JOHANNES KEPLER UNIVERSITY LINZ

Institute of Computational Mathematics

The Ciarlet-Raviart Method for Biharmonic Problems on General Polygonal Domains: Mapping Properties and Preconditioning

Walter Zulehner

Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

NuMa-Report No. 2013-07 October 2013
Rev. 1 April 2015

A–4040 LINZ, Altenbergerstraße 69, Austria

Technical Reports before 1998:

1995

95-1	Hedwig Brandstetter *Was ist neu in Fortran 90?*	March 1995
95-2	G. Haase, B. Heise, M. Kuhn, U. Langer *Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations.*	August 1995
95-3	Joachim Schöberl *An Automatic Mesh Generator Using Geometric Rules for Two and Three Space Dimensions.*	August 1995

1996

96-1	Ferdinand Kickinger *Automatic Mesh Generation for 3D Objects.*	February 1996
96-2	Mario Goppold, Gundolf Haase, Bodo Heise und Michael Kuhn *Preprocessing in BE/FE Domain Decomposition Methods.*	February 1996
96-3	Bodo Heise *A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation.*	February 1996
96-4	Bodo Heise und Michael Jung *Robust Parallel Newton-Multilevel Methods.*	February 1996
96-5	Ferdinand Kickinger *Algebraic Multigrid for Discrete Elliptic Second Order Problems.*	February 1996
96-6	Bodo Heise *A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation.*	May 1996
96-7	Michael Kuhn *Benchmarking for Boundary Element Methods.*	June 1996

1997

97-1	Bodo Heise, Michael Kuhn and Ulrich Langer *A Mixed Variational Formulation for 3D Magnetostatics in the Space* $H(rot) \cap H(div)$	February 1997
97-2	Joachim Schöberl *Robust Multigrid Preconditioning for Parameter Dependent Problems I: The Stokes-type Case.*	June 1997
97-3	Ferdinand Kickinger, Sergei V. Nepomnyaschikh, Ralf Pfau, Joachim Schöberl *Numerical Estimates of Inequalities in* $H^{\frac{1}{2}}$.	August 1997
97-4	Joachim Schöberl *Programmbeschreibung NAOMI 2D und Algebraic Multigrid.*	September 1997

From 1998 to 2008 technical reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`

THE CIARLET-RAVIART METHOD FOR BIHARMONIC PROBLEMS ON GENERAL POLYGONAL DOMAINS: MAPPING PROPERTIES AND PRECONDITIONING*

WALTER ZULEHNER†

Abstract. For biharmonic boundary value problems, the Ciarlet-Raviart mixed method is considered on polygonal domains without additional convexity assumptions. Mapping properties of the involved operators on the continuous as well as on the discrete level are studied. Based on this, efficient preconditioners are constructed and numerical experiments are shown.

Key words. biharmonic equation, Ciarlet-Raviart method, mixed methods, mapping properties, preconditioning

AMS subject classifications. 65N22, 65F08, 65F10

1. Introduction. We consider the first biharmonic boundary value problem: Find y such that

$$\Delta^2 y = f \quad \text{in } \Omega, \qquad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where Ω is an open and bounded set in $\mathbb{R}^2$ with a polygonal Lipschitz boundary Γ, Δ and $\partial/\partial n$ denote the Laplace operator and the derivative in the direction normal to the boundary, respectively, and $f \in H^{-1}(\Omega)$. Here and throughout the paper we use $L^2(\Omega)$, $H^m(\Omega)$, and $H_0^m(\Omega)$ with its dual space $H^{-m}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces with corresponding norms $\|.\|_0$, $\|.\|_m$, $|.|_m$, and $\|.\|_{-m}$ for positive integers m; see, e.g., [1]. Problems of this type occur, for example, in fluid mechanics, where y is the stream function of a two-dimensional Stokes flow (see, e.g., [12]), and in elasticity, where y is the vertical deflection of a clamped Kirchhoff plate (see, e.g., [8]).

The standard (primal) variational formulation of (1.1) reads as follows: Find $y \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta y \, \Delta z \; dx = \langle f, z \rangle \quad \text{for all } z \in H_0^2(\Omega), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product in $H^* \times H$ for a Hilbert space H with dual H^*, here for $H = H_0^1(\Omega)$. (If $H = \mathbb{R}^n$, we use $\langle \cdot, \cdot \rangle$ for the Euclidean inner product.) Existence and uniqueness of a solution to (1.2) is guaranteed by the Lax-Milgram theorem; see, e.g., [22], [19].

Conforming finite element methods based on (1.2) require approximation spaces of continuously differentiable functions, which are not so easy to construct for unstructured meshes. Another challenging issue for (1.2) is the construction of efficient preconditioners for iterative methods for solving a discretized version of (1.2). Standard techniques which might help to resolve these difficulties are discontinuous Galerkin methods or mixed methods. We focus here on the well-known mixed method

*This research was supported by the Austrian Science Fund (FWF), W1214-N15, project DK12.
†Institute of Computational Mathematics, Johannes Kepler University Linz, 4040 Linz, Austria (zulehner@numa.uni-linz.ac.at).

by Ciarlet and Raviart (see [10]), for which an auxiliary variable

$$u = -\Delta y$$

is introduced. For the Stokes problem u is the vorticity of the flow, and for plate bending problems u can be interpreted as bending moment. With this auxiliary variable the fourth-order differential equation in (1.1) can be rewritten as a system of two second-order equations leading to the boundary value problem: Find y and u such that

$$-\Delta y = u, \quad -\Delta u = f \quad \text{in } \Omega, \qquad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma. \tag{1.3}$$

Finite element methods for (1.3) were studied on convex domains Ω by many authors; see, e.g., [10], [26], [11], [4], [12]. The equivalence of variational formulations for (1.1) and for (1.3) is a subtle issue, which, for the first biharmonic problem, was already addressed in the pioneering paper [10] for convex domains and was essentially settled in [5] for domains without convexity assumptions.

Strongly related to the Ciarlet-Raviart mixed method is a boundary operator formulation for another auxiliary variable

$$\lambda = u\big|_\Gamma$$

on the continuous as well as on the discrete level; see [9], [13]. On the discrete level, this approach can be seen as a reduction of the mixed problem to a Schur complement problem.

For convex domains and the more for nonconvex domains, preconditioning the mixed method is still a challenging issue because the mapping properties of the involved linear operators are by far not trivial. One possible approach is the use of mesh-dependent norms for the mixed method; see [4]. However, the analysis was restricted to convex domains and, more severely, the resulting preconditioner requires a preconditioner for a matrix which can be interpreted as a discretization of a differential operator of order 4. In [27] preconditioners were studied which require only standard components, motivated by a reasonable trade-off between optimality (in the sense of mesh-independent convergence rates) and practicability. For the boundary operator formulation preconditioning was studied in [24] quite in the spirit of operator preconditioning; see, e.g., [18] and [20] for a general discussion of this concept. The preconditioner proposed in [24] leads to mesh-independent convergence rates for convex domains.

The aim of this paper is to construct an efficient preconditioner with mesh-independent convergent rates without convexity assumptions. We will extend results from [24] for the reduced problem in λ and show some preliminary results on a class of preconditioners for the original (nonreduced) mixed formulation in y and u.

The paper is organized as follows. In Sections 2 and 3 the mapping properties are analyzed for the mixed and the reduced formulation, respectively. After discussing the discretized problems quite in the spirit of the analysis of the corresponding continuous problems in Section 4, the main results on preconditioning are developed in Section 5. A few numerical experiments are presented in Section 6 for illustrating the theoretical results, followed by concluding remarks in Section 7. Some technical details on harmonic extension operators needed for the analysis in the previous sections are collected in an appendix.

2. The Ciarlet-Raviart method. Here we briefly recall known results on the original mixed formulation and its modification in [5].

2.1. The original method. We consider the following standard mixed variational formulation for (1.3): For $f \in H^{-1}(\Omega)$, find $u \in H^1(\Omega)$ and $y \in H^1_0(\Omega)$ such that

$$\begin{aligned} \int_\Omega u\, v \, dx \qquad & - \int_\Omega \nabla v \cdot \nabla y \, dx = 0 && \text{for all } v \in H^1(\Omega), \\ - \int_\Omega \nabla u \cdot \nabla z \, dx \qquad & \qquad\qquad = -\langle f, z\rangle && \text{for all } z \in H^1_0(\Omega), \end{aligned} \tag{2.1}$$

where ∇ denotes the gradient. This problem has the typical structure of a saddle point problem:

$$\begin{aligned} a(u, v) + b(v, y) &= 0 && \text{for all } v \in V, \\ b(u, z) &= -\langle f, z\rangle && \text{for all } z \in Q \end{aligned}$$

for the Hilbert spaces

$$V = H^1(\Omega) \quad \text{and} \quad Q = H^1_0(\Omega)$$

and the bilinear forms

$$a(u, v) = \int_\Omega u\, v \, dx \quad \text{and} \quad b(v, z) = - \int_\Omega \nabla v \cdot \nabla z \, dx. \tag{2.2}$$

If the linear operator $\mathcal{A}\colon X \longrightarrow X^*$ with $X = V \times Q$ is introduced by

$$\left\langle \mathcal{A} \begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} v \\ z \end{bmatrix} \right\rangle = a(u, v) + b(v, y) + b(u, z),$$

the mixed variational problem (2.1) can be rewritten as a linear operator equation

$$\mathcal{A} \begin{bmatrix} u \\ y \end{bmatrix} = - \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Observe that the bilinear form a is symmetric, i.e., $a(u, v) = a(v, u)$, and nonnegative, i.e., $a(v, v) \geq 0$. In this case it is well-known that $\mathcal{A}$ is an isomorphism from X onto X^* iff the following conditions are satisfied (see, e.g., [7]):

1. a is bounded: There is a constant $\|a\| > 0$ such that

$$|a(u, v)| \leq \|a\| \, \|u\|_V \, \|v\|_V \quad \text{for all } u,\ v \in V.$$

2. b is bounded: There is a constant $\|b\| > 0$ such that

$$|b(v, z)| \leq \|b\| \, \|v\|_V \|z\|_Q \quad \text{for all } v \in V,\ z \in Q.$$

3. a is coercive on the kernel of b: There is a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha \, \|v\|_V^2 \quad \text{for all } v \in \ker B$$

with $\ker B = \{w \in V : b(w,z) = 0 \text{ for all } z \in Q\}$.

4. b satisfies the inf-sup condition: There is a constant $\beta > 0$ such that

$$\inf_{0 \neq z \in Q} \sup_{0 \neq v \in V} \frac{b(v,z)}{\|v\|_V \, \|z\|_Q} \geq \beta.$$

Here $\|\cdot\|_V$ and $\|\cdot\|_Q$ denote the norms in V and Q, respectively. We will refer to these conditions as Brezzi's conditions with constants $\|a\|$, $\|b\|$, α, and β.

For (2.1) one of these conditions is not satisfied: the bilinear form a is not coercive on $\ker B$. Nevertheless, for convex domains Ω, existence of a unique solution and error estimates could be established; see, e.g. [10], [26], [11], [4], and many others. But even for convex domains, and more so for nonconvex domains, not having an isomorphism makes it hard to develop efficient preconditioners.

2.2. The modified method. In [5] it was proposed to replace the space $H^1(\Omega)$ for the unknown u by the Hilbert space of less regularity

$$H^{-1}(\Delta,\Omega) = \{v \in L^2(\Omega) \colon \Delta v \in H^{-1}(\Omega)\},$$

equipped with the norm

$$\|v\|_{-1,\Delta} = \left(\|v\|_0^2 + \|\Delta v\|_{-1}^2\right)^{1/2}.$$

Here Δv denotes the application of the Laplace operator to v in the distributional sense. The original space $H^1(\Omega)$ is a proper subset of the new space $H^{-1}(\Delta,\Omega)$. This requires extending the definition of the bilinear form b accordingly by

$$b(v,z) = \langle \Delta v, z\rangle,$$

which, of course, coincides with the original definition for $v \in H^1(\Omega)$. Then the extended version of (2.1) for this larger primal space reads as follows: For $f \in H^{-1}(\Omega)$, find $u \in H^{-1}(\Delta,\Omega)$ and $y \in H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega u\, v \, dx + \langle \Delta v, y\rangle &= 0 && \text{for all } v \in H^{-1}(\Delta,\Omega), \\ \langle \Delta u, z\rangle &= -\langle f, z\rangle && \text{for all } z \in H_0^1(\Omega). \end{aligned} \tag{2.3}$$

We recall the following result from [5].

Theorem 2.1. *The bilinear forms a and b given by*

$$a(u,v) = \int_\Omega u\, v\, dx \quad \text{and} \quad b(v,z) = \langle \Delta v, z\rangle$$

for $V = H^{-1}(\Delta,\Omega)$ and $Q = H_0^1(\Omega)$ with the norms $\|v\|_V = \|v\|_{-1,\Delta}$ and $\|z\|_Q = |q|_1$ satisfy Brezzi's conditions with the constants

$$\|a\| = \|b\| = \alpha = 1 \quad \text{and} \quad \beta = (1 + c_F^2)^{-1/2},$$

where c_F *denotes the constant in Friedrichs' inequality:* $\|v\|_0 \leq c_F\, |v|_1$ *for all* $v \in H_0^1(\Omega)$.

Problems (1.2) and (2.3) are fully equivalent for convex as well as for nonconvex polygonal domains, since both problems are uniquely solvable and it is easy to see that (u, y) with $u = -\Delta y$ solves (2.3) if $y \in H_0^2(\Omega)$ solves (1.2). This has already been recognized in [5] in the context of the Stokes problem.

REMARK 1. *The space* $H^{-1}(\Delta, \Omega)$ *is related to the space*

$$\Sigma = \{v \in L^2(\Omega) \colon \operatorname{curl} v \in H_0(\operatorname{div}, \Omega)^*\},$$

which was used for the analysis of the Ciarlet-Raviart method in [3]. If Ω *is simply connected, then these two spaces are identical; otherwise they are not.*

3. Reduction to a boundary operator equation. Next we want to reduce the variational problem (2.3) for y and u to a variational problem for the trace λ of u only. For this we need two decomposition results. The first decomposition is closely related to results in [2]. The focus here is the formulation in the framework of space decompositions.

LEMMA 3.1. $H^{-1}(\Delta, \Omega) = H_0^1(\Omega) \oplus \mathscr{H}(\Omega)$ *with*

$$\mathscr{H}(\Omega) = \{v \in L^2(\Omega) \colon \Delta v = 0\},$$

where $\oplus$ *denotes the direct sum of Hilbert spaces, whose canonical norm is given here by*

$$\|(v_0, v_1)\|^2_{H_0^1(\Omega) \oplus \mathscr{H}(\Omega)} = |v_0|_1^2 + \|v_1\|_0^2.$$

In detail, for each $v \in H^{-1}(\Delta, \Omega)$, *there is a unique decomposition*

$$v = v_0 + v_1 \quad \text{with} \quad v_0 \in H_0^1(\Omega) \quad \text{and} \quad v_1 \in \mathscr{H}(\Omega),$$

and there are positive constants $\underline{c}$ *and* $\overline{c}$ *such that*

$$\underline{c}\, \left(|v_0|_1^2 + \|v_1\|_0^2\right) \leq \|v\|_{-1,\Delta}^2 \leq \overline{c}\, \left(|v_0|_1^2 + \|v_1\|_0^2\right) \quad \text{for all } v \in H^{-1}(\Delta, \Omega).$$

The constants $\underline{c}$ *and* $\overline{c}$ *depend only on the constant* c_F *of Friedrichs' inequality.*

Proof. For $v \in H^{-1}(\Delta, \Omega)$, let $v_0 \in H_0^1(\Omega)$ be the unique solution to the variational problem

$$\int_\Omega \nabla v_0 \cdot \nabla z \, dx = -\langle \Delta v, z \rangle \quad \text{for all } y \in H_0^1(\Omega). \tag{3.1}$$

By taking the supremum over all $z \in H_0^1(\Omega)$ we obtain $|v_0|_1 = \|\Delta v\|_{-1}$.

For $v_1 = v - v_0$, we have $\Delta v_1 = \Delta v - \Delta v_0 = 0$ in the distributional sense. Hence $v_1 \in \mathscr{H}(\Omega)$. On the other hand, if $v = v_0 + v_1$ with $v_0 \in H_0^1(\Omega)$ and $v_1 \in \mathscr{H}(\Omega)$, then $-\Delta v_0 = -\Delta v + \Delta v_1 = -\Delta v$, which is equivalent to the variational problem (3.1). So, v_0 is the unique solution of (3.1).

Furthermore, we have

$$\begin{aligned} \|v\|_{-1,\Delta}^2 &= \|v\|_0^2 + \|\Delta v\|_{-1}^2 = \|v_0 + v_1\|_0^2 + |v_0|_1^2 \\ &\leq 2\, \|v_0\|_0^2 + 2\, \|v_1\|_0^2 + |v_0|_1^2 \leq (2c_F^2 + 1)\, |v_0|_1^2 + 2\, \|v_1\|_0^2 \end{aligned}$$

and

$$|v_0|_1^2 + \|v_1\|_0^2 = |v_0|_1^2 + \|v - v_0\|_0^2 \le |v_0|_1^2 + 2\,\|v\|_0^2 + 2\,\|v_0\|_0^2$$
$$\le 2\,\|v\|_0^2 + (2c_F^2 + 1)\,|v_0|_1^2 = 2\,\|v\|_0^2 + (2c_F^2 + 1)\,\|\Delta v\|_{-1}^2.$$

Then the estimates immediately follow with $1/\underline{c} = \overline{c} = \max(2, 2c_F^2 + 1)$. □

Observe that $\mathscr{H}(\Omega) = \ker B$ with the notation introduced in Section 2.

Notation 1. *For estimates of the form*

$$\underline{c}\, f_2(x) \le f_1(x) \le \overline{c}\, f_2(x) \quad \text{for all } x \in H,$$

where f_1 and f_2 are nonnegative functions, with some positive constants $\underline{c}$, $\overline{c}$ independent of $x \in H$ and, later on for discretized problems, also independent of the mesh size, we briefly write

$$f_1(x) \sim f_2(x) \quad \text{for all } x \in H.$$

If $f_1(x) = \langle M_1 x, x\rangle$, $f_2(x) = \langle M_2 x, x\rangle$ for symmetric and positive definite matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ with $H = \mathbb{R}^n$, we use the simplified notation $M_1 \sim M_2$.

With this notation the estimates in the last lemma can be written as

$$\|v\|_{-1,\Delta}^2 \sim |v_0|_1^2 + \|v_1\|_0^2 \quad \text{for all } v \in H^{-1}(\Delta, \Omega).$$

Corollary 3.2. *For all $v \in H^{-1}(\Delta, \Omega)$, we have $v\big|_{\Omega'} \in H^1(\Omega')$ for all open sets Ω' with $\overline{\Omega'} \subset \Omega$.*

Proof. We use Weyl's lemma to conclude that $\mathscr{H}(\Omega) \subset C^\infty(\Omega)$. Then the statement immediately follows from the decomposition $\varphi\, v = \varphi\, v_0 + \varphi\, v_1$ for a test function $\varphi \in C_0^\infty(\Omega)$ with φ identical to 1 on Ω'. □

For the description of a further decomposition of $\mathscr{H}(\Omega)$ we need trace and extension operators for $H^{-1}(\Delta, \Omega)$.

The properties for the trace operator are well-known and summarized here; they easily follow from the results in [16], [15]. The boundary Γ of the polygonal domain Ω can be written as

$$\Gamma = \Gamma_C \cup \Gamma_E \quad \text{with} \quad \Gamma_E = \bigcup_{k=1}^K \Gamma_k,$$

where Γ_C denotes the set of all corners of Γ and Γ_k, $k = 1, 2, \ldots, K$, are the edges of Γ, considered as open line segments. The trace operator γ^0, given by

$$\gamma^0 v = \left(\gamma_k^0 v\right)_{k=1,\ldots,K} \quad \text{with} \quad \gamma_k^0 v = v|_{\Gamma_k}$$

for smooth functions on $\overline{\Omega}$, has a unique continuous extension as an operator

$$\gamma^0 \colon H^{-1}(\Delta, \Omega) \longrightarrow H_{pw}^{-1/2}(\Gamma) = \Pi_{k=1}^K H^{-1/2}(\Gamma_k),$$

where $H^{-1/2}(\Gamma_k)$ is the dual of $\widetilde{H}^{1/2}(\Gamma_k)$; see [21] for details. (Another widely used notation for $\widetilde{H}^{1/2}(\Gamma_k)$ is $H_{00}^{1/2}(\Gamma_k)$; see [19].) The standard norm in $H^{-1/2}(\Gamma_k)$ is denoted by $\|.\|_{-1/2,\Gamma_k}$. The norm in $H_{pw}^{-1/2}(\Gamma)$, denoted by $\|\cdot\|_{-1/2,\Gamma}$, is the canonical product norm of its factor spaces, given by

$$\|\mu\|_{-1/2,\Gamma}^2 = \sum_{k=1}^{K} \|\mu_k\|_{-1/2,\Gamma_k}^2 \quad \text{for } \mu = (\mu_k)_{k=1,\dots,K} \in H_{pw}^{-1/2}(\Gamma).$$

NOTATION 2. *For the notation of norms or duality products for functions on the boundary Γ or some edge Γ_k, we explicitly use Γ or Γ_k as subscripts. A subscript pw (piecewise) is used for spaces of functions on Γ which are products of spaces of functions defined on the edges Γ_k for $k \geq 1$. For simplicity we omit this subscript for the corresponding norm. A subscript h (mesh size) is used for mesh-dependent norms.*

The intersection of the kernel of γ^0 and $\mathscr{H}(\Omega)$, given by

$$N = \ker \gamma^0 \cap \mathscr{H}(\Omega) = \{v \in L^2(\Omega) \colon \Delta v = 0 \text{ and } \gamma^0 v = 0\}, \tag{3.2}$$

is known to be finite-dimensional. The dimension of N is equal to the number of reentrant corners of Ω; see [16], [15].

The existence of an extension operator and its properties, which are well-known for convex or smooth domains, will be extended to general polygonal domains in the next theorem; see the appendix for the proof.

THEOREM 3.3. *There is a linear operator*

$$E^0 \colon H_{pw}^{-1/2}(\Gamma) \longrightarrow H^{-1}(\Delta, \Omega)$$

which is a right inverse of γ^0 with the following properties:

1. $\operatorname{im} E^0 \subset \mathscr{H}(\Omega)$*, where* $\operatorname{im} L$ *denotes the image of a linear operator* L*.*
2. $\mathscr{H}(\Omega) = \operatorname{im} E^0 \perp N$*, where the symbol* $\perp$ *denotes the* L^2*-orthogonal decomposition.*
3. $\|E^0\mu\|_0^2 \sim \|\mu\|_{-1/2,\Gamma}^2$ *for all* $\mu \in H_{pw}^{-1/2}(\Gamma)$*.*

The first part means that E^0 can be viewed as a harmonic extension operator, the second part contains the required decomposition result, and the last part shows that E^0 is an isomorphism between the trace space and its image. The existence of the right inverse E^0 immediately implies that the trace operator γ^0 maps from $H^{-1}(\Delta, \Omega)$ onto $H_{pw}^{-1/2}(\Gamma)$.

Lemma 3.1 and Theorem 3.3 allow the following reduction of (2.3).

THEOREM 3.4. *Let $u \in H^{-1}(\Delta, \Omega)$ and $y \in H_0^1(\Omega)$ be the unique solution of (2.3). Then $\lambda = \gamma^0 u \in H_{pw}^{-1/2}(\Gamma)$ is the unique solution of the variational problem*

$$\int_\Omega E^0\lambda\, E^0\mu \, dx = -\int_\Omega u_0\, E^0\mu \, dx \quad \textit{for all } \mu \in H_{pw}^{-1/2}(\Gamma), \tag{3.3}$$

where $u_0 \in H_0^1(\Omega)$ is the unique weak solution of the Dirichlet problem for the Laplace operator, i.e.,

$$\int_\Omega \nabla u_0 \cdot \nabla z \, dx = \langle f, z\rangle \quad \textit{for all } z \in H_0^1(\Omega). \tag{3.4}$$

Proof. From Lemma 3.1 and the second part of Theorem 3.3 it follows that there is a unique element $u_0 \in H_0^1(\Omega)$ such that $u = u_0 + E^0\lambda + n$ for some $n \in N$. The second line of (2.3) simplifies to (3.4), since $\Delta(E^0\lambda+n) = 0$ according to the first part of Theorem 3.3. Problem (3.3) follows from the first line of (2.3) for test functions of the form $v = E^0\mu$ with $\mu \in H_{pw}^{-1/2}(\Gamma)$, since $E^0\mu$ is orthogonal to n according to the second part of Theorem 3.3. Using the third part of Theorem 3.3 the well-posedness of (3.3) follows from the Lax-Milgram theorem. ∎

This generalizes the boundary operator equation, formulated in [13] for smooth domains Ω, to the case of general polygonal domains Ω.

4. Discretization. Let $\mathcal{T}_h$ be an admissible triangulation of the domain Ω. We proceed as usual to construct a conforming finite element space for approximating $H^{-1}(\Delta,\Omega)$ by choosing piecewise linear functions which lie in this space. From Corollary 3.2 it immediately follows that a piecewise smooth function lies in $H^{-1}(\Delta,\Omega)$ iff it is continuous. This leads to the standard finite element space

$$\mathcal{S}_h(\Omega) = \left\{v_h \in C(\overline{\Omega})\colon v_h|_T \in P_1 \text{ for all } T \in \mathcal{T}_h\right\},$$

where P_1 denotes the set of linear polynomials. Additionally we introduce

$$\mathcal{S}_{h,0}(\Omega) = \mathcal{S}_h(\Omega) \cap H_0^1(\Omega).$$

Using $\mathcal{S}_h(\Omega)$ and $\mathcal{S}_{h,0}(\Omega)$ as approximation spaces for $H^{-1}(\Delta,\Omega)$ and $H_0^1(\Omega)$, respectively, we obtain the following conforming finite element method for (2.3): Find $u_h \in \mathcal{S}_h(\Omega)$ and $y_h \in \mathcal{S}_{h,0}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega u_h\, v_h \, dx \quad - \int_\Omega \nabla v_h \cdot \nabla y_h \, dx = 0 && \text{for all } v_n \in \mathcal{S}_h(\Omega), \\ &-\int_\Omega \nabla u_h \cdot \nabla z_h \, dx \qquad\qquad\quad = -\langle f, z_h\rangle && \text{for all } z_h \in \mathcal{S}_{h,0}(\Omega). \end{aligned} \tag{4.1}$$

Observe that $\mathcal{S}_h(\Omega) \subset H^1(\Omega)$. Therefore, the definition (2.2) of b can be used. This is exactly the original discrete problem studied in [10]. So, on the discrete level, there is no direct influence of the use of $H^{-1}(\Delta,\Omega)$ for u, v instead of $H^1(\Omega)$.

Analogously to Theorem 2.1 the well-posedness of the discrete problem can be shown.

THEOREM 4.1. *Brezzi's conditions are satisfied for (4.1) on the discrete spaces* $V = \mathcal{S}_h(\Omega)$ *and* $Q = \mathcal{S}_{h,0}(\Omega)$ *and the norms* $\|v\|_V = \|v\|_{-1,\Delta,h}$ *and* $\|z_h\|_Q = |z_h|_1$, *where*

$$\|v_h\|_{-1,\Delta,h} = \left(\|v_h\|_0^2 + \|\Delta v_h\|_{-1,h}^2\right)^{1/2} \quad \textit{with} \quad \|\ell\|_{-1,h} = \sup_{z_h \in \mathcal{S}_{h,0}(\Omega)} \frac{|\langle \ell, z_h\rangle|}{|z_h|_1},$$

with the same constants as in Theorem 2.1 for the continuous problem (2.3).

Proof. The proof follows the corresponding proof in [5] for the continuous problem.

1. Let $u_h, v_h \in \mathcal{S}_h(\Omega)$. Then

$$|a(u_h, v_h)| \le \|u_h\|_0\, \|v_h\|_0 \le \|u_h\|_{-1,\Delta,h} \|v_h\|_{-1,\Delta,h}.$$

2. Let $v_h \in \mathcal{S}_h(\Omega)$, $z_h \in \mathcal{S}_{h,0}(\Omega)$. Then

$$|b(v_h, z_h)| \leq \|\Delta v_h\|_{-1,h} \, |z_h|_1 \leq \|v_h\|_{-1,\Delta,h} \, |z_h|_1.$$

3. Let $v_h \in \ker B_h = \{w_h \in \mathcal{S}_h(\Omega) \colon b(w_h, z_h) = 0 \text{ for all } z_h \in \mathcal{S}_{h,0}(\Omega)\}$. Then

$$a(v_h, v_h) = \|v_h\|_0^2 = \|v_h\|_{-1,\Delta,h}^2.$$

4. Let $0 \neq z_h \in \mathcal{S}_{h,0}(\Omega)$. Then

$$\sup_{0 \neq v_h \in \mathcal{S}_h(\Omega)} \frac{b(v_h, z_h)}{\|v_h\|_{-1,\Delta,h}} \geq \frac{b(-z_h, z_h)}{\|z_h\|_{-1,\Delta,h}} = \frac{|z_h|_1^2}{\|z_h\|_1} \geq \left(c_F^2 + 1\right)^{-1/2} |z_h|_1. \qquad \square$$

Observe that the norms introduced for the space $H^{-1}(\Delta, \Omega)$ in (2.1) and its discrete counterpart $\mathcal{S}_h(\Omega)$ in (4.1) are similar but different. For the discrete problem the norm is mesh-dependent.

REMARK 2. *Following the same ideas as presented in [26], [11], and [3], where convex domains were considered, error estimates can be extended to general polygonal domains. For example, one can show that*

$$\|u - u_h\|_0 + |y - y_h|_1 \leq c \left(\|u - R_h u\|_0 + h^{-1/2} \, |y - R_{h,0} \, y|_{W^1_\infty(\Omega)}\right),$$

where R_h and $R_{h,0}$ denote the Ritz projections in $H^1(\Omega)$ and $H^1_0(\Omega)$, respectively. For the details on error estimates for the Ritz projections on general polygonal domains we refer to the literature, e.g., [25]. For convex domains, the estimates for $|y - y_h|_1$ have been improved by duality arguments. An extension to general polygonal domains is rather involved and beyond the scope of this paper.

The actual computations will be performed in matrix-vector notation. We will now rewrite (4.1) in this way. Let $\underline{v}_h$ and $\underline{z}_h$ be the coefficient vectors of $v_h \in \mathcal{S}_h(\Omega)$ and $z_h \in \mathcal{S}_{h,0}(\Omega)$ with respect to the nodal bases in these spaces, respectively. The splitting into interior nodes and nodes on the boundary Γ induces a corresponding block structure of $\underline{v}_h$:

$$\underline{v}_h = \begin{bmatrix} \underline{v}_{h,0} \\ \underline{\mu}_h \end{bmatrix}.$$

The mass matrix M_h and the stiffness matrix K_h representing $\|\cdot\|_0$ and $|\cdot|_1$ on $\mathcal{S}_h(\Omega)$, respectively, can be partitioned accordingly:

$$M_h = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \quad \text{and} \quad K_h = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}.$$

Then the variational problem (4.1) reads in matrix-vector notation

$$\begin{bmatrix} M_{00} & M_{01} & -K_{00} \\ M_{10} & M_{11} & -K_{10} \\ -K_{00} & -K_{01} & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_{h,0} \\ \underline{\lambda}_h \\ \underline{y}_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\underline{f}_h \end{bmatrix}. \tag{4.2}$$

A reduction of this block system to a single system for $\underline{\lambda}_h$ can be easily achieved by eliminating $\underline{u}_{h,0}$ and $\underline{y}_h$ using the third and first block lines, respectively. This leads to

$$S_h \, \underline{\lambda}_h = \underline{g}_h \tag{4.3}$$

with

$$S_h = M_{11} - M_{10}K_{00}^{-1}K_{01} - K_{10}K_{00}^{-1}M_{01} + K_{10}K_{00}^{-1}M_{00}K_{00}^{-1}K_{01}$$

and the right-hand side

$$\underline{g}_h = \left(K_{10}K_{00}^{-1}M_{00} - M_{10}\right) K_{00}^{-1}\underline{f}_h.$$

The matrix S_h is known as a Schur complement of the block system.

As in the continuous case the reduction to the boundary can also be done by a decomposition result for the finite element space $\mathcal{S}_h(\Omega)$, which reveals some extra structural information of the Schur complement matrix.

We start with the following discrete version of Lemma 3.1.

LEMMA 4.2. *$\mathcal{S}_h(\Omega) = \mathcal{S}_{h,0}(\Omega) \oplus \mathscr{H}_h(\Omega)$ with*

$$\mathscr{H}_h(\Omega) = \left\{ v_h \in \mathcal{S}_h(\Omega) \colon \int_\Omega \nabla v_h \cdot \nabla z_h \, dx = 0 \text{ for all } z_h \in \mathcal{S}_{h,0}(\Omega) \right\}.$$

In detail, for each $v_h \in \mathcal{S}_h(\Omega)$, we have the following unique decomposition:

$$v_h = \hat{v}_{h,0} + \hat{v}_{h,1} \quad \text{with} \quad \hat{v}_{h,0} \in \mathcal{S}_{h,0}(\Omega) \quad \text{and} \quad \hat{v}_{h,1} \in \mathscr{H}_h(\Omega)$$

and

$$\|v_h\|_{-1,\Delta,h}^2 \sim |\hat{v}_{h,0}|_1^2 + \|\hat{v}_{h,1}\|_0^2 \quad \text{for all } v_h \in \mathcal{S}_h(\Omega)$$

with the same constants as in Lemma 3.1.

The proof of this lemma is a complete copy of the proof in the continuous case and is therefore omitted.

Observe that for $v_h \in \mathcal{S}_h(\Omega)$, the decompositions in Lemmas 3.1 and 4.2 are different, in general. The space $\mathscr{H}_h(\Omega)$ is known as the space of discrete harmonic functions.

Next we introduce the trace space of functions from $\mathcal{S}_h(\Omega)$ by

$$\mathcal{S}_h(\Gamma) = \{\mu_h = v_h\big|_\Gamma \colon v_h \in \mathcal{S}_h(\Omega)\}.$$

For each $\mu_h \in \mathcal{S}_h(\Gamma)$, there is a unique element $v_h \in \mathcal{S}_h(\Omega)$ with $v_h\big|_\Gamma = \mu_h$ and

$$\int_\Omega \nabla v_h \cdot \nabla z_h \, dx = 0 \quad \text{for all } z_h \in \mathcal{S}_{h,0}(\Omega).$$

The associated mapping $E_h \colon \mathcal{S}_h(\Gamma) \longrightarrow \mathscr{H}_h(\Omega)$, $\mu_h \mapsto v_h$ is the well-known discrete harmonic extension. In matrix-vector notation, this mapping reads

$$\underline{\mu}_h \mapsto \underline{v}_h = \begin{bmatrix} E_{01} \\ I \end{bmatrix} \underline{\mu}_h \quad \text{with} \quad E_{01} = -K_{00}^{-1}K_{01}.$$

Here $\underline{\mu}_h$ denotes the coefficient vector of μ_h with respect to the nodal basis of $\mathcal{S}_h(\Gamma)$. It is easy to see that E_h is bijective.

Analogously to Theorem 3.4 we now obtain

THEOREM 4.3. *Let $u_h \in \mathcal{S}_h(\Omega)$ and $y_h \in \mathcal{S}_{h,0}(\Omega)$ be the unique solution of (4.1). Then $\lambda_h = u_h\big|_\Gamma \in \mathcal{S}_h(\Gamma)$ is the unique solution of the variational problem*

$$\int_\Omega E_h\lambda_h \, E_h\mu_h \; dx = -\int_\Omega \hat{u}_{h,0} \, E_h\mu \; dx \quad \text{for all } \mu \in \mathcal{S}_h(\Gamma), \tag{4.4}$$

where $\hat{u}_{h,0} \in \mathcal{S}_{h,0}(\Omega)$ is the unique solution of the discrete variational problem

$$\int_\Omega \nabla \hat{u}_{h,0} \cdot \nabla z_h \; dx = \langle f, z_h \rangle \quad \text{for all } z_h \in \mathcal{S}_{h,0}(\Omega).$$

The proof, which is completely analogous to the continuous case, is omitted.

Moreover, as already observed in [24], it is easy to show that the matrix representation of the bilinear form on the left-hand side in (4.4) is the Schur complement S_h:

$$\int_\Omega E_h\lambda_h \, E_h\mu_h \; dx = \langle S_h \, \underline{\lambda}_h, \underline{\mu}_h \rangle \quad \text{for all } \lambda_h, \;\mu_h \in \mathcal{S}_h(\Gamma). \tag{4.5}$$

5. Preconditioning. The starting point for the construction of a preconditioner for (4.2) is Theorem 4.1, which shows the well-posedness of the discrete problem with respect to the norms $\|\cdot\|_{-1,\Delta,h}$ and $|\cdot|_1$, whose matrix representations are given by

$$\|v_h\|_{-1,\Delta,h} = \|\underline{v}_h\|_{P_h} \text{ with } P_h = M_h + \begin{bmatrix} K_{00} \\ K_{10} \end{bmatrix} K_{00}^{-1} \begin{bmatrix} K_{00} & K_{01} \end{bmatrix} \text{ and } |z_h|_1 = \|\underline{z}_h\|_{K_{00}}.$$

Here the following notation is used.

NOTATION 3. *For a positive definite matrix $M \in \mathbb{R}^{n\times n}$ the associated inner product is given by $\langle x, y\rangle_M = \langle Mx, y\rangle$. Both the vector norm and the matrix norm associated with the inner product $\langle \cdot,\cdot\rangle_M$ are denoted by $\|\cdot\|_M$.*

Therefore, as a consequence of Theorem 4.1 the spectrum of the preconditioned matrix $\mathcal{P}_h^{-1}\mathcal{A}_h$ with

$$\mathcal{P}_h = \begin{bmatrix} P_h & 0 \\ 0 & K_{00} \end{bmatrix} \quad \text{and} \quad \mathcal{A}_h = \begin{bmatrix} M_{00} & M_{01} & -K_{00} \\ M_{10} & M_{11} & -K_{10} \\ -K_{00} & -K_{01} & 0 \end{bmatrix} \tag{5.1}$$

is bounded away from 0 and ∞ uniformly with respect to the mesh size h. The application of this preconditioner requires an efficient method for multiplying $\mathcal{P}_h^{-1}$ with a vector, which in general is too costly. In practice, the blocks P_h and K_{00} are replaced by efficient preconditioners $\hat{P}_h$ and $\hat{K}_{00}$, leading to a practical preconditioner,

$$\hat{\mathcal{P}}_h = \begin{bmatrix} \hat{P}_h & 0 \\ 0 & \hat{K}_{00} \end{bmatrix}. \tag{5.2}$$

Standard multilevel or multigrid methods are available for $\hat{K}_{00}$. Therefore, we will concentrate on the construction of an efficient preconditioner $\hat{P}_h$ for P_h.

Lemma 4.2 gives a first hint for preconditioning P_h. In matrix-vector notation it states that

$$\|\underline{v}_h\|_{P_h}^2 \sim \|\underline{\hat{v}}_{h,0}\|_{K_{00}}^2 + \|\underline{\hat{v}}_{h,1}\|_{M_h}^2 \quad \text{for all } \underline{v}_h = \begin{bmatrix} \underline{v}_{h,0} \\ \underline{\mu}_h \end{bmatrix}$$

with

$$\underline{\hat{v}}_{h,0} = \begin{bmatrix} I & K_{00}^{-1} K_{01} \end{bmatrix} \begin{bmatrix} \underline{v}_{h,0} \\ \underline{\mu}_h \end{bmatrix} \quad \text{and} \quad \underline{\hat{v}}_{h,1} = \begin{bmatrix} -K_{00}^{-1} K_{01} \\ I \end{bmatrix} \underline{\mu}_h,$$

which, by elementary calculations, leads to

$$P_h \sim \begin{bmatrix} I & 0 \\ K_{10} K_{00}^{-1} & I \end{bmatrix} \begin{bmatrix} K_{00} & 0 \\ 0 & S_h \end{bmatrix} \begin{bmatrix} I & K_{00}^{-1} K_{01} \\ 0 & I \end{bmatrix}.$$

This motivates the use of preconditioners of the form

$$\hat{P}_h = \begin{bmatrix} I & 0 \\ -\hat{E}_{01}^T & I \end{bmatrix} \begin{bmatrix} \hat{K}_{00} & 0 \\ 0 & \hat{S}_h \end{bmatrix} \begin{bmatrix} I & -\hat{E}_{01} \\ 0 & I \end{bmatrix}$$

with three essential components, $\hat{K}_{00}$, $\hat{S}_h$, and $\hat{E}_{01}$. It is reasonable to choose the same preconditioner for K_{00} in $\hat{P}_h$ as in $\hat{\mathcal{P}}_h$. Candidates for $\hat{S}_h$ are preconditioners for S_h. The third component $\hat{E}_{01}$ is considered as an approximation of $E_{01} = -K_{00}^{-1} K_{01}$. The associated mapping $\hat{E}_h \colon \mathcal{S}_h(\Gamma) \longrightarrow \mathcal{S}_h(\Omega)$, given by

$$\underline{\mu}_h \mapsto \begin{bmatrix} \hat{E}_{01} \\ I \end{bmatrix} \underline{\mu}_h,$$

can be seen as an approximation to the discrete harmonic extension E_h.

Preconditioners of this type have been intensively studied in the context of domain decomposition methods. A typical result reads as follows; see [17] for the proof.

Theorem 5.1. *Assume that $\hat{K}_{00} \sim K_{00}$, $\hat{S}_h \sim S_h$, and that there is a positive constant such that*

$$\left\| \begin{bmatrix} \hat{E}_{01} \\ I \end{bmatrix} \underline{\mu}_h \right\|_{P_h}^2 \le c\, \|\underline{\mu}_h\|_{S_h}^2 \quad \textit{for all } \underline{\mu}_h.$$

Then $\hat{P}_h \sim P_h$.

Observe that the last condition translates to

$$\|\hat{E}_h\, \mu_h\|_{-1,\Delta,h}^2 \le c\, \|E_h\, \mu_h\|_0^2 \quad \text{for all } \mu_h \in \mathcal{S}_h(\Gamma), \tag{5.3}$$

i.e., the approximate harmonic extension has to be bounded with respect to the given norms.

Next we discuss the choice of the two remaining components $\hat{S}_h$ and $\hat{E}_{01}$, which then completes the construction of the preconditioner $\hat{\mathcal{P}}_h$ of $\mathcal{A}_h$, leading to the main result of this section, summarized in Theorem 5.6.

5.1. Schur complement preconditioning. The mapping property of S_h is contained in the following theorem, which does not rely on any convexity assumption. This generalizes a result in [24], where the convex case was considered.

THEOREM 5.2. *For the norm* $\|\cdot\|_{-1/2,\Gamma,h}$ *in* $\mathcal{S}_h(\Omega)$, *given by*

$$\|\mu_h\|^2_{-1/2,\Gamma,h} = \|\mu_h\|^2_{-1/2,\Gamma} + h\,\|\mu_h\|^2_{0,\Gamma},$$

we have

$$\langle S_h\,\underline{\mu}_h, \underline{\mu}_h\rangle = \|E_h\mu_h\|_0^2 \sim \|\mu_h\|^2_{-1/2,\Gamma,h} + \|\Pi_N E_h\,\mu_h\|_0^2 \quad \text{for all } \mu_h \in \mathcal{S}_h(\Gamma), \tag{5.4}$$

where Π_N *is the* L^2*-orthogonal projection onto* N*; see (3.2).*

Proof. We closely follow the proof in [24] and denote the classical harmonic extension operator as a mapping from $H^{1/2}(\Gamma)$ onto $H^1(\Omega)$ by E^1; see the appendix for details. The symbol c is used as a generic constant, which might change its value at each appearance.

For $\mu_h \in \mathcal{S}_h(\Omega) \subset H^{1/2}(\Gamma)$, let $v = E^1\mu_h \in H^1(\Omega)$ and $v_h = E_h\mu_h \in \mathcal{S}_h(\Omega)$ be its harmonic and the discrete harmonic extension, respectively.

For $v^* = (I - \Pi_N)\,v$ and $v_h^* = (I - \Pi_N)\,v_h$, there exists $z \in H^2(\Omega) \cap H^1_0(\Omega)$ with $v^* - v_h^* = \Delta z$, since $\operatorname{im}(I - \Pi_N) = \operatorname{im}\Delta$ for the Laplace operator $\Delta\colon H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$; see (A.3) in the appendix. Then we have

$$\begin{aligned}\|v_h^* - v^*\|_0^2 &= -\int_\Omega (v_h^* - v^*)\,\Delta z\;dx = -\int_\Omega (v_h - v)\,\Delta z\;dx\\ &= \int_\Omega \nabla(v_h - v)\cdot\nabla z\;dx = \int_\Omega \nabla(v_h - v)\cdot\nabla(z - z_h)\;dx\end{aligned}$$

for an arbitrary element $z_h \in \mathcal{S}_{h,0}(\Omega)$. The last identity follows from the Galerkin orthogonality. Therefore, for the pointwise interpolant z_h of z, we obtain

$$\|v_h^* - v^*\|_0^2 \le |v_h - v|_1\,|z - z_h|_1 \le c\,h\,|v|_1\,\|z\|_2 \le c\,h\,\|\mu_h\|_{1/2,\Gamma}\,\|v_h^* - v^*\|_0,$$

using the approximation property of $\mathcal{S}_h(\Omega)$ for functions in $H^2(\Omega)$ and the mapping properties of E^1 and Δ. This implies

$$\|v_h^* - v^*\|_0 \le c\,h\,\|\mu_h\|_{1/2,\Gamma} \le c\,h^{1/2}\,\|\mu_h\|_{0,\Gamma}$$

by using an inverse inequality for the second estimate. Furthermore,

$$\|v^*\|_0 = \|(I - \Pi_N)E^1\mu_h\|_0 = \|E^0\mu_h\|_0 \le c\,\|\mu_h\|_{-1/2,\Gamma};$$

see Theorem A.2 in the appendix, and Theorem 3.3, part 3. Therefore, we obtain

$$\|(I - \Pi_N)E_h\mu_h\|_0 = \|v_h^*\|_0 \le \|v^*\|_0 + \|v_h^* - v^*\|_0 \le \overline{c}\,\left(\|\mu_h\|_{-1/2,\Gamma} + h^{1/2}\,\|\mu_h\|_{0,\Gamma}\right).$$

With $\|E_h\mu_h\|_0^2 = \|(I - \Pi_N)E_h\mu_h\|_0^2 + \|\Pi_N E_h\mu_h\|_0^2$ the second estimate easily follows.

For the first estimate we start with

$$\|E_h\mu_h\|_0 = \|v_h\|_0 \ge \|v_h^*\|_0 \ge \|v^*\|_0 - \|v_h^* - v^*\|_0 \ge c\,\left(\|\mu_h\|_{-1/2,\Gamma} - h^{1/2}\,\|\mu_h\|_{0,\Gamma}\right)$$

Using the inverse inequality $\|v_h\|_0 \geq c\, h^{1/2}\, \|\mu_h\|_{0,\Gamma}$ and $\|E_h \mu_h\|_0 \geq \|\Pi_N E_h \mu_h\|_0$, the first inequality easily follows. □

In order to construct a preconditioner for S_h we start by first considering the term $\|\mu_h\|_{-1/2,\Gamma,h}$ in (5.4) only. A preconditioner for this norm, i.e. an easy to invert approximation to the matrix representing this norm, was already proposed in [24] based on preconditioners for $\|\cdot\|_{-1/2,\Gamma_k}$, $k \geq 1$. We follow this idea but replace the preconditioner for $\|\cdot\|_{-1/2,\Gamma_k}$, for which the FFT was used in [24], by a simpler standard multilevel preconditioner of the type as analyzed in [6].

For this, let $\mathcal{T}_\ell$, $\ell = 0, 1, 2, \ldots, L$, be a hierarchy of uniformly refined subdivisions of Ω of mesh size h_ℓ with $\mathcal{T}_L = \mathcal{T}_h$ with associated finite element spaces $\mathcal{S}_\ell(\Omega)$ of continuous and piecewise linear functions and their trace spaces $\mathcal{S}_\ell(\Gamma)$. Furthermore, let $\mathcal{S}_\ell(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$ be the linear span of all nodal basis functions from $\mathcal{S}_\ell(\Gamma)$ associated with nodes from Γ_C and from Γ_E, respectively.

Then the proposed preconditioner is of additive Schwarz type, given by

$$\left(\hat{S}_h^{(0)}\right)^{-1} = h_L\, R_{C,L}\, A_{C,L}\, R_{C,L}^T + \sum_{\ell=0}^{L} h_\ell\, R_{E,\ell}\, A_{E,\ell}\, R_{E,\ell}^T.$$

Here $R_{C,L}$ and $R_{E,\ell}$ denote the matrix representations of the canonical embeddings of $\mathcal{S}_L(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$ into $\mathcal{S}_L(\Gamma)$, respectively. The matrices $A_{C,L}$ and $A_{E,\ell}$ are given by

$$A_{C,L} = \bar{M}_{C,L}^{-1}\, K_{C,L}\, \bar{M}_{C,L}^{-1}, \quad A_{E,\ell} = \bar{M}_{E,\ell}^{-1}\, K_{E,\ell}\, \bar{M}_{E,\ell}^{-1},$$

where $\bar{M}_{C,L}$ and $\bar{M}_{E,\ell}$ are the matrix representations of the discrete version of the norm $\|\cdot\|_{0,\Gamma}$ which results from the elementwise use of the trapezoidal rule on $\mathcal{S}_L(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$, respectively. $\bar{K}_{C,L}$ and $\bar{K}_{E,\ell}$ are the matrix representations of the norm $|\cdot|_{1,\Gamma}$ on $\mathcal{S}_L(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$, respectively, where this norm is given by

$$|\mu_h|_{1,\Gamma}^2 = \sum_{k=1}^{K} \int_{\Gamma_k} |\nabla_{\Gamma_k} \mu_h|^2\, ds \quad \text{with the tangential gradient } \nabla_{\Gamma_k}.$$

Observe that the boundary mass matrices $\bar{M}_{C,L}$ and $\bar{M}_{E,\ell}$ are diagonal. So, the application of the preconditioner requires only the multiplication by boundary stiffness matrices $\bar{K}_{C,L}$ and $\bar{K}_{E,\ell}$ and some componentwise scaling on each refinement level.

Now we have the next theorem.

THEOREM 5.3. *$\|\mu_h\|_{-1/2,\Gamma,h}^2 \sim \langle \hat{S}_h^{(0)}\, \underline{\mu}_h, \underline{\mu}_h \rangle$ for all $\mu_h \in \mathcal{S}_h(\Gamma)$.*

Proof. Each part of the proof is based on fairly standard arguments from [24], [23], [6], and [14]. We just have to put known things together.

First of all, for the decomposition $\mathcal{S}_L(\Gamma) = \mathcal{S}_L(\Gamma_C) \oplus \mathcal{S}_L(\Gamma_E)$ we obtain

$$\|\mu_L\|_{-1/2,\Gamma,h}^2 \sim h_L\, \|\mu_C\|_{0,\Gamma}^2 + \|\mu_E\|_{-1/2,\Gamma}^2$$

for all $\mu_L \in \mathcal{S}_L(\Gamma)$ with $\mu_L = \mu_C + \mu_E$, $\mu_C \in \mathcal{S}_L(\Gamma_C)$, $\mu_E \in \mathcal{S}_L(\Gamma_E)$, see [24, Proposition 7.5].

Next, using the multiscale representation of the norm $\|\cdot\|_{-1/2,\Gamma_k}$ from Theorem 4 in [23], following the main idea from [6] and replacing the norm $\|\cdot\|_{0,\Gamma_k}$ in this representation by the norm $h_\ell^{-1}\,\|\cdot\|_{-1,\Gamma_k}$, and applying Proposition 3 from [14] one obtains

$$\|\mu_E\|^2_{-1/2,\Gamma} \sim \inf\left\{\sum_{\ell=0}^{L} h_\ell^{-1}\,\|\mu_\ell\|^2_{-1,\Gamma}\colon \mu_E = \sum_{\ell=0}^{L}\mu_\ell,\ \mu_\ell \in \mathcal{S}_\ell(\Gamma_E)\right\}$$

for all $\mu_E \in \mathcal{S}_L(\Gamma_E)$, where $\|\cdot\|_{-1,\Gamma}$ denotes the canonical norm in $H^{-1}_{pw}(\Gamma)$.

Finally, using

$$\|\mu_\ell\|^2_{-1,\Gamma} \sim \langle A^{-1}_{E,\ell}\,\underline{\mu_\ell}, \underline{\mu_\ell}\rangle \quad \text{for all } \mu_\ell \in \mathcal{S}_\ell(\Gamma_E)$$

from [6] and

$$h_L\,\|\mu_C\|^2_{0,\Gamma} \sim h_L^{-1}\,\langle A^{-1}_{C,L}\,\underline{\mu}_C, \underline{\mu}_C\rangle \quad \text{for all } \mu_C \in \mathcal{S}_L(\Gamma_C),$$

which follows from a simple scaling argument, we obtain a stable space decomposition, whose associated additive Schwarz operator is $\hat{S}_h^{(0)}$. ☐

For studying the term $\|\Pi_N E_h\,\mu_h\|_0^2$ in (5.4) we assume that a basis $\{s_1,\ldots,s_J\}$ of N is known. Then we have

$$\|\Pi_N v_h\|_0^2 = \langle M^T_{Nh} M_N^{-1}\, M_{Nh}\,\underline{v}_h, \underline{v}_h\rangle$$

with the mass matrices

$$M_N = \left(\int_\Omega s_i\, s_j\; dx\right) \quad \text{and} \quad M_{Nh} = \left(\int_\Omega s_i\,\varphi_j\; dx\right),$$

where $\{\varphi_1,\varphi_2,\ldots,\varphi_I\}$ denotes the nodal basis of $\mathcal{S}_h(\Omega)$. Hence

$$\|\Pi_N E_h\,\mu_h\|_0^2 = \left\langle M^T_{Nh} M_N^{-1}\, M_{Nh}\begin{bmatrix} E_{01} \\ I\end{bmatrix}\underline{\mu}_h, \begin{bmatrix} E_{01} \\ I\end{bmatrix}\underline{\mu}_h\right\rangle = \left\langle U_h M_N^{-1}\, U_h^T\,\underline{\mu}_h, \underline{\mu}_h\right\rangle$$

with $U_h = \begin{bmatrix} E_{01}^T & I\end{bmatrix} M^T_{Nh}$. Observe that the rank of $U_h M_N^{-1}\, U_h^T$ is equal to the dimension of N, i.e., the (fixed) number of reentrant corners.

So, in summary, we obtain

$$\langle S_h\,\underline{\mu}_h, \underline{\mu}_h\rangle = \|E_h\,\mu_h\|_0^2 \sim \left\langle \left[\hat{S}_h^{(0)} + U_h M_N^{-1} U_h^T\right]\,\underline{\mu}_h, \underline{\mu}_h\right\rangle \quad \text{for all } \mu \in \mathcal{S}_h(\Gamma),$$

which completes the proof of the next theorem.

Theorem 5.4. *$\hat{S}_h^{(1)} \sim S_h$ with $\hat{S}_h^{(1)} = \hat{S}_h^{(0)} + U_h M_N^{-1} U_h^T$.*

Furthermore, we have

$$\left(\hat{S}_h^{(1)}\right)^{-1}\underline{\mu}_h = \left[I - V_h(M_N + U_h^T V_h)^{-1} U_h^T\right]\left(\hat{S}_h^{(0)}\right)^{-1}\underline{\mu}_h \quad \text{with} \quad V_h = \left(\hat{S}_h^{(0)}\right)^{-1} U_h$$

by the Sherman-Morrison-Woodbury formula. The matrix $I - V_h(M_N + U_h^T V_h)^{-1} U_h^T$ has to be computed only once and the computational costs are rather low for domains with a small number of reentrant corners. This makes $\hat{S}_h^{(1)}$ an efficient preconditioner for S_h.

Remark 3. *A basis $\{s_1,\ldots,s_J\}$ of N is given by $s_j = s_{-j}(r_j,\theta_j) - w_j$ with*

$$s_{-j}(r_j,\theta_j) = \eta(r_j)\, r_j^{-\pi/\omega_j}\,\sin((\omega_j/\pi)\,\theta_j)$$

and $w_j \in H_0^1(\Omega)$*, given by the variational problem*

$$\int_\Omega \nabla w_j \cdot \nabla v \; dx = \int_\Omega (-\Delta s_{-j})\, v \; dx \quad \text{for all } v \in H_0^1(\Omega). \tag{5.5}$$

Here (r_j, θ_j) *denotes the polar coordinates centered at a reentrant corner with internal angle* ω_j *spanned by* $\theta_j = 0$ *and* $\theta_j = \omega_j$*, and* $\eta(r_j)$ *is a cutoff function which is identical to 1 in a neighborhood of the corner; see [16], [15].*

5.2. Approximate discrete harmonic extensions. The evaluation of $v_h = E_h\, \mu_h$, where E_h is the discrete harmonic extension, requires the exact solve of the linear system

$$K_{00}\, \underline{v}_{h,0} = -K_{01}\, \underline{\mu}_h.$$

If instead we use an inner iteration by performing r steps of the Richardson method with preconditioner $\hat{K}_{00}$ and initial guess 0, then we end up with an approximate harmonic extension $\hat{E}_h^{(r)}$, given by

$$\underline{\mu}_h \mapsto \begin{bmatrix} \hat{E}_{01}^{(r)} \\ I \end{bmatrix} \underline{\mu}_h \quad \text{with} \quad \hat{E}_{01}^{(r)} = \left[I - (I - \hat{K}_{00}^{-1} K_{00})^r \right] E_{01}. \tag{5.6}$$

(For $r = 1$ we simply get $\hat{E}_{01}^{(1)} = -\hat{K}_{00}^{-1} K_{01}$.) We will now show that this approximate discrete harmonic extension satisfies the third condition of Theorem 5.1 under reasonable assumptions.

LEMMA 5.5. *Assume that the inner iteration converges in the corresponding energy norm with a convergence rate* $q < 1$ *which is independent of* h*, i.e.,*

$$\|I - \hat{K}_{00}^{-1} K_{00}\|_{K_{00}} \le q.$$

Then, for $r = \mathcal{O}(|\ln h|)$*, there is a constant* c *such that*

$$\|\hat{E}_h^{(r)}\, \mu_h\|_{-1,\Delta,h} \le c\, \|E_h\, \mu_h\|_0 \quad \text{for all } \mu_h \in \mathcal{S}_h(\Gamma).$$

Proof. For all $\mu_h \in \mathcal{S}_h(\Gamma)$, we have

$$\begin{aligned} \|\hat{E}_h^{(r)}\, \mu_h\|_{-1,\Delta,h} &\le \|E_h\, \mu_h\|_{-1,\Delta,h} + \|(\hat{E}_h^{(r)} - E_h)\, \mu_h\|_{-1,\Delta,h} \\ &= \|E_h\, \mu_h\|_0 + \|(\hat{E}_h^{(r)} - E_h)\, \mu_h\|_1 \\ &\le \|E_h\, \mu_h\|_0 + (c_F^2 + 1)^{1/2}\, |(\hat{E}_h^{(r)} - E_h)\, \mu_h|_1. \end{aligned}$$

Now it easily follows that

$$\begin{aligned} |(\hat{E}_h^{(r)} - E_h)\, \mu_h|_1 &= \|(\hat{E}_{01}^{(r)} - E_{01})\, \underline{\mu}_h\|_{K_{00}} = \|(I - \hat{K}_{00}^{-1} K_{00})^r E_{01}\, \underline{\mu}_h\|_{K_{00}} \\ &\le q^r\, \|E_{01}\, \underline{\mu}_h\|_{K_{00}} \le q^r\, |E_h\, \mu_h|_1 \le c\, q^r h^{-1}\, \|E_h\, \mu_h\|_0. \end{aligned}$$

If $r = \mathcal{O}(|\ln h|)$, the factor $q^r h^{-1}$ is uniformly bounded, which completes the proof. □

To summarize the discussion on preconditioning, Theorem 5.1, Theorem 5.4, and Lemma 5.5 in connection with (5.3) lead to the following main result of this section.

THEOREM 5.6. *The spectrum of the preconditioned matrix* $\hat{\mathcal{P}}_h^{-1}\mathcal{A}_h$ *with* $\mathcal{A}_h$ *given by (5.1) and*

$$\hat{\mathcal{P}}_h = \begin{bmatrix} \hat{P}_h & 0 \\ 0 & \hat{K}_{00} \end{bmatrix} \quad \text{with} \quad \hat{P}_h = \begin{bmatrix} I & 0 \\ -\hat{E}_{01}^T & I \end{bmatrix} \begin{bmatrix} \hat{K}_{00} & 0 \\ 0 & \hat{S}_h \end{bmatrix} \begin{bmatrix} I & -\hat{E}_{01} \\ 0 & I \end{bmatrix},$$

where $\hat{K}_{00} \sim K_{00}$, $\hat{E}_{01} = \hat{E}_h^{(r)}$ *with* $r = \mathcal{O}(|\ln h|)$, *given by (5.6), and* $\hat{S}_h = \hat{S}_h^{(1)}$ *(see Theorem 5.4), is bounded away from 0 and* ∞ *uniformly with respect to the mesh size* h. *And, consequently, the condition number of* $\hat{\mathcal{P}}_h^{-1}\mathcal{A}_h$ *is bounded independently of the mesh size* h.

REMARK 4. *If* Ω *is simply connected, then the space* $H^{-1}(\Delta, \Omega)$ *is identical to the space* Σ *(see Remark 1), which was introduced in [3] not only for the Ciarlet-Raviart method but also for a mixed method of the vector Laplacian with Dirichlet boundary conditions and for the vorticity-velocity-pressure formulation of the Stokes problem. Therefore, the preconditioner* $\hat{P}_h$ *can also be used as an essential part of a block preconditioner for these problems.*

6. Numerical experiments. We consider the simple biharmonic test problem

$$\Delta^2 y = f \quad \text{in } \Omega, \qquad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma$$

on two domains, the square $\Omega = \Omega_S = (-1,1)^2$ (representing the convex case) and the L-shaped domain $\Omega = \Omega_L$ depicted in figures 1 and 2, where also the initial mesh (level $\ell = 0$) is shown. The right-hand side $f(x)$ is chosen such that

$$y(x) = \left[1 - \cos(2\pi x_1)\right] \left[1 - \cos(4\pi x_2)\right]$$

is the exact solution to the problem. The initial meshes are uniformly refined until the final level $\ell = L$.

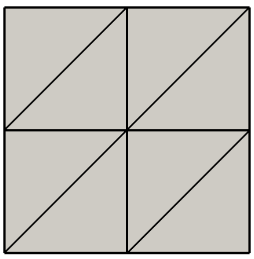

FIG. 1. $\Omega = \Omega_S$

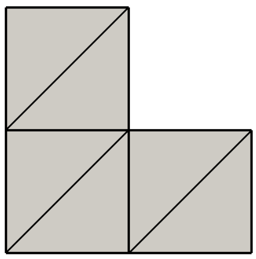

FIG. 2. $\Omega = \Omega_L$

We will present numerical results demonstrating the quality of the preconditioners $\hat{S}_h^{(0)}$ and (in the nonconvex case) $\hat{S}_h^{(1)}$ for S_h and the preconditioner $\hat{\mathcal{P}}_h$ for $\mathcal{A}_h$. The Schur complement preconditioners were tested by applying the preconditioned gradient (PG), the conjugate gradient method (CG), and its preconditioned variant (PCG) to (4.3); the method of choice for (4.2) was the preconditioned minimal residual method (PMINRES). In all experiments a reduction of the Euclidean norm of the initial residual by a factor of 10^{-8} was used as stopping criterion for the iterative methods, where the initial guess was chosen randomly out of the range spanned by the corresponding exact quantities.

For the preconditioner $\hat{K}_{00}$, which is used as the first diagonal block in $\hat{P}_h$, as the last diagonal block in $\hat{\mathcal{P}}_h$, and as a preconditioner in the inner iteration for the approximate discrete harmonic extension, we always choose one multigrid V-cycle with one step of forward and backward Gauss-Seidel smoothing. The action of the exact inverse K_{00}, as needed for S_h and for the exact discrete harmonic extension E_h, was realized by applying an inner iteration with 10 V-cycles. The preconditioner $\hat{S}_h^{(1)}$ was constructed as described in Remark 3 with the modification that the solution w_j to (5.5) was replaced by the corresponding approximate solution in $\mathcal{S}_{h,0}(\Omega)$.

Table 1 shows the observed number of iterations for (4.3) in the convex case $\Omega = \Omega_S$. The first column contains the level L of refinement. The next three columns show the results for CG and for PG and PCG, both with the preconditioner $\hat{S}_h^{(0)}$.

Table 1
Number of iterations for (4.3), $\Omega = \Omega_S$ (square)

L	CG	PG	PCG
6	66	77	20
7	91	77	21
8	120	75	21
9	160	71	20

As expected the number of iterations grows for CG without preconditioning if the mesh size decreases. The second column shows that the preconditioner alone with PG already leads to convergence rates which are uniformly bounded in h. Of course, the use of this preconditioner in PCG results in a further reduction of the number of iterations.

Table 2 shows the results for the L-shaped domain $\Omega = \Omega_L$ representing a non-convex case. The second and third columns contain the numbers of iterations for PG with the preconditioners $\hat{S}_h^{(0)}$ and $\hat{S}_h^{(1)}$, respectively, while in the next two columns the corresponding results for PCG are shown.

Table 2
Number of iterations for (4.3), $\Omega = \Omega_L$

L	PG($\hat{S}_h^{(0)}$)	PG($\hat{S}_h^{(1)}$)	PCG($\hat{S}_h^{(0)}$)	PCG($\hat{S}_h^{(1)}$)
6	337	77	24	21
7	427	81	25	22
8	385	81	26	22
9	457	81	25	21

By comparing the second and the third columns one sees that $\hat{S}_h^{(1)}$ performs significantly better as a preconditioner in PG than $\hat{S}_h^{(0)}$, as expected from the analysis. Nevertheless, as seen from the fourth column, PCG works well for the nonoptimal preconditioner $\hat{S}_h^{(0)}$. A relevant further improvement by using the better preconditioner $\hat{S}_h^{(0)}$ in PCG was not observed. This important feature was observed here experimentally and is not yet supported by analysis. If confirmed, this would considerably contribute to practicability, in particular, for a possible extension to three-dimensional problems; see the concluding remarks.

Finally, Table 3 shows some preliminary results for preconditioning the nonreduced system (4.2), whose total numbers n of unknowns are shown in the second column. PMINRES was applied to the L-shaped domain $\Omega = \Omega_L$; for preconditioning the Schur complement the nonoptimal preconditioner $\hat{S}_h^{(0)}$ was used. The third and the fourth columns contain the results if using the costly exact discrete harmonic extension E_h and the less costly approximate version $\hat{E}_h^{(r)}$ with r inner iterations, respectively. The chosen value of r is shown in parentheses in the fourth column.

TABLE 3
Number of iterations for (4.2), $\Omega = \Omega_L$

L	n	PMINRES(E_h)	PMINRES($\hat{E}_h^{(r)}$)
4	1,538	46	43 (3)
5	6,146	45	48 (3)
6	24,578	43	47 (4)
7	98,306	43	48 (4)
8	393,218	44	46 (5)
9	1,572,866	44	56 (5)

It can be seen that a modest increase in the number r of inner iterations keeps the number of iterations in the range of the observed number of iterations if using the costly exact discrete harmonic extension. This is in accordance with Lemma 5.5.

7. Concluding remarks. Efficient Schur complement preconditioners were derived and analyzed for convex and non-convex polygonal domains, respectively. There is experimental evidence that $\hat{S}_h^{(0)}$ also works fine for nonconvex polygonal domains in combination with a Krylov subspace method (PCG for (4.3) or PMINRES for (4.2)). This is especially advantageous for a possible extension to three-dimensional problems, where $\hat{S}_h^{(1)}$ would be much harder to construct. Preconditioning the reduced system (4.3) has, therefore, reached a satisfactory state. Observe, however, that the computational costs for evaluating one residual for (4.3) is relatively high, since it requires the application of (an accurate approximation of) the discrete harmonic extension E_h twice.

The situation is less clear for the nonreduced problem (4.2). Here the evaluation of the residual is computationally inexpensive. The computational costs for applying $\hat{\mathcal{P}}_h$ depend mainly on the choice for $\hat{E}_h$. If $\hat{E}_h = E_h$, the computational costs of one step of PCG for (4.3) and one step of PMINRES for (4.2) are roughly the same. The number of iterations differs by a factor of about 2 (see Table 1, third column and Table 3, last column), and so do the observed computing times, as expected. Possible improvements are to use symmetric indefinite preconditioners (see [27]), based on the same components as the proposed symmetric and positive definite block diagonal preconditioner, in particular with the same extension $\hat{E}_h = E_h$, in combination of Krylov subspace methods such as GMRES. Another possible improvement is the replacement of E_h by more efficient approximate discrete harmonic extensions. The few numerical experiments with an inner iteration as shown in Table 3 already lead to an improvement in computing time by almost a factor of 2. Efficient approximate harmonic extensions are well-developed and understood as bounded operators from $\mathcal{S}_h(\Gamma) \subset H^{1/2}(\Gamma)$ to $H^1(\Omega)$; see, e.g., [17]. Here the challenge of future work is the construction of efficient approximate harmonic extensions which satisfy (5.3).

Appendix. Harmonic extension operators. The trace space of functions from $H^1(\Omega)$ is $H^{1/2}(\Gamma)$. The well-known harmonic extension operator

$$E^1 \colon H^{1/2}(\Omega) \longrightarrow H^1(\Omega)$$

is given by the following variational problem: For $\mu \in H^{1/2}(\Gamma)$, find $v = E^1\lambda \in H^1(\Omega)$ such that $v\big|_\Gamma = \mu$ and

$$\int_\Omega \nabla v \cdot \nabla z \; dx = 0 \quad \text{for all } z \in H_0^1(\Omega). \tag{A.1}$$

An essential property of E^1 is

$$|E^1\mu|_1^2 \sim \|\mu\|_{1/2,\Gamma}^2 \quad \text{for all } \mu \in H^{1/2}(\Gamma),$$

where $\|\cdot\|_{1/2,\Gamma}$ denotes the standard norm in $H^{1/2}(\Gamma)$.

A harmonic extension operator

$$E^0 \colon H_{pw}^{-1/2}(\Gamma) \longrightarrow H^{-1}(\Delta, \Omega)$$

which is more appropriate in the context of this paper is given by the following variational problem: For $\mu \in H_{pw}^{-1/2}(\Gamma)$, find $v = E^0\mu \in \operatorname{im}\Delta$ such that $\gamma^0 v = \mu$ and

$$\int_\Omega v\, \Delta z \; dx = \langle \mu, \gamma^1 z\rangle_\Gamma \quad \text{for all } z \in H^2(\Omega) \cap H_0^1(\Omega) \tag{A.2}$$

with

$$\langle \mu, g\rangle_\Gamma = \sum_{k=1}^K \langle \mu_k, g_k\rangle_{\Gamma_k} \text{ for } \mu = (\mu_k)_{k=1,\dots,K} \in H_{pw}^{-1/2}(\Gamma),\ g = (g_k)_{k=1,\dots,K} \in H_{pw}^{1/2}(\Gamma),$$

where $\langle \cdot, \cdot \rangle_{\Gamma_k}$ denotes the duality product in $H_{pw}^{-1/2}(\Gamma) \times H_{pw}^{1/2}(\Gamma)$ and γ^1 is the trace operator, given by

$$\gamma^1 z = (\gamma_k^1 z)_{k=1,\dots,K} \quad \text{with} \quad \gamma_k^1 z = \frac{\partial z}{\partial n}\Big|_{\Gamma_k} \quad \text{for} \quad z \in H^2(\Omega) \cap H_0^1(\Omega).$$

Here, $H_{pw}^{1/2}(\Gamma)$ denotes the product space $\prod_{k=1}^K H^{1/2}(\Gamma_k)$ and $\operatorname{im}\Delta$ is the image of the Laplace operator $\Delta \colon H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$.

For this definition of Δ, it is known that

$$L^2(\Omega) = \operatorname{im}\Delta \perp N, \tag{A.3}$$

which immediately implies that $\operatorname{im}\Delta = N^\perp = \operatorname{im}(I - \Pi_N)$; see [16], [15].

Remark 5. *For smooth functions on $\overline{\Omega}$ the right-hand side in (A.2) can be rewritten in a more traditional fashion as*

$$\langle \mu, \gamma^1 z\rangle_\Gamma = \int_\Gamma \mu \, \frac{\partial z}{\partial n} \; dS.$$

Variants of (A.2) are often called very weak formulations of the corresponding Dirichlet problem for the Laplace operator.

In [12, page 184] this harmonic extension operator was studied for smooth and for convex polygonal domains Ω, although on a not-yet-appropriate trace space in the case of convex polygonal domains. We will show now that E^0 is well-defined on general polygonal domains.

THEOREM A.1. *For each $\mu \in H_{pw}^{-1/2}(\Gamma)$, there is a unique solution $v = E^0\mu \in \operatorname{im}(\Delta)$ to (A.2) and*

$$\|E^0\mu\|_0^2 \sim \|\mu\|_{-1/2,\Gamma}^2 \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma).$$

Proof. For $v = \Delta w$ with $w \in W = H^2(\Omega) \cap H_0^1(\Omega)$, problem (A.2) coincides with the second biharmonic boundary value problem for w, which is known to be well-posed. For given $\mu \in H_{pw}^{-1/2}(\Gamma)$, let w_μ be its solution. Then

$$\|w_\mu\|_2 \sim \|\ell_\mu\|_{W^*} \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma),$$

where ℓ_μ denotes the linear functional on the right-hand side of (A.2), given by $z \mapsto \langle \mu, \gamma^1 z\rangle_\Gamma$.

The trace operator $\gamma^1 \colon W \longrightarrow \widetilde{H}_{pw}^{1/2}(\Gamma)$ is well-defined, bounded and surjective; see [16]. Therefore, the bilinear form $(z, \mu) \mapsto \langle \mu, \gamma^1 z\rangle_\Gamma$ is well-defined and bounded on $W \times H_{pw}^{-1/2}(\Gamma)$, and it satisfies an inf-sup condition. Therefore,

$$\|\ell_\mu\|_{W^*} \sim \|\mu\|_{-1/2,\Gamma} \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma).$$

Using $\|\Delta w\|_0 \sim \|w\|_2$ for all $w \in H^2(\Omega) \cap H_0^1(\Omega)$, we finally obtain

$$\|E^0\mu\|_0 = \|\Delta w_\mu\|_0 \sim \|\ell_\mu\|_{W^*} \sim \|\mu\|_{-1/2,\Gamma} \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma). \qquad \square$$

We have the following relation between E^1 and E^0 on the domain $H^{1/2}(\Gamma)$, where both extension operators exist.

THEOREM A.2. *$E^0\mu = (I - \Pi_N)E^1\mu$ for all $\mu \in H^{1/2}(\Gamma)$.*

Proof. Both $E^0\mu$ and $E^1\lambda$ are harmonic and have the same trace μ. Therefore, $E^1\mu - E^0\mu \in N$, i.e., there is an $n \in N$ with $E^1\mu = E^0\mu + n$. Moreover, $E^0\mu \in \operatorname{im}\Delta$ by definition. From (A.3) it follows that $n = \Pi_N E^1\mu$, which implies $E^0\mu = E^1\mu - n = E^1\mu - \Pi_N E^1\mu$. $\square$

Theorem 3.3 is a simple consequence of the last two theorems.

REMARK 6. *For convex domains, N is trivial. Only in this case the two harmonic extension operators coincide.*

Acknowledgment. The author would like to thank the anonymous referees for their valuable comments and suggestions, which helped to improve this manuscript. In particular, the suggestion to discuss the relation to the space Σ (see Remark 1) and its implications for other mixed methods was very helpful.

REFERENCES

[1] R. A. ADAMS AND J. J. F. FOURNIER, *Sobolev spaces. 2nd ed.*, Pure and Applied Mathematics 140. New York, NY: Academic Press, 2003.

[2] M. Amara and F. Dabaghi, *An optimal C^0 finite element algorithm for the 2D biharmonic problem: Theoretical analysis and numerical results*, Numer. Math., 90 (2001), pp. 19–46.
[3] D. N. Arnold, R. S. Falk, and J. Gopalakrishnan, *Mixed finite element approximation of the vector Laplacian with Dirichlet boundary conditions*, Math. Models Methods Appl. Sci., 22 (2012), pp. 1250024, 26.
[4] I. Babuška, J. Osborn, and J. Pitkaeranta, *Analysis of mixed methods using mesh dependent norms*, Math. Comput., 35 (1980), pp. 1039–1062.
[5] C. Bernardi, V. Girault, and Y. Maday, *Mixed spectral element approximation of the Navier-Stokes equations in the stream-function and vorticity formulation*, IMA J. Numer. Anal., 12 (1992), pp. 565–608.
[6] J. H. Bramble, Z. Leyk, and J. E. Pasciak, *The analysis of multigrid algorithms for pseudodifferential operators of order minus one*, Math. Comput., 63 (1994), pp. 461–478.
[7] D. Boffi; F. Brezzi and M. Fortin, *Mixed Finite Element Methods and Applications*, Berlin: Springer, 2013.
[8] P. G. Ciarlet, *The Finite Element Methods for Elliptic Problems*, Classics in Applied Mathematics. 40. Philadelphia, PA: SIAM, 2002.
[9] P. G. Ciarlet and R. Glowinski, *Dual iterative techniques for solving a finite element approximation of the biharmonic equation*, Computer Methods Appl. Mech. Engin., 5 (1975), pp. 277–295.
[10] P. G. Ciarlet and P.-A. Raviart, *A mixed finite element method for the biharmonic equation*, in Math. Aspects of Finite Elements in Partial Differential Equations, Proc. Symp. Madison 1974, Carl de Boor, ed., Academic Press, New York, 1974, pp. 125 – 145.
[11] R. S. Falk and J. E. Osborn, *Error estimates for mixed methods*, RAIRO, Anal. Numér., 14 (1980), pp. 249–277.
[12] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. (Extended version of the 1979 publ.)*, Springer Series in Computational Mathematics, 5. Berlin etc.: Springer-Verlag, 1986.
[13] R. Glowinski and O. Pironneau, *Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem*, SIAM Rev., 21 (1979), pp. 167–212.
[14] M. Griebel and P. Oswald, *Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems*, Adv. Comput. Math., 4 (1995), pp. 171–206.
[15] P. Grisvard, *Singularities in Boundary Value Problems*, Recherches en Mathématiques Appliquées. 22. Paris: Masson. Berlin: Springer-Verlag, 1992.
[16] ———, *Elliptic Problems in Nonsmooth Domains. Reprint of the 1985 hardback ed.*, Classics in Applied Mathematics 69. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2011.
[17] G. Haase, U. Langer, A. Meyer, and S. V. Nepomnyaschikh, *Hierarchical extension operators and local multigrid methods in domain decomposition preconditioners*, East-West J. Numer. Math., 2 (1994), pp. 173–193.
[18] R. Hiptmair, *Operator preconditioning*, Comput. Math. Appl., 52 (2006), pp. 699–706.
[19] J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications. Vol. I*, Die Grundlehren der mathematischen Wissenschaften. Band 181. Berlin-Heidelberg-New York: Springer-Verlag, 1972.
[20] K.-A. Mardal and R. Winther, *Preconditioning discretizations of systems of partial differential equations*, Numer. Linear Algebra Appl., 18 (2011), pp. 1–40.
[21] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge: Cambridge University Press, 2000.
[22] J. Nečas, *Les Méthodes directes en Théorie des Équations elliptiques*, Paris: Masson et Cie; Prague: Academia, 1967.
[23] P. Oswald, *Multilevel norms for $H^{-1/2}$*, Computing, 61 (1998), pp. 235–255.
[24] P. Peisker, *On the numerical solution of the first biharmonic equation*, RAIRO, Anal. Numér., 22 (1988), pp. 655–676.
[25] A.H. Schatz and L.B. Wahlbin, *Maximum norm estimates in the finite element method on plane polygonal domains. Part 1*, Math. Comput., 32 (1978), pp. 73–109.
[26] R. Scholz, *A mixed method for 4th order problems using linear finite elements*, RAIRO, Anal. Numér., 12 (1978), pp. 85–90.
[27] D. J. Silvester and M. D. Mihajlović, *A black-box multigrid preconditioner for the biharmonic equation*, BIT, 44 (2004), pp. 151–163.

Latest Reports in this series

2009 - 2011

[..]

2012

[..]		
2012-07	Helmut Gfrerer *On Directional Metric Subregularity and Second-Order Optimality Conditions for a Class of Nonsmooth Mathematical Programs*	August 2012
2012-08	Michael Kolmbauer and Ulrich Langer *Efficient Solvers for Some Classes of Time-Periodic Eddy Current Optimal Control Problems*	November 2012
2012-09	Clemens Hofreither, Ulrich Langer and Clemens Pechstein *FETI Solvers for Non-Standard Finite Element Equations Based on Boundary Integral Operators*	November 2012
2012-10	Helmut Gfrerer *On Metric Pseudo-(sub)Regularity of Multifunctions and Optimality Conditions for Degenerated Mathematical Programs*	December 2012
2012-11	Clemens Pechstein and Clemens Hofreither *A Rigorous Error Analysis of Coupled FEM-BEM Problems with Arbitrary Many Subdomains*	December 2012
2012-12	Markus Eslitzbichler, Clemens Pechstein and Ronny Ramlau *An H^1-Kaczmarz Reconstructor for Atmospheric Tomography*	December 2012
2012-13	Clemens Pechstein *On Iterative Substructuring Methods for Multiscale Problems*	December 2012

2013

2013-01	Ulrich Langer and Monika Wolfmayr *Multiharmonic Finite Element Analysis of a Time-Periodic Parabolic Optimal Control Problem*	January 2013
2013-02	Helmut Gfrerer *Optimality Conditions for Disjunctive Programs Based on Generalized Differentiation with Application to Mathematical Programs with Equilibrium Constraints*	March 2013
2013-03	Clemens Hofreither, Ulrich Langer and Clemens Pechstein *BEM-based Finite Element Tearing and Interconnecting Methods*	May 2013
2013-04	Irina Georgieva and Clemens Hofreither *Cubature Rules for Harmonic Functions Based on Radon Projections*	June 2013
2013-05	Astrid Pechstein and Clemens Pechstein *A FETI Method For A TDNNS Discretization of Plane Elasticity*	August 2013
2013-06	Peter Gangl and Ulrich Langer *Topology Optimization of Electric Machines Based on Topological Sensitivity Analysis*	September 2013
2013-07	Walter Zulehner *The Ciarlet-Raviart Method for Biharmonic Problems on General Polygonal Domains: Mapping Properties and Preconditioning*	October 2013

From 1998 to 2008 reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`