The Ciarlet-Raviart Method for Biharmonic Problems on General Polygonal Domains: Mapping Properties and Preconditioning

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THE CIARLET-RAVIART METHOD FOR BIHARMONIC PROBLEMS ON GENERAL POLYGONAL DOMAINS: MAPPING PROPERTIES AND PRECONDITIONING

WALTER ZULEHNER

Abstract. For biharmonic boundary value problems, the Ciarlet-Raviart mixed method is considered on polygonal domains without additional convexity assumptions. Mapping properties of the involved operators on the continuous as well as on the discrete level are studied. Based on this, efficient preconditioners are constructed and numerical experiments are shown.

Key words. biharmonic equation, Ciarlet-Raviart method, mixed methods, mapping properties, preconditioning

AMS subject classifications. 65N22, 65F08, 65F10

1. Introduction. We consider the first biharmonic boundary value problem: Find \( y \) such that

\begin{equation}
\Delta^2 y = f \quad \text{in} \ \Omega, \quad y = \frac{\partial y}{\partial n} = 0 \quad \text{on} \ \Gamma,
\end{equation}

where \( \Omega \) is an open and bounded set in \( \mathbb{R}^2 \) with a polygonal Lipschitz boundary \( \Gamma \), \( \Delta \) and \( \partial / \partial n \) denote the Laplace operator and the derivative in the direction normal to the boundary, respectively, and \( f \in H^{-1}(\Omega) \). Here and throughout the paper we use \( L^2(\Omega), H^m(\Omega), \text{and } H^0_0(\Omega) \) with its dual space \( H^{-m}(\Omega) \) to denote the standard Lebesgue and Sobolev spaces with corresponding norms \( \| \cdot \|_0, \| \cdot \|_m, | \cdot |_m, \text{and } \| \cdot \|_{-m} \) for positive integers \( m \); see, e.g., [1]. Problems of this type occur, for example, in fluid mechanics, where \( y \) is the stream function of a two-dimensional Stokes flow (see, e.g., [12]), and in elasticity, where \( y \) is the vertical deflection of a clamped Kirchhoff plate (see, e.g., [8]).

The standard (primal) variational formulation of (1.1) reads as follows: Find \( y \in H^2_0(\Omega) \) such that

\begin{equation}
\int_{\Omega} \Delta y \Delta z \ dx = \langle f, z \rangle \quad \text{for all } z \in H^2_0(\Omega),
\end{equation}

where \( \langle \cdot, \cdot \rangle \) denotes the duality product in \( H^* \times H \) for a Hilbert space \( H \) with dual \( H^* \); here for \( H = H^1_0(\Omega) \). (If \( H = \mathbb{R}^n \), we use \( \langle \cdot, \cdot \rangle \) for the Euclidean inner product.) Existence and uniqueness of a solution to (1.2) is guaranteed by the Lax-Milgram theorem; see, e.g., [22], [19].

Conforming finite element methods based on (1.2) require approximation spaces of continuously differentiable functions, which are not so easy to construct for unstructured meshes. Another challenging issue for (1.2) is the construction of efficient preconditioners for iterative methods for solving a discretized version of (1.2). Standard techniques which might help to resolve these difficulties are discontinuous Galerkin methods or mixed methods. We focus here on the well-known mixed method

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by Ciarlet and Raviart (see [10]), for which an auxiliary variable
\[ u = -\Delta y \]
is introduced. For the Stokes problem \( u \) is the vorticity of the flow, and for plate bending problems \( u \) can be interpreted as bending moment. With this auxiliary variable the fourth-order differential equation in (1.1) can be rewritten as a system of two second-order equations leading to the boundary value problem: Find \( y \) and \( u \) such that
\[ -\Delta y = u, \quad -\Delta u = f \quad \text{in } \Omega, \quad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma. \]

Finite element methods for (1.3) were studied on convex domains \( \Omega \) by many authors; see, e.g., [10], [26], [11], [4], [12]. The equivalence of variational formulations for (1.1) and for (1.3) is a subtle issue, which, for the first biharmonic problem, was already addressed in the pioneering paper [10] for convex domains and was essentially settled in [5] for domains without convexity assumptions.

Strongly related to the Ciarlet-Raviart mixed method is a boundary operator formulation for another auxiliary variable
\[ \lambda = u_{|\Gamma} \]
on the continuous as well as on the discrete level; see [9], [13]. On the discrete level, this approach can be seen as a reduction of the mixed problem to a Schur complement problem.

For convex domains and the more for nonconvex domains, preconditioning the mixed method is still a challenging issue because the mapping properties of the involved linear operators are by far not trivial. One possible approach is the use of mesh-dependent norms for the mixed method; see [4]. However, the analysis was restricted to convex domains and, more severely, the resulting preconditioner requires a preconditioner for a matrix which can be interpreted as a discretization of a differential operator of order 4. In [27] preconditioners were studied which require only standard components, motivated by a reasonable trade-off between optimality (in the sense of mesh-independent convergence rates) and practicability. For the boundary operator formulation preconditioning was studied in [24] quite in the spirit of operator preconditioning; see, e.g., [18] and [20] for a general discussion of this concept. The preconditioner proposed in [24] leads to mesh-independent convergence rates for convex domains.

The aim of this paper is to construct an efficient preconditioner with mesh-independent convergent rates without convexity assumptions. We will extend results from [24] for the reduced problem in \( \lambda \) and show some preliminary results on a class of preconditioners for the original (nonreduced) mixed formulation in \( y \) and \( u \).

The paper is organized as follows. In Sections 2 and 3 the mapping properties are analyzed for the mixed and the reduced formulation, respectively. After discussing the discretized problems quite in the spirit of the analysis of the corresponding continuous problems in Section 4, the main results on preconditioning are developed in Section 5. A few numerical experiments are presented in Section 6 for illustrating the
Theoretical results, followed by concluding remarks in Section 7. Some technical details on harmonic extension operators needed for the analysis in the previous sections are collected in an appendix.

2. The Ciarlet-Raviart method. Here we briefly recall known results on the original mixed formulation and its modification in [5].

2.1. The original method. We consider the following standard mixed variational formulation for (1.3): For \( f \in H^{-1}(\Omega) \), find \( u \in H^1(\Omega) \) and \( y \in H^1_0(\Omega) \) such that

\[
\begin{align*}
\int_{\Omega} uv \, dx - \int_{\Omega} \nabla v \cdot \nabla y \, dx &= 0 \quad \text{for all } v \in H^1(\Omega), \\
- \int_{\Omega} \nabla u \cdot \nabla z \, dx &= -\langle f, z \rangle \quad \text{for all } z \in H^1_0(\Omega),
\end{align*}
\]

where \( \nabla \) denotes the gradient. This problem has the typical structure of a saddle point problem:

\[
\begin{align*}
a(u, v) + b(v, y) &= 0 \quad \text{for all } v \in V, \\
b(u, z) &= -\langle f, z \rangle \quad \text{for all } z \in Q
\end{align*}
\]

for the Hilbert spaces

\[
V = H^1(\Omega) \quad \text{and} \quad Q = H^1_0(\Omega)
\]

and the bilinear forms

\[
a(u, v) = \int_{\Omega} uv \, dx \quad \text{and} \quad b(v, z) = -\int_{\Omega} \nabla v \cdot \nabla z \, dx.
\]

If the linear operator \( \mathcal{A} : X \rightarrow X^* \) with \( X = V \times Q \) is introduced by

\[
\left\langle \mathcal{A} \begin{bmatrix} u \\ y \end{bmatrix}, \begin{bmatrix} v \\ z \end{bmatrix} \right\rangle = a(u, v) + b(v, y) + b(u, z),
\]

the mixed variational problem (2.1) can be rewritten as a linear operator equation

\[
\mathcal{A} \begin{bmatrix} u \\ y \end{bmatrix} = - \begin{bmatrix} 0 \\ f \end{bmatrix}.
\]

Observe that the bilinear form \( a \) is symmetric, i.e., \( a(u, v) = a(v, u) \), and nonnegative, i.e., \( a(v, v) \geq 0 \). In this case it is well-known that \( \mathcal{A} \) is an isomorphism from \( X \) onto \( X^* \) iff the following conditions are satisfied (see, e.g., [7]):

1. \( a \) is bounded: There is a constant \( \|a\| > 0 \) such that

\[
|a(u, v)| \leq \|a\| \|u\|_V \|v\|_V \quad \text{for all } u, v \in V.
\]

2. \( b \) is bounded: There is a constant \( \|b\| > 0 \) such that

\[
|b(v, z)| \leq \|b\| \|v\|_V \|z\|_Q \quad \text{for all } v \in V, z \in Q.
\]
3. $a$ is coercive on the kernel of $b$: There is a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2_V$$

for all $v \in \ker B$

with $\ker B = \{w \in V : b(w, z) = 0 \text{ for all } z \in Q\}$.

4. $b$ satisfies the inf-sup condition: There is a constant $\beta > 0$ such that

$$\inf_{0 \neq z \in Q} \sup_{0 \neq v \in V} \frac{b(v, z)}{\|v\|_V \|z\|_Q} \geq \beta.$$  

Here $\|\cdot\|_V$ and $\|\cdot\|_Q$ denote the norms in $V$ and $Q$, respectively. We will refer to these conditions as Brezzi’s conditions with constants $\|a\|$, $\|b\|$, $\alpha$, and $\beta$.

For (2.1) one of these conditions is not satisfied: the bilinear form $a$ is not coercive on $\ker B$. Nevertheless, for convex domains $\Omega$, existence of a unique solution and error estimates could be established; see, e.g., [10], [26], [11], [4], and many others. But even for convex domains, and more so for nonconvex domains, not having an isomorphism makes it hard to develop efficient preconditioners.

2.2. The modified method. In [5] it was proposed to replace the space $H^1(\Omega)$ for the unknown $u$ by the Hilbert space of less regularity

$$H^{-1}(\Delta, \Omega) = \{v \in L^2(\Omega) : \Delta v \in H^{-1}(\Omega)\},$$

equipped with the norm

$$\|v\|_{-1, \Delta} = (\|v\|_0^2 + \|\Delta v\|_{-1,1}^2)^{1/2}.$$  

Here $\Delta v$ denotes the application of the Laplace operator to $v$ in the distributional sense. The original space $H^1(\Omega)$ is a proper subset of the new space $H^{-1}(\Delta, \Omega)$. This requires extending the definition of the bilinear form $b$ accordingly by

$$b(v, z) = \langle \Delta v, z \rangle,$$

which, of course, coincides with the original definition for $v \in H^1(\Omega)$. Then the extended version of (2.1) for this larger primal space reads as follows: For $f \in H^{-1}(\Omega)$, find $u \in H^{-1}(\Delta, \Omega)$ and $y \in H_0^1(\Omega)$ such that

$$\begin{align*}
\int_{\Omega} u v \, dx + \langle \Delta v, y \rangle &= 0 \quad \text{for all } v \in H^{-1}(\Delta, \Omega), \\
\langle \Delta u, z \rangle &= -\langle f, z \rangle \quad \text{for all } z \in H_0^1(\Omega).
\end{align*}$$

(2.3)

We recall the following result from [5].

**Theorem 2.1.** The bilinear forms $a$ and $b$ given by

$$a(u, v) = \int_{\Omega} u v \, dx \quad \text{and} \quad b(v, z) = \langle \Delta v, z \rangle$$

for $V = H^{-1}(\Delta, \Omega)$ and $Q = H_0^1(\Omega)$ with the norms $\|v\|_V = \|v\|_{-1, \Delta}$ and $\|z\|_Q = |q|_1$ satisfy Brezzi’s conditions with the constants

$$\|a\| = \|b\| = \alpha = 1 \quad \text{and} \quad \beta = (1 + c_F^2)^{-1/2},$$

where $c_F$ is the constant from (2.2).

where \( c_F \) denotes the constant in Friedrichs’ inequality: \(|v|_0 \leq c_F |v|_1\) for all \( v \in H^1_0(\Omega)\).

Problems (1.2) and (2.3) are fully equivalent for convex as well as for nonconvex polygonal domains, since both problems are uniquely solvable and it is easy to see that \((u, y)\) with \( u = -\Delta y \) solves (2.3) if \( y \in H^1_0(\Omega) \) solves (1.2). This has already been recognized in [5] in the context of the Stokes problem.

**Remark 1.** The space \( H^{-1}(\Delta, \Omega) \) is related to the space

\[
\Sigma = \{ v \in L^2(\Omega) : \text{curl} \, v \in H_0(\text{div}, \Omega)^* \},
\]

which was used for the analysis of the Ciarlet-Raviart method in [3]. If \( \Omega \) is simply connected, then these two spaces are identical; otherwise they are not.

3. **Reduction to a boundary operator equation.** Next we want to reduce the variational problem (2.3) for \( y \) and \( u \) to a variational problem for the trace \( \lambda \) of \( u \) only. For this we need two decomposition results. The first decomposition is closely related to results in [2]. The focus here is the formulation in the framework of space decompositions.

**Lemma 3.1.** \( H^{-1}(\Delta, \Omega) = H^1_0(\Omega) \oplus \mathcal{H}(\Omega) \) with

\[
\mathcal{H}(\Omega) = \{ v \in L^2(\Omega) : \Delta v = 0 \},
\]

where \( \oplus \) denotes the direct sum of Hilbert spaces, whose canonical norm is given here by

\[
\|(v_0, v_1)\|_{H^1_0(\Omega) \oplus \mathcal{H}(\Omega)} = |v_0|_1^2 + |v_1|_0^2.
\]

In detail, for each \( v \in H^{-1}(\Delta, \Omega) \), there is a unique decomposition

\[
v = v_0 + v_1 \quad \text{with} \quad v_0 \in H^1_0(\Omega) \quad \text{and} \quad v_1 \in \mathcal{H}(\Omega),
\]

and there are positive constants \( \xi \) and \( \tau \) such that

\[
\xi \left( |v_0|_1^2 + |v_1|_0^2 \right) \leq \|v\|^2_{-1, \Delta} \leq \tau \left( |v_0|_1^2 + |v_1|_0^2 \right) \quad \text{for all} \quad v \in H^{-1}(\Delta, \Omega).
\]

The constants \( \xi \) and \( \tau \) depend only on the constant \( c_F \) of Friedrichs’ inequality.

**Proof.** For \( v \in H^{-1}(\Delta, \Omega) \), let \( v_0 \in H^1_0(\Omega) \) be the unique solution to the variational problem

\[
\int_{\Omega} \nabla v_0 \cdot \nabla z \, dx = -\langle \Delta v, z \rangle \quad \text{for all} \quad y \in H^1_0(\Omega).
\]

By taking the supremum over all \( z \in H^1_0(\Omega) \) we obtain \(|v_0|_1 = \|\Delta v\|_{-1}\).

For \( v_1 = v - v_0 \), we have \( \Delta v_1 = \Delta v - \Delta v_0 = 0 \) in the distributional sense. Hence \( v_1 \in \mathcal{H}(\Omega) \). On the other hand, if \( v = v_0 + v_1 \) with \( v_0 \in H^1_0(\Omega) \) and \( v_1 \in \mathcal{H}(\Omega) \), then \(-\Delta v_0 = -\Delta v + \Delta v_1 = -\Delta v \), which is equivalent to the variational problem (3.1). So, \( v_0 \) is the unique solution of (3.1).

Furthermore, we have

\[
\|v\|^2_{-1, \Delta} = \|v\|^2_1 + \|\Delta v\|^2_{-1} = \|v_0 + v_1\|^2_{0} + |v_0|^2_1 \\
\leq 2 \|v_0\|^2_0 + 2 \|v_1\|^2_0 + |v_0|^2_1 \leq (2c_F^2 + 1) |v_0|^2_1 + 2 \|v_1\|^2_0
\]
and
\[ |v_0|^2 + |v_1|^2 = |v_0|^2 + |v - v_0|^2 \leq |v_0|^2 + 2|v|^2 + 2|v_0|^2 \]
\[ \leq 2|v|^2 + (2c_F^2 + 1)|v_0|^2 = 2|v|^2 + (2c_F^2 + 1)\|\Delta v\|_{-1}^2. \]

Then the estimates immediately follow with \(1/c = \pi = \max(2, 2c_F^2 + 1). \]

Observe that \(\mathcal{H}(\Omega) = \ker B\) with the notation introduced in Section 2.

**Notation 1.** For estimates of the form
\[ c f_2(x) \leq f_1(x) \leq \bar{c} f_2(x) \quad \text{for all } x \in H, \]
where \(f_1\) and \(f_2\) are nonnegative functions, with some positive constants \(c, \bar{c}\) independent of \(x \in H\) and, later on for discretized problems, also independent of the mesh size, we briefly write
\[ f_1(x) \sim f_2(x) \quad \text{for all } x \in H. \]

If \(f_1(x) = \langle M_1 x, x \rangle, f_2(x) = \langle M_2 x, x \rangle\) for symmetric and positive definite matrices \(M_1, M_2 \in \mathbb{R}^{n \times n}\) with \(H = \mathbb{R}^n\), we use the simplified notation \(M_1 \sim M_2\).

With this notation the estimates in the last lemma can be written as
\[ \|v\|_{-1,\Delta}^2 \sim |v_0|^2 + |v_1|^2_0 \quad \text{for all } v \in H^{-1}(\Delta, \Omega). \]

**Corollary 3.2.** For all \(v \in H^{-1}(\Delta, \Omega)\), we have \(v|_{\Omega'} \in H^1(\Omega')\) for all open sets \(\Omega' \subset \Omega\).

**Proof.** We use Weyl’s lemma to conclude that \(\mathcal{H}(\Omega) \subset C^\infty(\Omega)\). Then the statement immediately follows from the decomposition \(\varphi v = \varphi v_0 + \varphi v_1\) for a test function \(\varphi \in C_0^\infty(\Omega)\) with \(\varphi\) identical to 1 on \(\Omega'\).

For the description of a further decomposition of \(\mathcal{H}(\Omega)\) we need trace and extension operators for \(H^{-1}(\Delta, \Omega)\).

The properties for the trace operator are well-known and summarized here; they easily follow from the results in [16], [15]. The boundary \(\Gamma\) of the polygonal domain \(\Omega\) can be written as
\[ \Gamma = \Gamma_C \cup \Gamma_E \quad \text{with} \quad \Gamma_E = \bigcup_{k=1}^K \Gamma_k, \]
where \(\Gamma_C\) denotes the set of all corners of \(\Gamma\) and \(\Gamma_k, k = 1, 2, \ldots, K\), are the edges of \(\Gamma\), considered as open line segments. The trace operator \(\gamma^0\), given by
\[ \gamma^0 v = (\gamma^0_k v)_{k=1,\ldots,K} \quad \text{with} \quad \gamma^0_k v = v|_{\Gamma_k} \]
for smooth functions on \(\overline{\Omega}\), has a unique continuous extension as an operator
\[ \gamma^0 : H^{-1}(\Delta, \Omega) \rightarrow H^{-1/2}(\Gamma) = \bigcap_{k=1}^K H^{-1/2}(\Gamma_k), \]
where \(H^{-1/2}(\Gamma_k)\) is the dual of \(\tilde{H}^{1/2}(\Gamma_k)\); see [21] for details. (Another widely used notation for \(\tilde{H}^{1/2}(\Gamma_k)\) is \(H^{1/2}_{00}(\Gamma_k)\); see [19].) The standard norm in \(H^{-1/2}(\Gamma_k)\) is
denoted by $\| \cdot \|_{-1/2, \Gamma_k}$. The norm in $H_{pw}^{-1/2}(\Gamma)$, denoted by $\| \cdot \|_{-1/2, \Gamma}$, is the canonical product norm of its factor spaces, given by

$$\| \mu \|_{-1/2, \Gamma}^2 = \sum_{k=1}^{K} \| \mu_k \|_{-1/2, \Gamma_k}^2 \quad \text{for} \ \mu = (\mu_k)_{k=1,...,K} \in H_{pw}^{-1/2}(\Gamma).$$

**Notation 2.** For the notation of norms or duality products for functions on the boundary $\Gamma$ or some edge $\Gamma_k$, we explicitly use $\Gamma$ or $\Gamma_k$ as subscripts. A subscript $pw$ (piecewise) is used for spaces of functions on $\Gamma$ which are products of spaces of functions defined on the edges $\Gamma_k$ for $k \geq 1$. For simplicity we omit this subscript for the corresponding norm. A subscript $h$ (mesh size) is used for mesh-dependent norms.

The intersection of the kernel of $\gamma^0$ and $H(\Omega)$, given by

$$N = \ker \gamma^0 \cap H(\Omega) = \{ v \in L^2(\Omega) : \Delta v = 0 \text{ and } \gamma^0 v = 0 \},$$

is known to be finite-dimensional. The dimension of $N$ is equal to the number of reentrant corners of $\Omega$; see [16], [15].

The existence of an extension operator and its properties, which are well-known for convex or smooth domains, will be extended to general polygonal domains in the next theorem; see the appendix for the proof.

**Theorem 3.3.** There is a linear operator $E^0 : H_{pw}^{-1/2}(\Gamma) \longrightarrow H^{-1}(\Delta, \Omega)$ which is a right inverse of $\gamma^0$ with the following properties:

1. $\text{im } E^0 \subset H(\Omega)$, where $\text{im } L$ denotes the image of a linear operator $L$.
2. $H(\Omega) = \text{im } E^0 \perp N$, where the symbol $\perp$ denotes the $L^2$-orthogonal decomposition.
3. $\| E^0 \mu \|_0^2 \sim \| \mu \|_{-1/2, \Gamma}^2$ for all $\mu \in H_{pw}^{-1/2}(\Gamma)$.

The first part means that $E^0$ can be viewed as a harmonic extension operator, the second part contains the required decomposition result, and the last part shows that $E^0$ is an isomorphism between the trace space and its image. The existence of the right inverse $E^0$ immediately implies that the trace operator $\gamma^0$ maps from $H^{-1}(\Delta, \Omega)$ onto $H_{pw}^{-1/2}(\Gamma)$.

Lemma 3.1 and Theorem 3.3 allow the following reduction of (2.3).

**Theorem 3.4.** Let $u \in H^{-1}(\Delta, \Omega)$ and $y \in H^1_0(\Omega)$ be the unique solution of (2.3). Then $\lambda = \gamma^0 u \in H_{pw}^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\int_{\Omega} E^0 \lambda E^0 \mu \; dx = - \int_{\Omega} u_0 E^0 \mu \; dx \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma),$$

where $u_0 \in H^1_0(\Omega)$ is the unique weak solution of the Dirichlet problem for the Laplace operator, i.e.,

$$\int_{\Omega} \nabla u_0 \cdot \nabla z \; dx = \langle f, z \rangle \quad \text{for all } z \in H^1_0(\Omega).$$
Proof. From Lemma 3.1 and the second part of Theorem 3.3 it follows that there is a unique element \( u_0 \in H^1_0(\Omega) \) such that \( u = u_0 + E^0 \lambda + n \) for some \( n \in N \). The second line of (2.3) simplifies to (3.4), since \( \Delta(E^0 \lambda + n) = 0 \) according to the first part of Theorem 3.3. Problem (3.3) follows from the first line of (2.3) for test functions of the form \( v = E^0 \mu \) with \( \mu \in H^{-1/2}(\Gamma) \), since \( E^0 \mu \) is orthogonal to \( n \) according to the second part of Theorem 3.3. Using the third part of Theorem 3.3 the well-posedness of (3.3) follows from the Lax-Milgram theorem. □

This generalizes the boundary operator equation, formulated in [13] for smooth domains \( \Omega \), to the case of general polygonal domains \( \Omega \).

4. Discretization. Let \( T_h \) be an admissible triangulation of the domain \( \Omega \). We proceed as usual to construct a conforming finite element space for approximating \( H^{-1}(\Delta, \Omega) \) by choosing piecewise linear functions which lie in this space. From Corollary 3.2 it immediately follows that a piecewise smooth function lies in \( H^{-1}(\Delta, \Omega) \) iff it is continuous. This leads to the standard finite element space

\[
S_h(\Omega) = \{ v_h \in C(\overline{\Omega}) : v_h|_T \in P_1 \text{ for all } T \in T_h \},
\]

where \( P_1 \) denotes the set of linear polynomials. Additionally we introduce

\[
S_{h,0}(\Omega) = S_h(\Omega) \cap H^1_0(\Omega).
\]

Using \( S_h(\Omega) \) and \( S_{h,0}(\Omega) \) as approximation spaces for \( H^{-1}(\Delta, \Omega) \) and \( H^1_0(\Omega) \), respectively, we obtain the following conforming finite element method for (2.3): Find \( u_h \in S_h(\Omega) \) and \( y_h \in S_{h,0}(\Omega) \) such that

\[
\int_\Omega u_h v_h \, dx - \int_\Omega \nabla v_h \cdot \nabla y_h \, dx = 0 \quad \text{for all } v_n \in S_h(\Omega),
\]

\[
- \int_\Omega \nabla u_h \cdot \nabla z_h \, dx = -\langle f, z_h \rangle \quad \text{for all } z_h \in S_{h,0}(\Omega).
\]

Observe that \( S_h(\Omega) \subset H^1(\Omega) \). Therefore, the definition (2.2) of \( b \) can be used. This is exactly the original discrete problem studied in [10]. So, on the discrete level, there is no direct influence of the use of \( H^{-1}(\Delta, \Omega) \) for \( u, v \) instead of \( H^1(\Omega) \).

Analogously to Theorem 2.1 the well-posedness of the discrete problem can be shown.

**Theorem 4.1.** Brezzi’s conditions are satisfied for (4.1) on the discrete spaces \( V = S_h(\Omega) \) and \( Q = S_{h,0}(\Omega) \) and the norms \( \| v \|_V = \| v \|_{-1, \Delta, h} \) and \( \| z_h \|_Q = \| z_h \|_1 \), where

\[
\| v_h \|_{-1, \Delta, h} = (\| v_h \|_0^2 + \| \Delta v_h \|_{-1, h}^2)^{1/2} \quad \text{with} \quad \| \ell \|_{-1, h} = \sup_{z_h \in S_{h,0}(\Omega)} \frac{|\langle \ell, z_h \rangle|}{\| z_h \|_1},
\]

with the same constants as in Theorem 2.1 for the continuous problem (2.3).

**Proof.** The proof follows the corresponding proof in [5] for the continuous problem.

1. Let \( u_h, v_h \in S_h(\Omega) \). Then

\[
|a(u_h, v_h)| \leq \| u_h \|_0 \| v_h \|_0 \leq \| u_h \|_{-1, \Delta, h} \| v_h \|_{-1, \Delta, h}.
\]
2. Let \( v_h \in S_h(\Omega) \), \( z_h \in S_{h,0}(\Omega) \). Then
\[
|b(v_h, z_h)| \leq \|\Delta v_h\|_{-1,h} \|z_h\|_1 \leq \|v_h\|_{-1,\Delta,h} \|z_h\|_1.
\]

3. Let \( v_h \in \ker B_h = \{w_h \in S_h(\Omega) : b(w_h, z_h) = 0 \text{ for all } z_h \in S_{h,0}(\Omega)\} \). Then
\[
a(v_h, v_h) = \|v_h\|_0^2 = \|v_h\|^2_{-1,\Delta,h}.
\]

4. Let \( 0 \neq z_h \in S_{h,0}(\Omega) \). Then
\[
\sup_{0 \neq v_h \in S_h(\Omega)} \frac{b(v_h, z_h)}{\|v_h\|_{-1,\Delta,h}} \geq \frac{b(-z_h, z_h)}{\|z_h\|_{-1,\Delta,h}} = \frac{|z_h|^2}{\|z_h\|_1} \geq \left(\frac{c_F^2}{2} + 1\right)^{-1/2} |z_h|_1.
\]

Observe that the norms introduced for the space \( H^{-1}(\Delta, \Omega) \) in (2.1) and its discrete counterpart \( S_h(\Omega) \) in (4.1) are similar but different. For the discrete problem the norm is mesh-dependent.

**Remark 2.** Following the same ideas as presented in [26], [11], and [3], where convex domains were considered, error estimates can be extended to general polygonal domains. For example, one can show that
\[
\|u - u_h\|_0 + |y - y_h|_1 \leq c \left(\|u - R_h u\|_0 + h^{-1/2} |y - R_{h,0} y|_{W^2_2(\Omega)}\right),
\]
where \( R_h \) and \( R_{h,0} \) denote the Ritz projections in \( H^1(\Omega) \) and \( H^1_{0,0}(\Omega) \), respectively. For the details on error estimates for the Ritz projections on general polygonal domains we refer to the literature, e.g., [25]. For convex domains, the estimates for \( |y - y_h|_1 \) have been improved by duality arguments. An extension to general polygonal domains is rather involved and beyond the scope of this paper.

The actual computations will be performed in matrix-vector notation. We will now rewrite (4.1) in this way. Let \( \underline{v}_h \) and \( \underline{z}_h \) be the coefficient vectors of \( v_h \in S_h(\Omega) \) and \( z_h \in S_{h,0}(\Omega) \) with respect to the nodal bases in these spaces, respectively. The splitting into interior nodes and nodes on the boundary \( \Gamma \) induces a corresponding block structure of \( \underline{v}_h \):
\[
\underline{v}_h = \begin{bmatrix} v_{h,0} \\ \mu_h \end{bmatrix}.
\]
The mass matrix \( M_h \) and the stiffness matrix \( K_h \) representing \( \|\cdot\|_0 \) and \( \|\cdot\|_1 \) on \( S_h(\Omega) \), respectively, can be partitioned accordingly:
\[
M_h = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \quad \text{and} \quad K_h = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}.
\]

Then the variational problem (4.1) reads in matrix-vector notation
\[
\begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \begin{bmatrix} v_{h,0} \\ \mu_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

A reduction of this block system to a single system for \( \underline{\Delta}_h \) can be easily achieved by eliminating \( v_{h,0} \) and \( \mu_h \) using the third and first block lines, respectively. This leads to
\[
S_h \underline{\Delta}_h = \underline{g}_h
\]
with
\[ S_h = M_{11} - M_{10}K_{00}^{-1}K_{01} - K_{10}K_{00}^{-1}M_{01} + K_{10}K_{00}^{-1}M_{00}K_{00}^{-1}K_{01} \]
and the right-hand side
\[ g_h = (K_{10}K_{00}^{-1}M_{00} - M_{10})K_{00}^{-1}f_h. \]
The matrix \( S_h \) is known as a Schur complement of the block system.

As in the continuous case the reduction to the boundary can also be done by a decomposition result for the finite element space \( S_h(\Omega) \), which reveals some extra structural information of the Schur complement matrix.

We start with the following discrete version of Lemma 3.1.

**Lemma 4.2.** \( S_h(\Omega) = S_{h,0}(\Omega) \oplus H_h(\Omega) \) with
\[ H_h(\Omega) = \left\{ v_h \in S_h(\Omega) : \int_{\Omega} \nabla v_h \cdot \nabla z_h \, dx = 0 \text{ for all } z_h \in S_{h,0}(\Omega) \right\}. \]
In detail, for each \( v_h \in S_h(\Omega), \) we have the following unique decomposition:
\[ v_h = \hat{v}_{h,0} + \hat{v}_{h,1} \quad \text{with} \quad \hat{v}_{h,0} \in S_{h,0}(\Omega) \quad \text{and} \quad \hat{v}_{h,1} \in H_h(\Omega) \]
and
\[ \|v_h\|_{-1,h}^2 \sim |\hat{v}_{h,0}|_1^2 + |\hat{v}_{h,1}|_0^2 \quad \text{for all } v_h \in S_h(\Omega) \]
with the same constants as in Lemma 3.1.

The proof of this lemma is a complete copy of the proof in the continuous case and is therefore omitted.

Observe that for \( v_h \in S_h(\Omega), \) the decompositions in Lemmas 3.1 and 4.2 are different, in general. The space \( H_h(\Omega) \) is known as the space of discrete harmonic functions.

Next we introduce the trace space of functions from \( S_h(\Omega) \) by
\[ S_h(\Gamma) = \{ \mu_h = v_h|_\Gamma : v_h \in S_h(\Omega) \}. \]
For each \( \mu_h \in S_h(\Gamma), \) there is a unique element \( v_h \in S_h(\Omega) \) with \( v_h|_\Gamma = \mu_h \) and
\[ \int_{\Omega} \nabla v_h \cdot \nabla z_h \, dx = 0 \quad \text{for all } z_h \in S_{h,0}(\Omega). \]
The associated mapping \( E_h : S_h(\Gamma) \to H_h(\Omega), \) \( \mu_h \mapsto v_h \) is the well-known discrete harmonic extension. In matrix-vector notation, this mapping reads
\[ \mu_h \mapsto \mu_h = \begin{bmatrix} E_{01} \\ I \end{bmatrix} \mu_h \quad \text{with} \quad E_{01} = -K_{00}^{-1}K_{01}. \]
Here \( \mu_h \) denotes the coefficient vector of \( \mu_h \) with respect to the nodal basis of \( S_h(\Gamma). \) It is easy to see that \( E_h \) is bijective.
Analogously to Theorem 3.4 we now obtain

**Theorem 4.3.** Let \( u_h \in \mathcal{S}_h(\Omega) \) and \( y_h \in \mathcal{S}_{h,0}(\Omega) \) be the unique solution of (4.1). Then \( \lambda_h = u_h |_\Gamma \in \mathcal{S}_h(\Gamma) \) is the unique solution of the variational problem

\[
\int_{\Omega} E_h \lambda_h E_h \mu_h \, dx = - \int_{\Omega} \hat{u}_{h,0} E_h \mu_h \, dx \quad \text{for all } \mu \in \mathcal{S}_h(\Gamma),
\]

where \( \hat{u}_{h,0} \in \mathcal{S}_{h,0}(\Omega) \) is the unique solution of the discrete variational problem

\[
\int_{\Omega} \nabla \hat{u}_{h,0} : \nabla z_h \, dx = \langle f, z_h \rangle \quad \text{for all } z_h \in \mathcal{S}_{h,0}(\Omega).
\]

The proof, which is completely analogous to the continuous case, is omitted.

Moreover, as already observed in [24], it is easy to show that the matrix representation of the bilinear form on the left-hand side in (4.4) is the Schur complement \( S_h \):

\[
\int_{\Omega} E_h \lambda_h E_h \mu_h \, dx = \langle S_h \lambda_h, \mu_h \rangle \quad \text{for all } \lambda_h, \mu_h \in \mathcal{S}_h(\Gamma).
\]

**5. Preconditioning.** The starting point for the construction of a preconditioner for (4.2) is Theorem 4.1, which shows the well-posedness of the discrete problem with respect to the norms \( \| \cdot \|_{-1,\Delta,h} \) and \( | \cdot |_1 \), whose matrix representations are given by

\[
\| v_h \|_{-1,\Delta,h} = \| P_h v_h \|_{P_h} \text{ with } P_h = M_h + \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}^{-1} K_{00} \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix} \text{ and } | z_h |_1 = \| z_h \|_{K_{00}}.
\]

Here the following notation is used.

**Notation 3.** For a positive definite matrix \( M \in \mathbb{R}^{n \times n} \) the associated inner product is given by \( \langle x, y \rangle_M = \langle Mx, y \rangle \). Both the vector norm and the matrix norm associated with the inner product \( \langle \cdot, \cdot \rangle_M \) are denoted by \( \| \cdot \|_M \).

Therefore, as a consequence of Theorem 4.1 the spectrum of the preconditioned matrix \( P_h^{-1} A_h \) with

\[
P_h = \begin{bmatrix} P_h & 0 \\ 0 & K_{00} \end{bmatrix} \quad \text{and} \quad A_h = \begin{bmatrix} M_{00} & M_{01} & -K_{00} \\ M_{10} & M_{11} & -K_{10} \\ -K_{00} & -K_{01} & 0 \end{bmatrix}
\]

is bounded away from 0 and \( \infty \) uniformly with respect to the mesh size \( h \). The application of this preconditioner requires an efficient method for multiplying \( P_h^{-1} \) with a vector, which in general is too costly. In practice, the blocks \( P_h \) and \( K_{00} \) are replaced by efficient preconditioners \( \hat{P}_h \) and \( \hat{K}_{00} \), leading to a practical preconditioner,

\[
\hat{P}_h = \begin{bmatrix} \hat{P}_h & 0 \\ 0 & \hat{K}_{00} \end{bmatrix}.
\]

Standard multilevel or multigrid methods are available for \( \hat{K}_{00} \). Therefore, we will concentrate on the construction of an efficient preconditioner \( \hat{P}_h \) for \( P_h \).
Lemma 4.2 gives a first hint for preconditioning $P_h$. In matrix-vector notation it states that

$$\|v_h\|_{P_h}^2 \sim \|\hat{v}_{h,0}\|_{K_{00}}^2 + \|\hat{v}_{h,1}\|_{M_h}^2$$

for all $v_h = \begin{bmatrix} v_{h,0} \\ \mu_h \end{bmatrix}$ with

$$\hat{v}_{h,0} = \begin{bmatrix} I & K_{00}^{-1}K_{01} \\ K_{10}K_{00}^{-1} & I \end{bmatrix} \begin{bmatrix} v_{h,0} \\ \mu_h \end{bmatrix}$$

and

$$\hat{v}_{h,1} = \begin{bmatrix} -K_{00}^{-1}K_{01} \\ I \end{bmatrix} \mu_h,$$

which, by elementary calculations, leads to

$$P_h \sim \begin{bmatrix} I & 0 \\ K_{10}K_{00}^{-1} & I \end{bmatrix} \begin{bmatrix} \hat{K}_{00} & 0 \\ 0 & S_h \end{bmatrix} \begin{bmatrix} I & K_{00}^{-1}K_{01} \\ 0 & I \end{bmatrix}.$$

This motivates the use of preconditioners of the form

$$\hat{P}_h = \begin{bmatrix} I & 0 \\ -\hat{E}_{01}^T & I \end{bmatrix} \begin{bmatrix} \hat{K}_{00} & 0 \\ 0 & \hat{S}_h \end{bmatrix} \begin{bmatrix} I & -\hat{E}_{01} \\ 0 & I \end{bmatrix}$$

with three essential components, $\hat{K}_{00}$, $\hat{S}_h$, and $\hat{E}_{01}$. It is reasonable to choose the same preconditioner for $K_{00}$ in $\hat{P}_h$ as in $P_h$. Candidates for $\hat{S}_h$ are preconditioners for $S_h$. The third component $\hat{E}_{01}$ is considered as an approximation of $E_{01} = -K_{00}^{-1}K_{01}$. The associated mapping $E_h : S_h(\Gamma) \rightarrow S_h(\Omega)$, given by

$$\mu_h \mapsto \begin{bmatrix} \hat{E}_{01} \\ I \end{bmatrix} \mu_h,$$

can be seen as an approximation to the discrete harmonic extension $E_h$.

Preconditioners of this type have been intensively studied in the context of domain decomposition methods. A typical result reads as follows; see [17] for the proof.

**Theorem 5.1.** Assume that $\hat{K}_{00} \sim K_{00}$, $\hat{S}_h \sim S_h$, and that there is a positive constant such that

$$\|E_{01}\|_{P_h} \leq c \|\mu_h\|_{S_h}$$

for all $\mu_h$. Then $\hat{P}_h \sim P_h$.

Observe that the last condition translates to

$$\|E_{h}\|_{P_h}^2 \leq c \|\mu_h\|_{S_h}^2$$

for all $\mu_h$. This is the main result of this section, summarized in Theorem 5.6.
5.1. Schur complement preconditioning. The mapping property of $S_h$ is contained in the following theorem, which does not rely on any convexity assumption. This generalizes a result in [24], where the convex case was considered.

**Theorem 5.2.** For the norm $\| \cdot \|_{-1/2,\Gamma,h}$ in $S_h(\Omega)$, given by

$$
\| \mu_h \|_{-1/2,\Gamma,h}^2 = \| \mu_h \|_{-1/2,\Gamma}^2 + h \| \mu_h \|_{0,\Gamma}^2,
$$

we have

$$
\langle S_h \mu_h; E_h \mu_h \rangle = \| E_h \mu_h \|_0^2 \sim \| \mu_h \|_{-1/2,\Gamma,h}^2 + \| \Pi_N E_h \mu_h \|_0^2 \quad \text{for all } \mu_h \in S_h(\Gamma),
$$

where $\Pi_N$ is the $L^2$-orthogonal projection onto $N$; see (3.2).

**Proof.** We closely follow the proof in [24] and denote the classical harmonic extension operator as a mapping from $H^{1/2}(\Gamma)$ onto $H^1(\Omega)$ by $E^1$; see the appendix for details. The symbol $c$ is used as a generic constant, which might change its value at each appearance.

For $\mu_h \in S_h(\Omega) \subset H^{1/2}(\Gamma)$, let $v = E^1 \mu_h \in H^1(\Omega)$ and $v_h = E_h \mu_h \in S_h(\Omega)$ be its harmonic and the discrete harmonic extension, respectively.

For $v^* = (I - \Pi_N) v$ and $v^*_h = (I - \Pi_N) v_h$, there exists $z \in H^2(\Omega) \cap H^1_0(\Omega)$ with $v^* - v^*_h = \Delta z$, since $\text{im}(I - \Pi_N) = \text{im} \Delta$ for the Laplace operator $\Delta : H^2(\Omega) \cap H^1_0(\Omega) \rightarrow L^2(\Omega)$; see (A.3) in the appendix. Then we have

$$
\| v^*_h - v^* \|_0^2 = - \int_\Omega (v^*_h - v^*) \Delta z \, dx = - \int_\Omega (v_h - v) \Delta z \, dx
$$

$$
= \int_\Omega \nabla (v_h - v) \cdot \nabla z \, dx = \int_\Omega \nabla (v_h - v) \cdot \nabla (z - z_h) \, dx
$$

for an arbitrary element $z_h \in S_{h,0}(\Omega)$. The last identity follows from the Galerkin orthogonality. Therefore, for the pointwise interpolant $z_h$ of $z$, we obtain

$$
\| v^*_h - v^* \|_0^2 \leq |v_h - v|_1 |z - z_h| \leq c h |v|_1 \| z \|_2 \leq c h \| \mu_h \|_{1/2,\Gamma} \| v^*_h - v^* \|_0,
$$

using the approximation property of $S_h(\Omega)$ for functions in $H^2(\Omega)$ and the mapping properties of $E^1$ and $\Delta$. This implies

$$
\| v^*_h - v^* \|_0 \leq c h \| \mu_h \|_{1/2,\Gamma} \leq c h^{1/2} \| \mu_h \|_{0,\Gamma}
$$

by using an inverse inequality for the second estimate. Furthermore,

$$
\| v^* \|_0 = \| (I - \Pi_N) E^1 \mu_h \|_0 = \| E^0 \mu_h \|_0 \leq c \| \mu_h \|_{-1/2,\Gamma};
$$

see Theorem A.2 in the appendix, and Theorem 3.3, part 3. Therefore, we obtain

$$
\| (I - \Pi_N) E_h \mu_h \|_0 = \| v^*_h \|_0 \leq \| v^* \|_0 + \| v^*_h - v^* \|_0 \leq c \left( \| \mu_h \|_{-1/2,\Gamma} + h^{1/2} \| \mu_h \|_{0,\Gamma} \right).
$$

With $\| E_h \mu_h \|_0^2 = \| (I - \Pi_N) E_h \mu_h \|_0^2 + \Pi_N E_h \mu_h \|_0^2$ the second estimate easily follows.

For the first estimate we start with

$$
\| E_h \mu_h \|_0 = \| v_h \|_0 \geq \| v^*_h \|_0 \geq \| v^* \|_0 - \| v^*_h - v^* \|_0 \geq c \left( \| \mu_h \|_{-1/2,\Gamma} - h^{1/2} \| \mu_h \|_{0,\Gamma} \right)
$$
Using the inverse inequality \( \|v_h\|_0 \geq c h^{1/2} \|\mu_h\|_{0,\Gamma} \) and \( \|E_h\mu_h\|_0 \geq \|N E_h\mu_h\|_0 \), the first inequality easily follows.

In order to construct a preconditioner for \( S_h \) we start by first considering the term \( \|\mu_h\|_{-1/2,\Gamma,h} \) in (5.4) only. A preconditioner for this norm, i.e. an easy to invert approximation to the matrix representing this norm, was already proposed in [24] based on preconditioners for \( \|\cdot\|_{-1/2,\Gamma_k} \), \( k \geq 1 \). We follow this idea but replace the preconditioner for \( \|\cdot\|_{-1/2,\Gamma_k} \), for which the FFT was used in [24], by a simpler standard multilevel preconditioner of the type as analyzed in [6].

For this, let \( T_\ell, \ell = 0, 1, 2, \ldots, L \), be a hierarchy of uniformly refined subdivisions of \( \Omega \) of mesh size \( h_\ell \) with \( T_L = T_h \) with associated finite element spaces \( S_\ell(\Omega) \) of continuous and piecewise linear functions and their trace spaces \( S_\ell(\Gamma) \). Furthermore, let \( S_\ell(\Gamma_C) \) and \( S_\ell(\Gamma_E) \) be the linear span of all nodal basis functions from \( S_\ell(\Gamma) \) associated with nodes from \( \Gamma_C \) and from \( \Gamma_E \), respectively.

Then the proposed preconditioner is of additive Schwarz type, given by

\[
\left( \hat{S}_h^{(0)} \right)^{-1} = h_L R_{C,L} A_{C,L} R_{C,L}^T + \sum_{\ell=0}^{L} h_\ell R_{E,\ell} A_{E,\ell} R_{E,\ell}^T.
\]

Here \( R_{C,L} \) and \( R_{E,\ell} \) denote the matrix representations of the canonical embeddings of \( S_\ell(\Gamma_C) \) and \( S_\ell(\Gamma_E) \) into \( S_\ell(\Gamma) \), respectively. The matrices \( A_{C,L} \) and \( A_{E,\ell} \) are given by

\[
A_{C,L} = \hat{M}_{C,L}^{-1} K_{C,L} \hat{M}_{C,L}^{-1}, \quad A_{E,\ell} = \hat{M}_{E,\ell}^{-1} K_{E,\ell} \hat{M}_{E,\ell}^{-1},
\]

where \( \hat{M}_{C,L} \) and \( \hat{M}_{E,\ell} \) are the matrix representations of the discrete version of the norm \( \|\cdot\|_{0,\Gamma} \) which results from the elementwise use of the trapezoidal rule on \( S_\ell(\Gamma_C) \) and \( S_\ell(\Gamma_E) \), respectively. \( \hat{K}_{C,L} \) and \( \hat{K}_{E,\ell} \) are the matrix representations of the norm \( \|\cdot\|_{1,\Gamma} \) on \( S_\ell(\Gamma_C) \) and \( S_\ell(\Gamma_E) \), respectively, where this norm is given by

\[
\|\mu_h\|^2_{1,\Gamma} = \sum_{k=1}^{K} \int_{\Gamma_k} |\nabla_{\Gamma_k} \mu_h|^2 \, ds \quad \text{with the tangential gradient} \quad \nabla_{\Gamma_k}.
\]

Observe that the boundary mass matrices \( \hat{M}_{C,L} \) and \( \hat{M}_{E,\ell} \) are diagonal. So, the application of the preconditioner requires only the multiplication by boundary stiffness matrices \( \hat{K}_{C,L} \) and \( \hat{K}_{E,\ell} \) and some componentwise scaling on each refinement level.

Now we have the next theorem.

**Theorem 5.3.** \( \|\mu_h\|^2_{-1/2,\Gamma,h} \sim \langle \hat{S}_h^{(0)} \mu_h; \mu_h \rangle \) for all \( \mu_h \in S_h(\Gamma) \).

**Proof.** Each part of the proof is based on fairly standard arguments from [24], [23], [6], and [14]. We just have to put known things together.

First of all, for the decomposition \( S_\ell(\Gamma) = S_\ell(\Gamma_C) \oplus S_\ell(\Gamma_E) \) we obtain

\[
\|\mu_L\|^2_{-1/2,\Gamma,h} \sim h_L \|\mu_C\|^2_{0,\Gamma} + \|\mu_E\|^2_{-1/2,\Gamma}
\]

for all \( \mu_L \in S_\ell(\Gamma) \) with \( \mu_L = \mu_C + \mu_E \), \( \mu_C \in S_\ell(\Gamma_C) \), \( \mu_E \in S_\ell(\Gamma_E) \), see [24, Proposition 7.5].

Next, using the multiscale representation of the norm \( \|\cdot\|_{-1/2,\Gamma_k} \) from Theorem 4 in [23], following the main idea from [6] and replacing the norm \( \|\cdot\|_{0,\Gamma_k} \) in this
representation by the norm $h^{-1}_e \| \cdot \|_{-1, \Gamma}$, and applying Proposition 3 from [14] one obtains
\[
\| \mu_E \|_{-1, \Gamma}^2 \sim \inf \left\{ \sum_{\ell=0}^L h^{-1}_e \| \mu_\ell \|_{-1, \Gamma}^2 : \mu_E = \sum_{\ell=0}^L \mu_\ell, \mu_\ell \in \mathcal{S}_\ell(\Gamma_E) \right\}
\]
for all $\mu_E \in \mathcal{S}_E(\Gamma_E)$, where $\| \cdot \|_{-1, \Gamma}$ denotes the canonical norm in $H^{-1}_p(\Gamma)$.

Finally, using
\[
\| \mu_\ell \|_{-1, \Gamma}^2 \sim \langle A^{-1}_E \mu_\ell, \mu_\ell \rangle \quad \text{for all } \mu_\ell \in \mathcal{S}_E(\Gamma_E)
\]
from [6] and
\[
\inf_\| \mu_C \|_{0, \Gamma} \sim h^{-1}_L \langle A^{-1}_{C, L} \mu_C, \mu_C \rangle \quad \text{for all } \mu_C \in \mathcal{S}_L(\Gamma_C),
\]
which follows from a simple scaling argument, we obtain a stable space decomposition, whose associated additive Schwarz operator is $\hat{S}_h^{(0)}$.

For studying the term $\| \Pi_N E_h \|_0^2$ in (5.4) we assume that a basis $\{s_1, \ldots, s_J\}$ of $N$ is known. Then we have
\[
\| \Pi_N v_h \|_0^2 = \langle M^T_N M_N^{-1} M_{Nh} \psi_h, \psi_h \rangle
\]
with the mass matrices
\[
M_N = \left( \int \Omega s_i s_j \, dx \right) \quad \text{and} \quad M_{Nh} = \left( \int \Omega s_i \varphi_j \, dx \right),
\]
where $\{\varphi_1, \varphi_2, \ldots, \varphi_I\}$ denotes the nodal basis of $S_h(\Omega)$. Hence
\[
\| \Pi_N E_h \|_0^2 = \langle M^T_N M_N^{-1} M_{Nh} \left[ E_{01} \right] \mu_h, \left[ E_{01} \right] \mu_h \rangle = \langle U_h M_N^{-1} U^T_h \mu_h, \mu_h \rangle
\]
with $U_h = \left[ E_{01} \right] M^T_N$. Observe that the rank of $U_h M_N^{-1} U^T_h$ is equal to the dimension of $N$, i.e., the (fixed) number of reentrant corners.

So, in summary, we obtain
\[
\langle S_h \mu_h, \mu_h \rangle = \| E_h \mu_h \|_0^2 \sim \left\langle \left[ \hat{S}_h^{(0)} + U_h M_N^{-1} U^T_h \right] \mu_h, \mu_h \right\rangle \quad \text{for all } \mu \in \mathcal{S}_h(\Gamma),
\]
which completes the proof of the next theorem.

**Theorem 5.4.** $\hat{S}_h^{(1)} \sim S_h$ with $\hat{S}_h^{(1)} = \hat{S}_h^{(0)} + U_h M_N^{-1} U^T_h$.

Furthermore, we have
\[
\left( \hat{S}_h^{(1)} \right)^{-1} \mu_h = \left[ I - V_h (M_N + U_h^T V_h)^{-1} U_h^T \right] \left( \hat{S}_h^{(0)} \right)^{-1} \mu_h \quad \text{with} \quad V_h = \left( \hat{S}_h^{(0)} \right)^{-1} U_h
\]
by the Sherman-Morrison-Woodbury formula. The matrix $I - V_h (M_N + U_h^T V_h)^{-1} U_h^T$ has to be computed only once and the computational costs are rather low for domains with a small number of reentrant corners. This makes $\hat{S}_h^{(1)}$ an efficient preconditioner for $S_h$.

**Remark 3.** A basis $\{s_1, \ldots, s_J\}$ of $N$ is given by $s_j = s_{-j}(r_j, \theta_j) - w_j$ with
\[
s_{-j}(r_j, \theta_j) = \eta(r_j) r_j^{-\pi/\omega_j} \sin((\omega_j/\pi) \theta_j)
\]
and \( w_j \in H^1_0(\Omega) \), given by the variational problem

\[
\int_\Omega \nabla w_j \cdot \nabla v \, dx = \int_\Omega (\Delta s_j) v \, dx \quad \text{for all } v \in H^1_0(\Omega).
\]

Here \((r_j, \theta_j)\) denotes the polar coordinates centered at a reentrant corner with internal angle \( \omega_j \) spanned by \( \theta_j = 0 \) and \( \theta_j = \omega_j \), and \( \eta(r_j) \) is a cutoff function which is identical to 1 in a neighborhood of the corner; see [16], [15].

5.2. Approximate discrete harmonic extensions. The evaluation of \( v_h = E_h \mu_h \), where \( E_h \) is the discrete harmonic extension, requires the exact solve of the linear system

\[
K_{00} \mu_{h,0} = -K_{01} \mu_h.
\]

If instead we use an inner iteration by performing \( r \) steps of the Richardson method with preconditioner \( K_{00} \) and initial guess 0, then we end up with an approximate harmonic extension \( \hat{E}^{(r)}_h \), given by

\[
\mu_h \mapsto \left[ \hat{E}^{(1)}_{01} I \right] \mu_h \quad \text{with} \quad \hat{E}^{(r)}_{01} = \left[ I - (I - \hat{K}_{00}^{-1} K_{00})^r \right] E_{01}.
\]

(For \( r = 1 \) we simply get \( \hat{E}^{(1)}_{01} = -\hat{K}_{00}^{-1} K_{01} \).) We will now show that this approximate discrete harmonic extension satisfies the third condition of Theorem 5.1 under reasonable assumptions.

**Lemma 5.5.** Assume that the inner iteration converges in the corresponding energy norm with a convergence rate \( q < 1 \) which is independent of \( h \), i.e.,

\[
\| I - \hat{K}_{00}^{-1} K_{00} \|_{\mathcal{K}_{00}} \leq q.
\]

Then, for \( r = \mathcal{O}(\| \ln h \|) \), there is a constant \( c \) such that

\[
\| \hat{E}^{(r)}_h \mu_h \|_{-1,\Delta,h} \leq c \| E_h \mu_h \|_0 \quad \text{for all } \mu_h \in \mathcal{S}_h(\Gamma).
\]

**Proof.** For all \( \mu_h \in \mathcal{S}_h(\Gamma) \), we have

\[
\| \hat{E}^{(r)}_h \mu_h \|_{-1,\Delta,h} \leq \| E_h \mu_h \|_{-1,\Delta,h} + \| (\hat{E}^{(r)}_h - E_h) \mu_h \|_{-1,\Delta,h}
\]

\[
= \| E_h \mu_h \|_0 + \| (\hat{E}^{(r)}_h - E_h) \mu_h \|_1
\]

\[
\leq \| E_h \mu_h \|_0 + (c_F^2 + 1)^{1/2} \| (\hat{E}^{(r)}_h - E_h) \mu_h \|_1.
\]

Now it easily follows that

\[
\| (\hat{E}^{(r)}_h - E_h) \mu_h \|_1 = \| (\hat{E}^{(r)}_0 - E_{01}) \mu_h \|_{\mathcal{K}_{00}} = \| (I - \hat{K}_{00}^{-1} K_{00})^r E_{01} \mu_h \|_{\mathcal{K}_{00}}
\]

\[
\leq q^r \| E_{01} \mu_h \|_{\mathcal{K}_{00}} \leq q^r \| E_h \mu_h \|_1 \leq c q^r h^{-1} \| E_h \mu_h \|_0.
\]

If \( r = \mathcal{O}(\| \ln h \|) \), the factor \( q^r h^{-1} \) is uniformly bounded, which completes the proof. □

To summarize the discussion on preconditioning, Theorem 5.1, Theorem 5.4, and Lemma 5.5 in connection with (5.3) lead to the following main result of this section.
Theorem 5.6. The spectrum of the preconditioned matrix \( \hat{P}_h^{-1}A_h \) with \( A_h \) given by (5.1) and
\[
\hat{P}_h = \begin{bmatrix} \hat{P}_h & 0 \\ 0 & \hat{K}_{00} \end{bmatrix}
\]
with \( \hat{P}_h = \begin{bmatrix} I & 0 \\ -\hat{E}_{01}^T & I \end{bmatrix} \), \( \hat{K}_{00} = \begin{bmatrix} I & 0 \\ 0 & \hat{S}_h \end{bmatrix} \), \( \hat{E}_{01} \), \( \hat{S}_h \)
where \( \hat{K}_{00} \sim K_{00} \), \( \hat{E}_{01} = \hat{E}_h^{(r)} \) with \( r = O(\ln h) \), given by (5.6), and \( \hat{S}_h = \hat{S}_h^{(1)} \) (see Theorem 5.4), is bounded away from 0 and \( \infty \) uniformly with respect to the mesh size \( h \). And, consequently, the condition number of \( \hat{P}_h^{-1}A_h \) is bounded independently of the mesh size \( h \).

Remark 4. If \( \Omega \) is simply connected, then the space \( H^{-1}(\Delta, \Omega) \) is identical to the space \( \Sigma \) (see Remark 1), which was introduced in [3] not only for the Ciarlet-Raviart method but also for a mixed method of the vector Laplacian with Dirichlet boundary conditions and for the vorticity-velocity-pressure formulation of the Stokes problem. Therefore, the preconditioner \( \hat{P}_h \) can also be used as an essential part of a block preconditioner for these problems.

6. Numerical experiments. We consider the simple biharmonic test problem
\[
\Delta^2 y = f \quad \text{in } \Omega, \quad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma
\]
on two domains, the square \( \Omega = \Omega_S = (-1, 1)^2 \) (representing the convex case) and the \( L \)-shaped domain \( \Omega = \Omega_L \) depicted in figures 1 and 2, where also the initial mesh (level \( \ell = 0 \)) is shown. The right-hand side \( f(x) \) is chosen such that
\[
y(x) = \begin{bmatrix} 1 - \cos(2\pi x_1) \\ 1 - \cos(4\pi x_2) \end{bmatrix}
\]
is the exact solution to the problem. The initial meshes are uniformly refined until the final level \( \ell = L \).

We will present numerical results demonstrating the quality of the preconditioners \( \hat{S}_h^{(0)} \) and (in the nonconvex case) \( \hat{S}_h^{(1)} \) for \( S_h \) and the preconditioner \( \hat{P}_h \) for \( A_h \). The Schur complement preconditioners were tested by applying the preconditioned gradient (PG), the conjugate gradient method (CG), and its preconditioned variant (PCG) to (4.3); the method of choice for (4.2) was the preconditioned minimal residual method (PMINRES). In all experiments a reduction of the Euclidean norm of
the initial residual by a factor of $10^{-8}$ was used as stopping criterion for the iterative methods, where the initial guess was chosen randomly out of the range spanned by the corresponding exact quantities.

For the preconditioner $K_{00}$, which is used as the first diagonal block in $\hat P_h$, as the last diagonal block in $\hat P_h$, and as a preconditioner in the inner iteration for the approximate discrete harmonic extension, we always choose one multigrid V-cycle with one step of forward and backward Gauss-Seidel smoothing. The action of the exact inverse $K_{00}$, as needed for $S_h$ and for the exact discrete harmonic extension $E_h$, was realized by applying an inner iteration with 10 V-cycles. The preconditioner $\hat S_h^{(1)}$ was constructed as described in Remark 3 with the modification that the solution $w_j$ to (5.5) was replaced by the corresponding approximate solution in $S_{h,0}(\Omega)$.

Table 1 shows the observed number of iterations for (4.3) in the convex case $\Omega = \Omega_S$. The first column contains the level $L$ of refinement. The next three columns show the results for CG and for PG and PCG, both with the preconditioner $\hat S_h^{(0)}$.

<table>
<thead>
<tr>
<th>$L$</th>
<th>CG</th>
<th>PG</th>
<th>PCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>66</td>
<td>77</td>
<td>20</td>
</tr>
<tr>
<td>7</td>
<td>91</td>
<td>77</td>
<td>21</td>
</tr>
<tr>
<td>8</td>
<td>120</td>
<td>75</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>160</td>
<td>71</td>
<td>20</td>
</tr>
</tbody>
</table>

As expected the number of iterations grows for CG without preconditioning if the mesh size decreases. The second column shows that the preconditioner alone with PG already leads to convergence rates which are uniformly bounded in $h$. Of course, the use of this preconditioner in PCG results in a further reduction of the number of iterations.

Table 2 shows the results for the $L$-shaped domain $\Omega = \Omega_L$ representing a non-convex case. The second and third columns contain the numbers of iterations for PG with the preconditioners $\hat S_h^{(0)}$ and $\hat S_h^{(1)}$, respectively, while in the next two columns the corresponding results for PCG are shown.

<table>
<thead>
<tr>
<th>$L$</th>
<th>PG($\hat S_h^{(0)}$)</th>
<th>PG($\hat S_h^{(1)}$)</th>
<th>PCG($\hat S_h^{(0)}$)</th>
<th>PCG($\hat S_h^{(1)}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>337</td>
<td>77</td>
<td>24</td>
<td>21</td>
</tr>
<tr>
<td>7</td>
<td>427</td>
<td>81</td>
<td>25</td>
<td>22</td>
</tr>
<tr>
<td>8</td>
<td>385</td>
<td>81</td>
<td>26</td>
<td>22</td>
</tr>
<tr>
<td>9</td>
<td>457</td>
<td>81</td>
<td>25</td>
<td>21</td>
</tr>
</tbody>
</table>

By comparing the second and the third columns one sees that $\hat S_h^{(1)}$ performs significantly better as a preconditioner in PG than $\hat S_h^{(0)}$, as expected from the analysis. Nevertheless, as seen from the fourth column, PCG works well for the nonoptimal preconditioner $\hat S_h^{(0)}$. A relevant further improvement by using the better preconditioner $\hat S_h^{(0)}$ in PCG was not observed. This important feature was observed here experi-
mentally and is not yet supported by analysis. If confirmed, this would considerably contribute to practicability, in particular, for a possible extension to three-dimensional problems; see the concluding remarks.

Finally, Table 3 shows some preliminary results for preconditioning the nonreduced system (4.2), whose total numbers $n$ of unknowns are shown in the second column. PMINRES was applied to the $L$-shaped domain $\Omega = \Omega_L$; for preconditioning the Schur complement the nonoptimal preconditioner $\hat{S}_h^{(0)}$ was used. The third and the fourth columns contain the results if using the costly exact discrete harmonic extension $E_h$ and the less costly approximate version $\hat{E}_h^{(r)}$ with $r$ inner iterations, respectively. The chosen value of $r$ is shown in parentheses in the fourth column.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$n$</th>
<th>PMINRES($E_h$)</th>
<th>PMINRES($\hat{E}_h^{(r)}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1,538</td>
<td>46</td>
<td>43 (3)</td>
</tr>
<tr>
<td>5</td>
<td>6,146</td>
<td>45</td>
<td>48 (3)</td>
</tr>
<tr>
<td>6</td>
<td>24,578</td>
<td>43</td>
<td>47 (4)</td>
</tr>
<tr>
<td>7</td>
<td>98,306</td>
<td>43</td>
<td>48 (4)</td>
</tr>
<tr>
<td>8</td>
<td>393,218</td>
<td>44</td>
<td>46 (5)</td>
</tr>
<tr>
<td>9</td>
<td>1,572,866</td>
<td>44</td>
<td>56 (5)</td>
</tr>
</tbody>
</table>

It can be seen that a modest increase in the number $r$ of inner iterations keeps the number of iterations in the range of the observed number of iterations if using the costly exact discrete harmonic extension. This is in accordance with Lemma 5.5.

7. Concluding remarks. Efficient Schur complement preconditioners were derived and analyzed for convex and non-convex polygonal domains, respectively. There is experimental evidence that $\hat{S}_h^{(0)}$ also works fine for nonconvex polygonal domains in combination with a Krylov subspace method (PCG for (4.3) or PMINRES for (4.2)). This is especially advantageous for a possible extension to three-dimensional problems, where $\hat{S}_h^{(1)}$ would be much harder to construct. Preconditioning the reduced system (4.3) has, therefore, reached a satisfactory state. Observe, however, that the computational costs for evaluating one residual for (4.3) is relatively high, since it requires the application of (an accurate approximation of) the discrete harmonic extension $E_h$ twice.

The situation is less clear for the nonreduced problem (4.2). Here the evaluation of the residual is computationally inexpensive. The computational costs for applying $\mathcal{P}_h$ depend mainly on the choice for $E_h$. If $\hat{E}_h = E_h$, the computational costs of one step of PCG for (4.3) and one step of PMINRES for (4.2) are roughly the same. The number of iterations differs by a factor of about 2 (see Table 1, third column and Table 3, last column), and so do the observed computing times, as expected. Possible improvements are to use symmetric indefinite preconditioners (see [27]), based on the same components as the proposed symmetric and positive definite block diagonal preconditioner, in particular with the same extension $\hat{E}_h = E_h$, in combination of Krylov subspace methods such as GMRES. Another possible improvement is the replacement of $E_h$ by more efficient approximate discrete harmonic extensions. The few numerical experiments with an inner iteration as shown in Table 3 already lead
to an improvement in computing time by almost a factor of 2. Efficient approximate harmonic extensions are well-developed and understood as bounded operators from $S_h(\Gamma) \subset H^{1/2}(\Gamma)$ to $H^1(\Omega)$; see, e.g., [17]. Here the challenge of future work is the construction of efficient approximate harmonic extensions which satisfy (5.3).

**Appendix. Harmonic extension operators.** The trace space of functions from $H^1(\Omega)$ is $H^{1/2}(\Gamma)$. The well-known harmonic extension operator $E^1: H^{1/2}(\Omega) \longrightarrow H^1(\Omega)$ is given by the following variational problem: For $\mu \in H^{1/2}(\Gamma)$, find $v = E^1 \lambda \in H^1(\Omega)$ such that $v|_\Gamma = \mu$ and

$$\int_\Omega \nabla v \cdot \nabla z \, dx = 0 \quad \text{for all } z \in H^1_0(\Omega).$$

An essential property of $E^1$ is

$$|E^1 \mu|_1^2 \sim \|\mu\|^2_{1/2,\Gamma} \quad \text{for all } \mu \in H^{1/2}(\Gamma),$$

where $\| \cdot \|_{1/2,\Gamma}$ denotes the standard norm in $H^{1/2}(\Gamma)$.

A harmonic extension operator $E^0: H^{-1/2}_{pw}(\Gamma) \longrightarrow H^{-1}(\Delta, \Omega)$ which is more appropriate in the context of this paper is given by the following variational problem: For $\mu \in H^{-1/2}_{pw}(\Gamma)$, find $v = E^0 \mu \in \text{im} \Delta$ such that $\gamma^0 v = \mu$ and

$$\int_\Omega v \Delta z \, dx = \langle \mu, \gamma^1 z \rangle_\Gamma \quad \text{for all } z \in H^2(\Omega) \cap H^1_0(\Omega)$$

with

$$\langle \mu, g \rangle_\Gamma = \sum_{k=1}^K \langle \mu_k, g_k \rangle_{\Gamma_k} \quad \text{for } \mu = (\mu_k)_{k=1,...,K} \in H^{-1/2}(\Gamma), \ g = (g_k)_{k=1,...,K} \in H^{1/2}_{pw}(\Gamma),$$

where $\langle \cdot, \cdot \rangle_{\Gamma_k}$ denotes the duality product in $H^{-1/2}_{pw}(\Gamma) \times H^{1/2}_{pw}(\Gamma)$ and $\gamma^1$ is the trace operator, given by

$$\gamma^1 z = (\gamma^1_k z)_{k=1,...,K} \quad \text{with } \gamma^1_k z = \left. \frac{\partial z}{\partial n} \right|_{\Gamma_k} \quad \text{for } z \in H^2(\Omega) \cap H^1_0(\Omega).$$

Here, $H^{1/2}_{pw}(\Gamma)$ denotes the product space $\prod_{k=1}^K H^{1/2}(\Gamma_k)$ and $\text{im} \Delta$ is the image of the Laplace operator $\Delta: H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$.

For this definition of $\Delta$, it is known that

$$L^2(\Omega) = \text{im} \Delta \perp N,$$

which immediately implies that $\text{im} \Delta = N^\perp = \text{im}(I - \Pi_N)$; see [16], [15].

**Remark 5.** For smooth functions on $\Omega$ the right-hand side in (A.2) can be rewritten in a more traditional fashion as

$$\langle \mu, \gamma^1 z \rangle_\Gamma = \int_\Gamma \mu \frac{\partial z}{\partial n} \, dS.$$
Variants of (A.2) are often called very weak formulations of the corresponding Dirichlet problem for the Laplace operator.

In [12, page 184] this harmonic extension operator was studied for smooth and for convex polygonal domains $\Omega$, although on a not-yet-appropriate trace space in the case of convex polygonal domains. We will show now that $E^0$ is well-defined on general polygonal domains.

**Theorem A.1.** For each $\mu \in H_{pw}^{-1/2}(\Gamma)$, there is a unique solution $v = E^0\mu \in \text{im}(\Delta)$ to (A.2) and

$$\|E^0\mu\|_0^2 \sim \|\mu\|_{-1/2,\Gamma}^2 \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma).$$

**Proof.** For $v = \Delta w$ with $w \in W = H^2(\Omega) \cap H^1_0(\Omega)$, problem (A.2) coincides with the second biharmonic boundary value problem for $w$, which is known to be well-posed. For given $\mu \in H_{pw}^{-1/2}(\Gamma)$, let $w_\mu$ be its solution. Then

$$\|w_\mu\|_2 \sim \|\ell_\mu\|_{W^*} \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma),$$

where $\ell_\mu$ denotes the linear functional on the right-hand side of (A.2), given by $z \mapsto \langle \mu, \gamma^1 z \rangle_\Gamma$.

The trace operator $\gamma^1 : W \to \widetilde{H}^{1/2}(\Gamma)$ is well-defined, bounded and surjective; see [16]. Therefore, the bilinear form $(z, \mu) \mapsto \langle \mu, \gamma^1 z \rangle_\Gamma$ is well-defined and bounded on $W \times H_{pw}^{-1/2}(\Gamma)$, and it satisfies an inf-sup condition. Therefore,

$$\|\ell_\mu\|_{W^*} \sim \|\mu\|_{-1/2,\Gamma} \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma).$$

Using $\|\Delta w\|_0 \sim \|w\|_2$ for all $w \in H^2(\Omega) \cap H^1_0(\Omega)$, we finally obtain

$$\|E^0\mu\|_0 = \|\Delta w_\mu\|_0 \sim \|\ell_\mu\|_{W^*} \sim \|\mu\|_{-1/2,\Gamma} \quad \text{for all } \mu \in H_{pw}^{-1/2}(\Gamma). \quad \square$$

We have the following relation between $E^1$ and $E^0$ on the domain $H^{1/2}(\Gamma)$, where both extension operators exist.

**Theorem A.2.** $E^0\mu = (I - \Pi_N)E^1\mu$ for all $\mu \in H^{1/2}(\Gamma)$.

**Proof.** Both $E^0\mu$ and $E^1\mu$ are harmonic and have the same trace $\mu$. Therefore, $E^1\mu - E^0\mu \in N$, i.e., there is an $n \in N$ with $E^1\mu = E^0\mu + n$. Moreover, $E^0\mu \in \text{im } \Delta$ by definition. From (A.3) it follows that $n = \Pi_N E^1\mu$, which implies $E^0\mu = E^1\mu - n = E^1\mu - \Pi_N E^1\mu$. \sbox

Theorem 3.3 is a simple consequence of the last two theorems.

**Remark 6.** For convex domains, $N$ is trivial. Only in this case the two harmonic extension operators coincide.

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