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Efficient preconditioning for an optimal control problem with the time-periodic Stokes equations

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Abstract. For the optimal control problem with time-periodic Stokes equations a practical robust preconditioner is presented. The discretization of the corresponding optimality system leads to a linear system with a large, sparse and complex 4-by-4 block matrix in saddle point form. We present a decoupling strategy, which reduces the system to two linear systems with a real 4-by-4 block matrix. Based on analytic results on preconditioners for time-harmonic control problems in [4], a practical preconditioner is constructed, which is robust with respect to the mesh size h, the frequency ω and the control parameter ν. The result is illustrated by numerical examples with the preconditioned minimal residual method. Finally we discuss alternative stopping criteria.

1 The model problem

We consider the following problem: Find the velocity $\mathbf{u}(x,t)$, the pressure $p(x,t)$, and the force $\mathbf{f}(x,t)$ that minimize the cost functional

$$J(\mathbf{u},\mathbf{f}) = \frac{1}{2}\int_0^T \int_\Omega |\mathbf{u}(x,t) - \mathbf{u}_d(x,t)|^2 \; dx \; dt + \frac{\nu}{2}\int_0^T \int_\Omega |\mathbf{f}(x,t)|^2 \; dx \; dt$$

subject to the time-dependent Stokes problem

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{u}(x,t) - \Delta\mathbf{u}(x,t) + \nabla p(x,t) &= \mathbf{f}(x,t) \quad \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{u}(x,t) &= 0 \quad \text{in } \Omega \times (0,T), \\ \mathbf{u}(x,t) &= 0 \quad \text{on } \Gamma \times (0,T), \end{aligned}$$

with time-periodic conditions

$$\mathbf{u}(x,0) = \mathbf{u}(x,T), \; p(x,0) = p(x,T), \; \mathbf{f}(x,0) = \mathbf{f}(x,T) \quad \text{on } \Omega.$$

Here $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ is an open and bounded domain with Lipschitz boundary Γ, $\mathbf{u}_d(x,t)$ is a given target velocity, $\nu > 0$ is a cost or regularization

parameter, and $|.|$ denotes the Euclidean norm in $\mathbb{R}^d$. We assume that $u_d(x,t)$ is time-periodic.

For time discretization a truncated Fourier series expansion is used and for space discretization we choose appropriate finite element spaces $\mathbf{V}_h$ of dimension n and Q_h of dimension m for $\mathbf{u}$ and p, respectively, and the same finite element space $\mathbf{V}_h$ for $\mathbf{f}$ as well. The fully discretized problem can then be decoupled in systems, which only depend on one Fourier coefficient. For the Fourier coefficient corresponding to the frequency ω the system reads as follows:

$$J(\underline{\mathbf{u}}, \underline{\mathbf{f}}) = \frac{1}{2}(\underline{\mathbf{u}} - \underline{\mathbf{u}}_d)^* \mathbf{M}(\underline{\mathbf{u}} - \underline{\mathbf{u}}_d) + \frac{\nu}{2}\, \underline{\mathbf{f}}^* \mathbf{M} \underline{\mathbf{f}} \tag{1}$$

subject to

$$\begin{aligned} i\omega\, \mathbf{M}\, \underline{\mathbf{u}} + \mathbf{K}\underline{\mathbf{u}} - \mathbf{D}^T \underline{p} &= \mathbf{M}\underline{\mathbf{f}}, \\ \mathbf{D}\underline{\mathbf{u}} &= 0. \end{aligned}$$

Here the symbol $*$ denotes the conjugate transpose of a vector or a matrix and the real matrices $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{D}$ are the mass matrix, representing the L^2-inner product in $\mathbf{V}_h$, the discretized negative vector Laplacian, and the discretized divergence, respectively. The underlined quantities denote the coefficient vectors of finite element functions relative to a chosen basis.

The Lagrangian functional for this constrained optimization problem is given by

$$\mathcal{L}(\underline{u}, \underline{p}, \underline{f}, \underline{w}, \underline{r}) = J(\underline{\mathbf{u}}, \underline{\mathbf{f}}) + \underline{w}^* \left(i\omega\, M\, \underline{u} + K\underline{u} - \mathbf{D}^T \underline{p} - M\underline{f} \right) + \underline{r}^* \mathbf{D}\underline{u},$$

where $\underline{w}, \underline{r}$ denote the Lagrangian multipliers associated with the constraints. The first-order optimality conditions are $\nabla \mathcal{L}(\underline{u}, \underline{p}, \underline{f}) = 0$, and read in details:

$$\begin{bmatrix} M & 0 & 0 & K - i\omega\, M & -\mathbf{D}^T \\ 0 & 0 & 0 & -\mathbf{D} & 0 \\ 0 & 0 & \nu\, M & -M & 0 \\ K + i\omega\, M & -\mathbf{D}^T & -M & 0 & 0 \\ -\mathbf{D} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{p} \\ \underline{f} \\ \underline{w} \\ \underline{r} \end{bmatrix} = \begin{bmatrix} M\underline{u}_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2}$$

From the third row it follows that $\underline{f} = \nu^{-1}\underline{w}$. So the control $\underline{f}$ can be eliminated. After reordering we obtain the reduced optimality system:

$$\mathcal{M}\underline{x} = \underline{b}, \tag{3}$$

where

$$\mathcal{M} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} \underline{u} \\ \underline{w} \\ \underline{p} \\ \underline{r} \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} M\underline{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

with

$$A = \begin{bmatrix} \mathbf{M} & \sqrt{\nu}\,(\mathbf{K} - i\omega\, \mathbf{M}) \\ \sqrt{\nu}\,(\mathbf{K} + i\omega\, \mathbf{M}) & -\frac{1}{\nu}\mathbf{M} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -\mathbf{D} \\ -\mathbf{D} & 0 \end{bmatrix}.$$

2 Transformation to two systems with a real matrix

Elementary calculations show that:

$$\mathcal{M} = \mathbf{T}^* \mathcal{M}_{\mathbf{T}} \mathbf{T}, \tag{4}$$

where

$$\mathcal{M}_{\mathbf{T}} = \begin{bmatrix} (1+\nu\omega^2)^{1/2}\mathbf{M} & \mathbf{K} & 0 & -\mathbf{D}^T \\ \mathbf{K} & -\nu^{-1}(1+\nu\omega^2)^{1/2}\mathbf{M} & -\mathbf{D}^T & 0 \\ 0 & -\mathbf{D} & 0 & 0 \\ -\mathbf{D} & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathbf{T} = \begin{bmatrix} T \otimes I_n & 0 \\ 0 & T \otimes I_m \end{bmatrix} \quad \text{with} \quad T = (1+\nu\omega^2)^{-1/4} \begin{bmatrix} (1+\nu\omega^2)^{1/2} & -i \\ 0 & 1 \end{bmatrix}.$$

Here the symbol $\otimes$ denotes the Kronecker product and I_k denotes the identity matrix in $\mathbb{R}^k$. The original system (3) is equivalent to the two systems

$$\mathcal{M}_{\mathbf{T}} \underline{y}_1 = \underline{c}_1 \quad \text{and} \quad \mathcal{M}_{\mathbf{T}} \underline{y}_2 = \underline{c}_2, \tag{5}$$

with $\underline{c} = \underline{c}_1 + i\,\underline{c}_2 = \left(\mathbf{T}^{-1}\right)^* \underline{b}$ and $\underline{y} = \underline{y}_1 + i\,\underline{y}_2 = \mathbf{T}\underline{x}$. So instead of solving one linear system with a complex 4-by-4 block matrix, we have to solve two linear systems with the same 4-by-4 real block matrix $\mathcal{M}_{\mathbf{T}}$, which can be done in parallel.

3 Preconditioning

Our method of choice for solving (5) is the preconditioned MINRES method. As preconditioner $\mathcal{P}$ we consider the block preconditioner constructed in [4]:

$$\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & R \end{bmatrix}, \quad \text{where} \quad P = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \frac{1}{\nu}\mathbf{P} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \nu S & 0 \\ 0 & S \end{bmatrix}, \tag{6}$$

with real and symmetric positive matrices

$$\mathbf{P} = \mathbf{M} + \sqrt{\nu}\,(\mathbf{K} + \omega\,\mathbf{M}) \quad \text{and} \quad S = \mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T. \tag{7}$$

Definition 1. For a matrix N, we denote the eigenvalues of N with minimal and maximal modulus by $\lambda_{\min}(N)$ and $\lambda_{\max}(N)$, respectively.

We have the following estimates.

Theorem 2.

$$1/\sqrt{12} \le |\lambda_{\min}(\mathcal{P}^{-1}\mathcal{M}_{\boldsymbol{T}})| \quad \textit{and} \quad |\lambda_{\max}(\mathcal{P}^{-1}\mathcal{M}_{\boldsymbol{T}})| \le (1+\sqrt{5})/2.$$

For the proof, the detailed analysis and further structural spectral results, see [4].

Definition 3. We call a symmetric and positive definite matrix $\mathcal{Q}$ a robust preconditioner for $\mathcal{M}_{\mathbf{T}}$, if $\kappa(\mathcal{Q}^{-1}\mathcal{M}_{\mathbf{T}}) = \lambda_{\max}(\mathcal{Q}^{-1}\mathcal{M}_{\mathbf{T}})/\lambda_{\min}(\mathcal{Q}^{-1}\mathcal{M}_{\mathbf{T}}) \leq C$ with constant C independent of h, ω and ν.

The result of Theorem 2 implies that $\kappa(\mathcal{P}^{-1}\mathcal{M}_{\mathbf{T}}) \leq \sqrt{3}(1+\sqrt{5})$. Hence $\mathcal{P}$ is a robust preconditioner for $\mathcal{M}_{\mathbf{T}}$. Using well known convergence results for the preconditioned MINRES method (see [2]), it follows that the number of iterations, which is needed to decrease the relative error of the k-th residual measured in the $\|\cdot\|_{\mathcal{P}^{-1}}$-norm by a factor $\varepsilon > 0$, is independent of h, ω and ν. Thereby, for a symmetric and positive definite matrix M, the norm $\|\cdot\|_M$ is defined by $\langle M\cdot, \cdot\rangle^{1/2}$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product.

3.1 The practical preconditioner $\tilde{\mathcal{P}}$

The usage of $\mathcal{P}$ as preconditioner for $\mathcal{M}_{\mathbf{T}}$ requires the evaluation of $\mathbf{P}^{-1}\underline{d}$ and $S^{-1}\underline{e}$ for some given vectors $\underline{d}$ and $\underline{e}$ in every step of the MINRES method. These, especially the evaluation of $S^{-1}\underline{e}$, are nontrivial tasks, due to the potentially high number of involved unknowns. To decrease the computational costs we want to replace $\mathbf{P}$ and S by efficient approximations $\tilde{\mathbf{P}}$ and $\tilde{S}$, respectively. This leads to a preconditioner of the form:

$$\tilde{\mathcal{P}} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & \tilde{R} \end{bmatrix} \quad \text{with} \quad \tilde{P} = \begin{bmatrix} \tilde{\mathbf{P}} & 0 \\ 0 & \nu\tilde{\mathbf{P}} \end{bmatrix} \quad \text{and} \quad \tilde{R} = \begin{bmatrix} \tilde{S} & 0 \\ 0 & \frac{1}{\nu}\tilde{S} \end{bmatrix}. \tag{8}$$

Definition 4. For symmetric and positive definite matrices $M, N \in \mathbb{R}^{n\times n}$, we write $M \sim N$, if there exists positive constants γ_1 and γ_2 independent of h, ν and ω, such that $\gamma_1\langle M\underline{v}, \underline{v}\rangle \leq \langle N\underline{v}, \underline{v}\rangle \leq \gamma_2\langle M\underline{v}, \underline{v}\rangle$ for all $\underline{v} \in \mathbb{R}^n$.

Obviously we have that $\tilde{\mathcal{P}}$ is also a robust preconditioner for $\mathcal{M}_{\mathbf{T}}$, if $\tilde{\mathbf{P}} \sim \mathbf{P}$ and $\tilde{S} \sim S$. We will now present a possible choice for $\tilde{\mathbf{P}}$ and $\tilde{S}$ with $\tilde{\mathbf{P}} \sim \mathbf{P}$ and $\tilde{S} \sim S$, which guarantee the robustness of $\tilde{\mathcal{P}}$, and further an efficient evaluation of $\tilde{\mathcal{P}}^{-1}\underline{z}$ for a vector $\underline{z}$:

Choice for $\tilde{\mathbf{P}}$: We replace the evaluation of $\mathbf{P}^{-1}\underline{d}$ by one $V(1,1)$-cycle of a multigrid method with a symmetric Gauß-Seidel smoother as pre- and post-smoother applied to $\mathbf{P}\underline{v} = \underline{d}$, shortly denoted by $\tilde{\mathbf{P}}^{-1}\underline{d}$. In [8] it was shown that $\tilde{\mathbf{P}} \sim \mathbf{P}$.

Choice for $\tilde{\mathbf{S}}$: First we replace S by the so called *Cahouet-Chabard preconditioner* $S_{\mathrm{CH}} := (\sqrt{\nu}M_p^{-1} + (1+\sqrt{\nu}\omega)K_p^{-1})^{-1}$, where M_p and K_p denote the mass and stiffness matrices in the finite element space Q_h, respectively. For particular finite elements, e.g. the Taylor-Hood element, we have $S \sim S_{CH}$, see, e.g., [1], [5], [6] and [7]. In a second step of approximations, we replace the evaluation of $M_p^{-1}\underline{e}$ by one step of a symmetric Gauß-Seidel iteration applied to $M_p\underline{q} = \underline{e}$ and the evaluation of $K_p^{-1}\underline{e}$ by one $V(1,1)$-cycle

of the same multigrid method as before applied to $\mathbf{P}\underline{q} = \underline{e}$, shortly denoted by $\tilde{M}_p^{-1}\underline{e}$ and $\tilde{K}_p^{-1}\underline{e}$, respectively. Again from [8] we have $\tilde{M}_p \sim M_p$ and $\tilde{K}_p \sim K_p$. As result of this replacements we obtain

$$\tilde{S}_{\mathrm{CH}} := (\sqrt{\nu}\tilde{M}_p^{-1} + (1+\sqrt{\nu}\omega)\tilde{K}_p^{-1})^{-1}, \tag{9}$$

where $\tilde{S}_{CH} \sim S$ and the inverse of $\tilde{S}_{CH}$ can be applied efficiently. Now we replace the evaluation $S^{-1}\underline{e}$ by applying r-steps (typically $r = 1, 2, 3$) of the preconditioned Richardson method to the equation $S\underline{q} = \underline{e}$, with scaling parameters $\tau_i > 0$, the preconditioner $\tilde{S}_{CH}$ and the initial vector $q_0 = 0$. The corresponding preconditioner is given by

$$\tilde{S} = S\left(I_{2\,m} - \prod_{i=1}^{r}(I_{2\,m} - \tau_i\,\tilde{S}_{CH}^{-1}S)^i\right)^{-1}, \tag{10}$$

In order to guarantee that $\tilde{S}$ is positive definite, it is easy to see that the condition

$$1 - \prod_{i=1}^{r}(1-\tau_i\,\lambda)^i > 0 \quad \forall \lambda \in (0,1]. \tag{11}$$

suffices. In particular if we choose $\tau_1 > 0$ fixed and $\tau_i = 1$ for $i \geq 2$, then it follows that $\tilde{S}$ is symmetric, positive definite, and $\tilde{S} \sim S$.

In summary we obtain:

Theorem 5. *$\tilde{\mathcal{P}}$ defined in* (8) *with the previous presented choices for $\tilde{\mathbf{P}}$ and $\tilde{S}$, is a practical, symmetric and positive definite robust preconditioner for $\mathcal{M}_{\boldsymbol{T}}$.*

3.2 Numerical results

We present some numerical examples on the unit square domain $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$. Following Example 1 in [3] we choose the target velocity $\mathbf{u}_d(x,y) = [(U(x,y), V(x,y)]^T$, given by

$$U(x,y) = 10\,\varphi(x)\varphi'(y) \quad \text{and} \quad V(x,y) = -10\,\varphi'(x)\varphi(y),$$

with $\varphi(z) = \big(1-\cos(0.8\pi z)\big)(1-z)^2$. This target velocity $\mathbf{u}_d(x,y)$ is divergence free. The problem was discretized by the Taylor-Hood pair of finite element spaces consisting of continuous piecewise quadratic polynomials for the velocity $\mathbf{u}(x,y)$ and the force $\mathbf{f}(x,y)$, and continuous piecewise linear polynomials for the pressure $p(x,y)$ on a triangulation of Ω. The initial mesh contains four triangles obtained by connecting the two diagonals. The final mesh was constructed by applying ℓ uniform refinement steps to the initial mesh, leading to a mesh size $h = 2^{-\ell}$.

All presented numerical experiments refer to the first of the two systems from (5). The results for the second system are completely identical. Therefore, they are omitted. For each system, the total number of unknowns on the finest level $\ell = 7$ is 1 184 780.

Tables 1 and 2 contain the numerical results produced by the preconditioned MINRES method with the preconditioner $\tilde{\mathcal{P}}$ as described in (8), where we choose $r = 1$ with $\tau_1 = 1$ (i.e. $\tilde{S} = \tilde{S}_{CH}$). The considered values for the mesh size h, the frequency ω, and the regularization parameter ν are specified in the table captions, the first rows and first columns. The other entries of the tables contain the numbers of MINRES iterations that are required for reducing the initial errors in the $\tilde{\mathcal{P}}^{-1}$-norm by a factor of $\varepsilon = 10^{-8}$ with initial vector $x_0 = 0$, respectively.

Table 1. $\omega = 10^4$

	ν				
h	10^{-8}	10^{-4}	1	10^4	10^8
2^{-4}	44	46	46	46	46
2^{-5}	48	50	50	50	48
2^{-6}	50	52	52	52	52
2^{-7}	54	56	56	56	56

Table 2. $\nu = 10^{-4}$

	ω				
h	10^{-8}	10^{-4}	1	10^4	10^8
2^{-4}	87	87	87	46	38
2^{-5}	99	99	99	52	34
2^{-6}	101	101	101	51	30
2^{-7}	105	105	105	56	34

As expected from the results of Theorem 5, the condition numbers are bounded away from ∞ independent of h, ν and ω, leading to a uniform bound for the number of iterations.

Next, we compare the performance of the practical preconditioner $\tilde{\mathcal{P}}$ with the original (typically better but impractical) preconditioner $\mathcal{P}$ for the particular parameter choice $2^{-7}, \nu = 1$ and $\omega = 1$. In this case the number of iterations for $\tilde{\mathcal{P}}$ is 118, which is roughly four times higher than the expected number of iterations for $\mathcal{P}$, see Table 1 in [4]. Since the difference is relatively

high, it is worthwhile to consider other options for the inner iteration in order to reduce this gap. Table 3 shows the numbers of iterations $\tilde{k}$ and the computational costs, measured in the CPU-time, for $\tilde{S}$ with $r \in \{1, 2, 3\}$, for different values of $\tau_1 \in \{1, 4\}$ and $\tau_i = 1$ for $i \geq 2$. These and similar further numerical experiments show that a significant improvement of the numbers of iterations can be achieved by a proper choice for τ_1 and not so much by a higher number r of inner iterations. It turned out that $\tau_1 = 4$ is a very good choice, also for all other cases.

Table 3. $h = 2^{-7}$, $\nu = 1$, $\omega = 1$

r	scaling parameters	$\tilde{k}$	CPU-time [*sec*]
1	$\tau_1 = 1$ (i.e. $\tilde{S} = \tilde{S}_{CH}$)	118	577.65
1	$\tau_1 = 4$ (i.e. $\tilde{S} = 4\, S_{CH}$)	69	341.69
2	$\tau_1 = 4$, $\tau_2 = 1$	46	333.7
3	$\tau_1 = 4$, $\tau_2 = \tau_3 = 1$	41	402.73

4 Alternative stopping criteria

In our numerical examples the stopping criterion

$$\|r_k\|_{\tilde{\mathcal{P}}^{-1}} \leq \varepsilon \, \|r_0\|_{\tilde{\mathcal{P}}^{-1}} \tag{12}$$

was used. Another natural measure for the error is $\|x - x_k\|_{\tilde{\mathcal{P}}}$. This quantity is not directly computable but can be estimated by using the relation:

$$c\|x - x_k\|_{\tilde{\mathcal{P}}} \leq \|r_k\|_{\tilde{\mathcal{P}}^{-1}} \leq C\|x - x_k\|_{\tilde{\mathcal{P}}} \tag{13}$$

with $c = |\lambda_{\min}(\tilde{\mathcal{P}}^{-1}\mathcal{M}_{\mathbf{T}})|$ and $C = |\lambda_{\max}(\tilde{\mathcal{P}}^{-1}\mathcal{M}_{\mathbf{T}})|$. Approximations $\tilde{c}$ and $\tilde{C}$ for c and C, respectively, can be computed by using the so called harmonic Ritz values, see [9]. Therefore, the stopping criterion

$$\|x - x_k\|_{\mathcal{P}} \leq \varepsilon \, \|x - x_0\|_{\mathcal{P}} \tag{14}$$

is asymptotically satisfied, if we prescribe (12) with ε replaced by $\varepsilon_* = \tilde{c}/\tilde{C}\,\varepsilon$.

We test the use of the stopping criterion in (14) with a numerical example. For the parameter choice $h = 2^{-7}, \nu = 1$ and $\omega = 1$, we computed the numbers of iterations $\tilde{k}$ produced by the preconditioned MINRES method, for the two different stopping criteria (12) and (14). Thereby we choose for $\tilde{S}$, $r = 1$ with $\tau_1 = 1$ (i.e. $\tilde{S} = \tilde{S}_{\mathrm{CH}}$). As result we obtain $\tilde{k} = 118$ and $\tilde{k} = 130$, using (12) and (14), respectively. The computed approximations are $\tilde{c} = 0.152077$ and $\tilde{C} = 1.60562$.

Standard norm for stopping criterion: Finally we present an analytic convergence result, for the standard norm

$$\|(u,p,w,r)\|_{\mathcal{N}}^2 := \|u\|_{H^1(\Omega)}^2 + \|p\|_{L^2(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2 + \|r\|_{L^2(\Omega)}^2.$$

For $\nu \leq 1$, it is easy to see that:

$$\|x - x_k\|_{\mathcal{N}} / \|x - x_0\|_{\mathcal{N}} \leq 2\left(\max(2,\omega)/\nu\right)^2 \|x - x_k\|_{\mathcal{P}} / \|x - x_0\|_{\mathcal{P}}.$$

This allows to use this standard norm for the stopping criterion in an efficient manner via (14). Using this estimate in combination with the well known convergence results for the preconditioned MINRES method (see [2]), it follows that the number of iterations $k^\star$ which is needed to decrease the initial error by a factor $\varepsilon > 0$, depends only mildly on the parameters ω and ν, namely, logarithmically on $(\max(2,\omega)/\nu)^2$.

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