sometimes very difficult to compute. A weaker criterion, hopefully easier to verify, is given by the following proposition:

Proposition 6. *Let $G : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable at $\bar{x}$, let $D \subset \mathbb{R}^m$ be closed and suppose $G(\bar{x}) \in D$. Further assume that there is some neighborhood V of $G(\bar{x})$ and some $\eta > 0$, such that for every $d \in D \cap V$ there is some cone $K \subset T_D(d)$ with*

$$\inf\{\|\nabla G(\bar{x})^T\xi\| \mid \xi \in K^\circ, \|\xi\| = 1\} > \eta. \tag{32}$$

Then $G(\cdot) - D$ is metrically regular near $(\bar{x}, 0)$.

Proof. By contraposition. Assuming on the contrary that $G(\cdot) - D$ is not metrically regular near $(\bar{x}, 0)$, due to (31) there is some $0 \neq \xi \in N_D(G(\bar{x}))$ with $\nabla G(\bar{x})^T\xi = 0$. By the definition of the limiting normal cone there are sequences $(d_k) \xrightarrow{D} G(\bar{x})$ and $(\xi_k) \to \xi$ with $\xi_k \in \widehat{N}_D(d_k)$. Owing to [30, Theorem 6.28] we have $\widehat{N}_D(d_k) = T_D(d_k)^\circ$. By the assumption of the proposition, for every k there is some cone $K_k \subset T_D(d_k)$ such that (32) holds. This implies $K_k^\circ \supset T_D(d_k)^\circ = \widehat{N}_D(d_k)$ and

$$\|\nabla G(\bar{x})^T \frac{\xi_k}{\|\xi_k\|}\| \geq \inf\{\|\nabla G(\bar{x})^T\xi\| \mid \xi \in K_k^\circ, \|\xi\| = 1\} > \eta.$$

Thus $\|\nabla G(\bar{x})^T\xi\| = \lim_k \|\nabla G(\bar{x})^T\xi_k\| \geq \lim_k \eta\|\xi_k\| = \eta\|\xi\| > 0$ contradicting $\nabla G(\bar{x})^T\xi = 0$. $\square$

Consider now the mapping $\tilde{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$\tilde{S}(x) = \begin{cases} \{y \in \mathbb{R}^m \,|\, 0 \in F(x,y) + \widehat{N}_\Gamma(y)\} & \text{if } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable and $C \subset \mathbb{R}^n$ is a closed set.

Associated with $\tilde{S}$ is the perturbation mapping $\Psi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n$ given by

$$\Psi(x,y) := G(x,y) - D,\; G(x,y) := \begin{pmatrix} (y, -F(x,y)) \\ x \end{pmatrix},\; D := \operatorname{gph} \widehat{N}_\Gamma \times C, \tag{33}$$

so that $\operatorname{gph} \tilde{S} = \{(x,y) \,|\, 0 \in \Psi(x,y)\}$.

Theorem 5. *Consider $(\bar{x},\bar{y}) \in \operatorname{gph} \tilde{S}$. Assume that M is metrically subregular at $(\bar{y},0)$ and locally uniformly metrically regular except for $\bar{y}$. Further assume that the set-valued mapping Ψ given by (33) is metrically subregular at $((\bar{x},\bar{y})),(0,0,0))$, and suppose that there exists a subspace $P \subset T_C(\bar{x})$ with $T_C(\bar{x}) + P \subset T_C(\bar{x})$ and*

$$\nabla_x F(\bar{x},\bar{y})P + \operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} = \mathbb{R}^m. \tag{34}$$

Then one has, with $\bar{y}^* := -F(\bar{x},\bar{y})$, that

$$\begin{aligned} \widehat{N}_{\operatorname{gph}\tilde{S}}(\bar{x},\bar{y}) &= \nabla G(\bar{x},\bar{y})^T \widehat{N}_D(G(\bar{x},\bar{y})) \qquad (35) \\ &= \left\{ \begin{pmatrix} -\nabla_x F(\bar{x},\bar{y})^T w + c^* \\ -\nabla_y F(\bar{x},\bar{y})^T w + w^* \end{pmatrix} \,|\, c^* \in \widehat{N}_C(\bar{x}),\; (w^*,w) \in \widehat{N}_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*) \right\}. \end{aligned}$$

Proof. Set $\Omega := \{(x,y) \,|\, G(x,y) \in D\} = \operatorname{gph} \tilde{S}$. We will invoke Theorem 4 to prove that $\widehat{N}_\Omega(\bar{x},\bar{y}) = \nabla G(\bar{x},\bar{y})^T \widehat{N}_D(G(\bar{x},\bar{y}))$. Using Lemmas 1, 2 we obtain $\tilde{L} := \operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} \subset \widehat{N}_{\bar{K}}(v)\ \forall v \in \bar{K}$. Hence $T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*) + \{0_m\} \times \tilde{L} \subset T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ by virtue of Theorem 1. Defining the subspace L by $L := (\{0_m\} \times \tilde{L}) \times P$ and taking into account $T_D(G(\bar{x},\bar{y})) = T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*) \times T_C(\bar{x})$, we obtain $T_D(G(\bar{x},\bar{y})) + L \subset T_D(G(\bar{x},\bar{y}))$. Next we shall prove that

$$\nabla G(\bar{x},\bar{y})(\mathbb{R}^n \times \mathbb{R}^m) + L = (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n \tag{36}$$

holds true. By the assumptions for any $((v,v^*),u) \in (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n$ we can choose $p \in P$ and $\tilde{l} \in \tilde{L}$ with $\nabla_x F(\bar{x},\bar{y})p + \tilde{l} = v^* + \nabla_x F(\bar{x},\bar{y})u + \nabla_y F(\bar{x},\bar{y})v$. Then $l = ((0_m,\tilde{l}),p) \in L$ and

$$\nabla G(\bar{x},\bar{y}) \begin{pmatrix} u - p \\ v \end{pmatrix} + l = \begin{pmatrix} (v, -\nabla_x F(\bar{x},\bar{y})(u-p) - \nabla_y F(\bar{x},\bar{y})v + \tilde{l}) \\ u - p + p \end{pmatrix} = \begin{pmatrix} (v,v^*) \\ u \end{pmatrix}.$$

This verifies (36) and we can apply Theorem 4 to obtain the result. □

Corollary 1. *In the setting of Theorem 5 for any $v^* \in \mathbb{R}^m$ one has*

$$\widehat{D}^* \tilde{S}(\bar{x},\bar{y})(v^*) = \{\nabla_x F(\bar{x},\bar{y})^T w + \widehat{N}_C(\bar{x}) \,|\, 0 \in v^* + \nabla_y F(\bar{x},\bar{y})^T w + \widehat{D}^* \widehat{N}_\Gamma(\bar{y},\bar{y}^*)(w)\}. \tag{37}$$

There are various possibilities for verifying the metric subregularity of Ψ at $((\bar{x},\bar{y}),(0,0,0))$. Sometimes one can use even the following simple sufficient condition for metric regularity stated in Proposition 7 below. We think that this criterion is far away from being necessary, but it is easy to verify.

Proposition 7. *Let Ψ be given by (33), and let $0 \in \Psi(\bar{x},\bar{y})$. Assume that M is metrically subregular at $(\bar{y},0)$ and locally uniformly metrically regular except for $\bar{y}$. Further assume that for every $\lambda \in \mathcal{E}$ one has*

$$\nabla F_x(\bar{x},\bar{y})\hat{T}_C(\bar{x}) + \operatorname{span}\{\nabla q_i(\bar{y})^T \mid i \in I^+(\lambda)\} = \mathbb{R}^m,$$

where $\bar{y}^* = -F(\bar{x},\bar{y})$. Then Ψ is metrically regular near $((\bar{x},\bar{y}),(0,0,0))$.

Proof. For every $((y,y^*),x) \in D$ near $((\bar{y},\bar{y}^*),\bar{x})$ we have that M is at least metrically subregular at $(y,0)$. Thus, by Theorem 1 the cone $(\{0_m\} \times K(y,y^*)^\circ) \times T_C(x)$ is a subset of $T_D((y,y^*),x)$. Assuming now on the contrary that Ψ is not metrically regular near $((\bar{x},\bar{y}),(0,0,0))$, by Proposition 6 there is a sequence $((y_k,y_k^*),x_k) \xrightarrow{D} ((\bar{y},\bar{y}^*),\bar{x})$ with

$$\lim_{k\to\infty} \left(\inf\{\|\nabla G(\bar{x})^T\xi\| \mid \|\xi\| = 1, \xi \in ((\{0_m\} \times K(y_k,y_k^*)^\circ) \times T_C(x_k))^\circ\}\right) = 0.$$

Hence there is a sequence $\xi_k = ((v_k^*,v_k),x_k^*) \in (\mathbb{R}^m \times K(y_k,y_k^*)) \times \widehat{N}_C(x_k)$ with $\|\xi_k\| = 1$ and $\lim_k \nabla G(\bar{x},\bar{y})^T\xi_k = 0$. By passing to a subsequence if necessary we can assume that ξ_k converges to some $\xi = ((v^*,v),x^*)$ with $\|\xi\| = 1$ and

$$\nabla G(\bar{x},\bar{y})^T\xi = \begin{pmatrix} -\nabla_x F(\bar{x},\bar{y})^T v + x^* \\ -\nabla_y F(\bar{x},\bar{y})^T v + v^* \end{pmatrix} = 0. \tag{38}$$

By the defininition of the limiting normal cone together with [30, Exercise 6.38] we obtain $x^* \in N_C(\bar{x}) \subset \hat{T}_C(\bar{x})^\circ$. Using similar arguments as in the proof of Theorem 1 we can find a bounded sequence $(\lambda^k) \in \Lambda(y_k,y_k^*)$ and, by passing to a subsequence if necessary, we can assume that it converges to some $\tilde{\lambda} \in \bar{\Lambda}$. $\bar{\Lambda}$ is the sum of the convex hull of its extreme points and its recession cone and therefore there is some $\lambda \in \mathcal{E}$ with $I^+(\lambda) \subset I^+(\tilde{\lambda})$. Then $I^+(\lambda) \subset I^+(\lambda^k)$ for all k sufficiently large and from $v_k \in K(y_k,y_k^*)$ we deduce $\nabla q_i(y_k)v_k = 0$, $i \in I^+(\lambda_k)$ and, consequently, $\nabla q_i(\bar{y})v = 0$, $i \in I^+(\lambda)$. It can be easily seen from (38) that $v \neq 0$ since otherwise $((v^*,v),x^*)$ would be 0. By the assumption of the proposition there exists an element $s \in \hat{T}_C(\bar{x})$ and numbers μ_i, $i \in I^+(\lambda)$ with $\nabla F_x(\bar{x},\bar{y})s + \sum_{i\in I^+(\lambda)} \mu_i \nabla q_i(\bar{y})^T = v$ and therefore

$$0 < \|v\|^2 = v^T(\nabla F_x(\bar{x},\bar{y})s + \sum_{i\in I^+(\lambda)} \mu_i \nabla q_i(\bar{y})^T) = \langle x^*, s\rangle + \sum_{i\in I^+(\lambda)} \mu_i \nabla q_i(\bar{y})v \leq 0,$$

a contradiction. Hence Ψ is metrically regular near $((\bar{x},\bar{y}),(0,0,0))$. $\square$

The assumption (34) can be considerably weakened, if we strengthen our assumptions imposed on $\bar{\Lambda}(\cdot)$. To simplify the formulas in the statement below it is reasonable to introduce the Lagrangian associated with our generalized equation, i.e.,

$$\mathcal{L}(x,y,\lambda) = F(x,y) + \nabla q(y)^T\lambda.$$

Theorem 6. *Let* $(\bar{x}, \bar{y}) \in \tilde{\operatorname{gph}} S$ *and assume that* M *is metrically subregular at* $(\bar{y}, 0)$ *and locally uniformly metrically regular except for* $\bar{y}$*. Further assume that the set-valued mapping* Ψ *given by* (33) *is metrically subregular at* $((\bar{x}, \bar{y})), (0, 0, 0))$*, that* $\bar{\Lambda}(v_1) = \bar{\Lambda}(v_2)$ $\forall\, 0 \neq v_1, v_2 \in \bar{K}$ *and suppose that there exists a subspace* $P \subset T_C(\bar{x})$ *with* $T_C(\bar{x}) + P \subset T_C(\bar{x})$ *and*

$$\nabla_x F(\bar{x}, \bar{y})P + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})\bar{\mathcal{N}} + \operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} = \mathbb{R}^m, \tag{39}$$

where $\bar{y}^* := -F(\bar{x}, \bar{y})$ *and* $\bar{\lambda} \in \bar{\Lambda}(v)$ *for some* $0 \neq v \in \bar{\mathcal{N}}$ *is chosen arbitrary, if* $\bar{\mathcal{N}} \neq \{0\}$*, and* $\bar{\lambda} = 0$ *otherwise. Then* (35) *holds true and simplifies to*

$$\widehat{N}_{\operatorname{gph}\tilde{S}}(\bar{x}, \bar{y}) \;=\; \left\{ \begin{pmatrix} -\nabla_x F(\bar{x}, \bar{y})^T w \\ -\nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda}) w \end{pmatrix} \;|\; w \in \bar{K} \right\} + \widehat{N}_C(\bar{x}) \times \bar{K}^\circ. \tag{40}$$

Moreover, for any $v^* \in \mathbb{R}^m$ *one has*

$$\widehat{D}^* \tilde{S}(\bar{x}, \bar{y})(v^*) = \{\nabla_x F(\bar{x}, \bar{y})^T w + \widehat{N}_C(\bar{x}) \,|\, 0 \in v^* + \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda}) w + \bar{K}^\circ,\ -w \in \bar{K}\}. \tag{41}$$

Proof. The proof follows the same lines as the proof of Theorem 5 with the exception that we choose now $L = \hat{L} \times P$, where $\hat{L} = \{(w, \nabla^2(\bar{\lambda}^T q)(\bar{y}) w) \,|\, w \in \bar{\mathcal{N}}\} + \{0_m\} \times \tilde{L}$. In order to prove $T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) + L \subset T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$, choose any $(v, v^*) \in T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)$, $w \in \bar{\mathcal{N}}$ and $\xi \in \tilde{L}$. By Theorem 1 we have $v \in \bar{K}$ and there is some $\lambda \in \bar{\Lambda}(v)$ such that $v^* \in \nabla^2(\lambda^T q)(\bar{y}) v + \widehat{N}_{\bar{K}}(v)$. By Lemma 4 we have $\nabla^2(\lambda^T q)(\bar{y}) v \in \nabla^2(\bar{\lambda}^T q)(\bar{y}) v + \tilde{L}$, and, since $\widehat{N}_{\bar{K}}(v) + \tilde{L} = \widehat{N}_{\bar{K}}(v) = \widehat{N}_{\bar{K}}(v + w)$, we obtain

$$(v, v^*) + (w, \nabla^2(\bar{\lambda}^T q)(\bar{y}) w) + (0_m, \xi) \in (v + w, \nabla^2(\bar{\lambda}^T) q(\bar{y})(v + w)) + \{0_m\} \times \widehat{N}_{\bar{K}}(v + w),$$

which, together with $v + w \in \bar{K}$, shows the desired inclusion. To show (36), fix any $((v, v^*), u) \in (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n$ and choose $p \in P$, $w \in \bar{\mathcal{N}}$ and $\tilde{l} \in \tilde{L}$ with

$$\nabla_x F(\bar{x}, \bar{y}) p + (\nabla_y F(\bar{x}, \bar{y}) + \nabla^2(\bar{\lambda}^T q)(\bar{y})) w + \tilde{l} = v^* + \nabla_x F(\bar{x}, \bar{y}) u + \nabla_y F(\bar{x}, \bar{y}) v.$$

Then $l = ((w, \nabla^2(\bar{\lambda}^T q)(\bar{y})) w + \tilde{l}), p) \in L$ and

$$\begin{aligned}
&\nabla G(\bar{x}, \bar{y}) \begin{pmatrix} u - p \\ v - w \end{pmatrix} + l \\
&= \begin{pmatrix} (v - w + w, -\nabla_x F(\bar{x}, \bar{y})(u - p) - \nabla_y F(\bar{x}, \bar{y})(v - w) + \nabla^2(\bar{\lambda}^T q)(\bar{y}) w + \tilde{l}) \\ u - p + p \end{pmatrix} \\
&= \begin{pmatrix} (v, v^*) \\ u \end{pmatrix}.
\end{aligned}$$

This verifies (36) and then again (35) follows from Theorem 4. Theorem 3 yields now the assertion. $\square$

Remark 1. *In [12] the authors have derived* (41) *under the assumptions that* $C = \mathbb{R}^n$, $\nabla_x F(\bar{x}, \bar{y})$ *is surjective and MFCQ and CRCQ are fulfilled at* $\bar{y}$*. We conclude that this statement follows from Theorem 6 in a straightforward way.*

Remark 2. *If* $\bar{\mathcal{I}} = \bar{I}^+$, *then* $\operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} = \bar{\mathcal{N}}^\perp$. *If, in addition,* $\nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})$ *is positive definite on* $\bar{\mathcal{N}}$, *i.e.,* $v^T \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda}) v > 0 \ \forall\ 0 \neq v \in \bar{\mathcal{N}}$, *then for every* $v^* \in \mathbb{R}^m$ *the generalized equation*

$$0 \in \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda}) v + v^* + \widehat{N}_{\bar{\mathcal{N}}}(v)$$

has a solution, see e.g. [23, Theorem 4.6]. We conclude that in this case assumption (39) *is fulfilled with* $P = \{0\}$.

6 Applications

6.1 Isolated calmness

A multifunction $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ is said to have the *isolated calmness* property at $(\bar{u}, \bar{v}) \in \operatorname{gph} \Psi$, provided there exist neighborhoods $\mathcal{U}$ of $\bar{u}$ and $\mathcal{V}$ of $\bar{v}$ and a constant $\kappa \geq 0$ such that

$$\Psi(u) \cap \mathcal{V} \subset \{\bar{v}\} + \kappa \|u - \bar{u}\| \mathbb{B} \text{ when } u \in \mathcal{U}.$$

In [17], it has been proved that Ψ possesses the isolated calmness property at $(\bar{u}, \bar{v})$ if and only if

$$D\Psi(\bar{u}, \bar{v})(0) = \{0\}, \tag{42}$$

cf. also [5, Theorem 4C.1].

This result has been applied in [9, Theorem 4.1] to the solution map S given by (4). On the basis of Theorem 1 the latter result can be substantially generalized.

Theorem 7. *Let* $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ *and assume that* M *is metrically subregular at* $(\bar{y}, 0)$ *and uniformly metrically regular except for* $\bar{y}$. *Then* S *has the isolated calmness property provided for all* $v \in \mathbb{R}^m$ *one has the implication*

$$\left.\begin{array}{l} 0 \in \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \lambda) v + \widehat{N}_{\bar{K}}(v) \\ \lambda \in \bar{\Lambda}(v) \end{array}\right\} \Rightarrow v = 0. \tag{43}$$

Moreover, if $\nabla_x F(\bar{x}, \bar{y})$ *is surjective, than (43) is not just sufficient but also necessary for* S *to have the isolated calmness property at* $(\bar{x}, \bar{y})$.

Proof. By virtue of [30, Theorem 6.31] for all $h \in \mathbb{R}^n$

$$DS(\bar{x}, \bar{y})(h) \subset \{v \in \mathbb{R}^m | 0 \in \nabla_x F(\bar{x}, \bar{y}) h + \nabla_y F(\bar{x}, \bar{y}) v + D\widehat{N}_\Gamma(\bar{y}, -F(\bar{x}, \bar{y})(v))\}. \tag{44}$$

The first assertion thus follows from the combination of (42), (44) and Theorem 1.

The second assertion follows from the fact that inclusion (44) becomes equality whenever $\nabla_x F(\bar{x}, \bar{y})$ is surjective, cf [30, Exercise 6.32]. $\square$

Note that in the setting of Theorem 3 condition (43) can be simplified and attains the form:

$$0 \in \nabla_y \mathcal{L}(\bar{x}, \bar{y}, \bar{\lambda})v + \widehat{N}_{\bar{K}}(v) \Rightarrow v = 0,$$

where $\bar{\lambda}$ is an arbitrary multiplier from $\bar{\Lambda}(v)$ for some nonzero $v \in \bar{K}$. The case when $\bar{K} = \{0\}$ is, of course, trivial.

Example 4. *Consider the GE*

$$0 \in -x + \widehat{N}_\Gamma(y) \tag{45}$$

with Γ given in Example 3. Let $\bar{x} = (0,1)$ and $\bar{y} = (0,0)$ so that $\bar{y}^* = \bar{x}$, $\bar{K} = T_\Gamma(0) = \{v | v_1 \leq 0, v_2 = 0\}$,

$$\bar{\Lambda} = \{\lambda \in \mathbb{R}^3_+ | \lambda_1 - \lambda_2 = 1, \lambda_3 = 0\}$$

and

$$\bar{\Lambda}(v) = \begin{cases} \{\lambda \in \bar{\Lambda} | \lambda_1 = 1, \lambda_2 = 0\} & \text{if} \quad 0 \neq v \in \bar{K} \\ \bar{\Lambda} & \text{if} \quad v = 0. \end{cases}$$

As in Example 3 all conditions of Theorem 3 are fulfilled and so we may invoke Theorem 7 and conclude that the left-hand side of (43) attains for nonzero v the form of the system

$$0 \in \begin{bmatrix} -2v_1 \\ 0 \end{bmatrix} + \mathbb{R} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_1 \leq 0, \quad v_2 = 0.$$

This system clearly implies that $v = 0$ and so the solution map of GE (45) has the isolated calmness property at $(\bar{x}, \bar{y})$. $\triangle$

6.2 S-stationarity conditions for MPECs

Consider the mathematical program with equilibrium constraints

$$\min f(x,y) \quad \text{subject to} \quad 0 \in F(x,y) + \widehat{N}_\Gamma(y), x \in C \tag{46}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are continuously differentiable and $C \subset \mathbb{R}^n$ is a closed set.

Theorem 8. *Let $(\bar{x}, \bar{y})$ be a local solution of the MPEC* (46)*. Suppose that the assumptions of Theorem 5 are fulfilled. Then there exists a MPEC multiplier w such that*

$$0 \in \nabla_x f(\bar{x}, \bar{y})^T + \nabla_x F(\bar{x}, \bar{y})^T w + \widehat{N}_C(\bar{x}) \tag{47}$$

$$0 \in \nabla_y f(\bar{x}, \bar{y})^T + \nabla_y F(\bar{x}, \bar{y})^T w + \widehat{D}^* \widehat{N}_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)(w), \tag{48}$$

where $\bar{y}^* := -F(\bar{x}, \bar{y})$. In particular we have $w \in -\bigcap_{v \in \bar{\mathcal{N}}} \bar{\mathcal{W}}(v)$ and

$$\begin{aligned} 0 &\in \nabla_x f(\bar{x}, \bar{y})^T + \nabla_x F(\bar{x}, \bar{y})^T w + \widehat{N}_C(\bar{x}) \\ 0 &\in \nabla_y f(\bar{x}, \bar{y})^T + \nabla_y F(\bar{x}, \bar{y})^T w + \bigcap_{v \in \bar{\mathcal{N}}} \bar{L}(v; -w). \end{aligned}$$

Proof. Follows from Theorem 5 combined with the standard optimality condition $0 \in \nabla f(x,y) + \widehat{N}_{\mathrm{gph}\,\tilde{S}}(\bar{x},\bar{y})$. $\square$

Example 5. *Consider the MPEC* (46) *with* $x \in \mathbb{R}^3, y \in \mathbb{R}^3$

$$f(x,y) = -x_1 - y_1 + \tfrac{1}{2}y_2^2 + y_3^2,$$

$$F(x,y) = x,\ C = \{a \in \mathbb{R}^3 | a_1 \leq -1\}$$

and Γ *from Example 1. We claim that the pair* $(\bar{x},\bar{y}) = ((-1,0,0),(0,0,0)) \in C \times \Gamma$ *is a solution of this MPEC. Indeed,* $-\bar{x} \in \widehat{N}_\Gamma(\bar{y})$ *by Example 1 and for any feasible pair* $(\tilde{x},\tilde{y})$ *one has*

$$f(\tilde{x},\tilde{y}) \geq \inf_{\substack{x\in C\\ y\in\Gamma}} f(x,y) \geq \inf_{\substack{x_1\leq -1\\ (y_2,y_3)\in\mathbb{R}^2}} (-x_1 + \frac{1}{2}y_2^2 + y_3^2) = 1 = f(\bar{x},\bar{y}).$$

Next we verify the assumptions of Theorem 8. The required properties of the corresponding perturbation mapping M *hold by virtue of MFCQ. Put* $P = \{0\} \times \mathbb{R} \times \mathbb{R}$. *Since* $T_C(\bar{x}) = \mathbb{R}_- \times \mathbb{R} \times \mathbb{R}$, *one has* $T_C(\bar{x}) + P \subset T_C(\bar{x})$. *Further,* $\bar{y}^* = -\bar{x} = (1,0,0), \bar{\Lambda} = \{\lambda \in \mathbb{R}^2_+ | \lambda_1 + \lambda_2 = 1\}$ *and so*

$$V := \mathrm{span}\,\{\nabla q_i(\bar{y}) | i \in \bar{I}^+\} = \mathbb{R} \times \{0\} \times \{0\}.$$

Note that $\mathcal{E} = \{(0,1),(1,0)\}$ *and thus* $V = \mathrm{span}\,\{\nabla q_i(\bar{y}) | i \in I^+(\lambda)\}$ *holds for each* $\lambda \in \mathcal{E}$. *Since* $\hat{T}_C(\bar{x}) = T_C(\bar{x})$, *one has* $\nabla_x F(\bar{x},\bar{y})P = \{0\} \times \mathbb{R} \times \mathbb{R}$, $\nabla_x F(\bar{x},\bar{y})\hat{T}_C(\bar{x}) = \mathbb{R}_- \times \mathbb{R} \times \mathbb{R}$ *and*

$$\nabla_x F(\bar{x},\bar{y})\hat{T}_C(\bar{x}) + V = \nabla_x F(\bar{x},\bar{y})P + V = \mathbb{R}^3.$$

Thus all conditions of Proposition 7 and also of Theorem 8 have been verified. As derived in Example 1,

$$\widehat{N}_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*) = \bar{K}^\circ \times \{(0,0,0)\} = (\mathbb{R} \times \{0\} \times \{0\}) \times (0,0,0),$$

we may conclude that conditions (47), (48) *are fulfilled with* $w = 0$ *and the point* $(1,0,0)$ *belonging to* $\widehat{D}^* \widehat{N}_{\mathrm{gph}\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)(0)$. $\triangle$

7 Conclusion

It is well-known that, in contrast to metric regularity and some other stability notions, the property of metric subregularity at a point does not carry over to a neighborhood. This lack of stability does not cause any troubles in the first-order nonsmooth calculus, where qualification conditions based on metric subregularity have been developed for all basic calculus rules, cf. [13]. In the second-order calculus, however, more stable qualification conditions are needed. One uses typically a surjectivity/ nondegeneracy assumption ([20, 22]) or at least MFCQ. In this paper we suggest in this context to require, in addition to the metric subregularity, the uniform metric regularity except for the point in question introduced in Definition 2. At the first glance this combination may look somewhat cumbersome, but it turns out that at least in some second-order calculations (like the computation of generalized derivatives of the normal-cone mapping) it can very well be used. Moreover, as shown by examples, there are indeed realistic situations in which this combined property holds.

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