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NuMa-Report No. 2013-04 June 2013

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Cubature Rules for Harmonic Functions Based on Radon Projections

Irina Georgieva* Clemens Hofreither†

Abstract

We construct a class of cubature formulae for harmonic functions on the unit disk based on line integrals over $2n+1$ distinct chords. These chords are assumed to have constant distance t to the center of the disk, and their angles to be equispaced over the interval $[0, 2\pi]$. If t is chosen properly, these formulae integrate exactly all harmonic polynomials of degree up to $4n+1$, which is the highest achievable degree of precision for this class of cubature formulae. For more generally distributed chords, we introduce a class of interpolatory cubature formulae which we show to coincide with the previous formulae for the equispaced case. We give an error estimate for a particular cubature rule from this class.

1 Introduction

For univariate functions, quadrature is typically done by exactly integrating a polynomial which interpolates the integrand on a set of prescribed points which usually lie within the integration interval. This approach is met with well-known difficulties in the multivariate case. Most importantly, the Lagrangian interpolation problem for multivariate polynomials is not always solvable. In recent years, many researchers have proposed alternative schemes for multivariate interpolation and cubature where the given data comes not in the form of point evaluations, but rather as mean values over a set of prescribed hyperplanes. We point in particular to the results of Bojanov and Petrova [2, 3], who showed the existence of a unique cubature formula for the disk that uses n line integrals and is exact for all bivariate polynomials of degree up to $2n-1$, a result which is not possible using the same number of point evaluations.

Besides the more convincing mathematical theory, we point out that this kind of data appears naturally in the real world, e.g., in computer tomography with its many applications in medicine, radiology, geology, etc. These techniques have their mathematical foundation in the work of Johann Radon on

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the so-called Radon transform [15]. Reconstruction of functions from their line integrals can be formulated as an interpolation problem where not the function itself, but its Radon transform is sampled on a discrete set. Early major contributions on the topic of multivariate interpolation using integrals over hyperplanes are due to Marr [13] and Hakopian [12]. Research on this topic was continued in the previous decade by many researchers [1, 4, 9, 10, 8, 11].

In [14] a family of cubature rules for the unit disk using Radon projections along symmetrically located chords was found. In the present paper we derive cubature rules for harmonic functions using Radon projections. This subject turns out to be closely related to the interpolation of harmonic polynomials studied in [7], [6], [5]. However, we will find that the optimal choice of chords for the purpose of cubature is somewhat different from that for interpolation, and certain configurations which do not yield uniquely solvable interpolation problems still result in good cubature rules.

2 Preliminaries

Let $D \subset \mathbb{R}^2$ denote the open unit disk and ∂D the unit circle. By $I(\theta, t)$ we denote a chord of the unit circle at angle $\theta \in [0, 2\pi)$ and distance $t \in (-1, 1)$ from the origin (see Figure 1), parameterized by

$$s \mapsto (t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)^\top, \quad \text{where } s \in (-\sqrt{1-t^2}, \sqrt{1-t^2}).$$

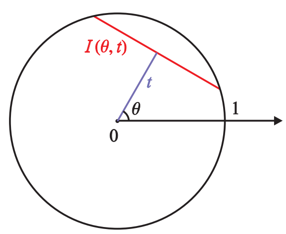

Figure 1: The chord $I(\theta, t)$ of the unit circle.

Definition 1. Let $u(x, y)$ be a real-valued bivariate function in the unit disk D. The *Radon projection* $\mathcal{R}_\theta(u; t)$ of u in direction θ is defined by the line integral

$$\mathcal{R}_\theta(u; t) := \int_{I(\theta,t)} u(\mathbf{x})\, d\mathbf{x} = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} u(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\, ds.$$

Johann Radon [15] showed in 1917 that a differentiable function u is uniquely determined by the values of its Radon transform,

$$u \;\mapsto\; \{\mathcal{R}_\theta(u; t) \,:\, -1 \le t \le 1,\; 0 \le \theta < \pi\}.$$

2.1 Radon projections of harmonic polynomials

Let Π_n^2 denote the space of real bivariate polynomials of total degree at most n. In the following, we will often work with the subspace

$$\mathcal{H}_n = \left\{p \in \Pi_n^2 : \Delta p = 0\right\}$$

of real bivariate harmonic polynomials of total degree at most n, which has dimension $2n+1$.

We use the basis of the harmonic polynomials

$$\phi_0(x,y) = 1, \quad \phi_{k,1}(x,y) = \mathrm{Re}(x+\mathrm{i}y)^k, \quad \phi_{k,2}(x,y) = \mathrm{Im}(x+\mathrm{i}y)^k. \tag{1}$$

In polar coordinates, they have the representation

$$\phi_{k,1}(r,\theta) = r^k\cos(k\theta), \quad \phi_{k,2}(r,\theta) = r^k\sin(k\theta).$$

The following result, which gives a closed formula for Radon projections of the basis harmonic polynomials, can be considered a harmonic analogue to the famous Marr's formula [13]. A special case of this harmonic version was first derived using tools from symbolic computation [7]. Later, Georgieva and Hofreither [6] have given an analytic proof in a more general setting.

Theorem 1 ([6])**.** *The Radon projections of the basis harmonic polynomials are given by*

$$\int_{I(\theta,t)} \phi_{k,1}\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\cos(k\theta),$$
$$\int_{I(\theta,t)} \phi_{k,2}\, d\mathbf{x} = \frac{2}{k+1}\sqrt{1-t^2}U_k(t)\sin(k\theta),$$

where $k \in \mathbb{N}$, $\theta \in \mathbb{R}$, $t \in (-1,1)$, and $U_k(t)$ is the k-th Chebyshev polynomial of second kind.

3 Cubature formulae on equispaced chords

For an integrable function u on D, we denote

$$I[u] := \int_D u(x)\,dx.$$

For the integrals of the basis harmonic polynomials over the unit disk D, it is easy to compute that

$$I[\phi_{k,1}] = \begin{cases} \pi, & k=0, \\ 0, & k \geq 1, \end{cases} \tag{2}$$
$$I[\phi_{k,2}] = 0, \quad k \geq 1.$$

In the following, we will construct cubature formulae for the unit disk D for harmonic functions using Radon projections along a set of chords

$$\mathcal{I} = \{I(\theta_j, t) : j = 1, \ldots, 2n+1\}$$

with a constant distance $t \in (-1, 1)$ to the origin.

Let $n \in \mathbb{N}_0$, fix some $c \in \mathbb{R}$ and $t \in (-1, 1)$ and define the cubature rule

$$Q[u] := c \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u, t).$$

Using Theorem 1, we obtain

$$Q[\phi_{k,1}] = c \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(\phi_{k,1}, t) = c \frac{2}{k+1} \sqrt{1-t^2} U_k(t) \sum_{j=1}^{2n+1} \cos(k\theta_j),$$

$$Q[\phi_{k,2}] = c \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(\phi_{k,2}, t) = c \frac{2}{k+1} \sqrt{1-t^2} U_k(t) \sum_{j=1}^{2n+1} \sin(k\theta_j).$$

We can rewrite the sums over trigonometric functions in the above formulas as the real and imaginary parts, respectively, of $\sum_{j=1}^{2n+1} e^{\mathrm{i}k\theta_j}$. With the special choice of equispaced angles

$$\theta_j = \frac{2j\pi}{2n+1}, \quad j = 1, 2, \ldots, 2n+1, \tag{3}$$

we can use known summation formulas for complex roots of unity to obtain

$$Q[\phi_{k,1}] = \begin{cases} \frac{2c}{k+1} \sqrt{1-t^2} U_k(t)(2n+1), & k \in \mathbb{N}_0 \cdot (2n+1), \\ 0, & \text{otherwise}. \end{cases}$$

$$Q[\phi_{k,2}] = 0, \quad k \geq 1.$$

Comparing with (2), for the cubature formula Q to be exact for all the harmonic polynomials of degree up to $2n$, i.e., $Q[\phi_{k,j}] = I[\phi_{k,j}]$, $k = 0, \ldots, 2n$, we only have to require

$$I[\phi_{0,1}] = \pi = 2c\sqrt{1-t^2}(2n+1) = Q[\phi_{0,1}].$$

This gives us the choice

$$c = \frac{\pi}{(4n+2)\sqrt{1-t^2}}.$$

If, in addition, we choose t to be a zero of U_{2n+1}, then

$$Q[\phi_{2n+1,1}] = 0 = I[\phi_{2n+1,1}]$$

and the formula Q is exact for all harmonic polynomials of degree up to $4n+1$, since the conditions for $k = 2n+2, \ldots, 4n+1$ are automatically satisfied. Since U_{2n+1} and U_{4n+2} share no roots, the degree of precision of Q is $4n+1$.

Thus we have proved the following theorem.

Theorem 2. *Consider the cubature formula*

$$I[u] \approx Q[u] = \frac{\pi}{(4n+2)\sqrt{1-t^2}} \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u,t) \tag{4}$$

with equispaced angles θ_j *as in* (3)*.*

- *For every* $t \in (-1,1)$*,* Q *is exact at least for all harmonic polynomials of degree up to* $2n$*.*
- *If* $t \in (-1,1)$ *is a zero of* U_{2n+1}*, then the cubature formula* Q *is exact for all harmonic polynomials of degree up to* $4n+1$*, and there exist harmonic polynomials of degree* $4n+2$ *for which it is not exact.*

We point out that, for properly chosen t, the cubature rule Q using $2n+1$ chords integrates exactly all harmonic polynomials from a space of dimension $\dim \mathcal{H}_{4n+1} = 8n+3$.

4 Interpolatory cubature

4.1 Interpolation by harmonic polynomials

For the sake of completeness, we state here some previous results from [6, 7] on interpolation of harmonic functions using Radon projections.

For prescribed chords

$$\mathcal{I} = \{I(\theta_j, t_j) : j = 1, \dots, 2n+1\}$$

of the unit circle and associated given values $\{\gamma_I\}$, we wish to find a harmonic polynomial $p \in \mathcal{H}_n$ such that

$$\int_I p(\mathbf{x})\, d\mathbf{x} = \gamma_I \qquad \forall I \in \mathcal{I}. \tag{5}$$

Expanding the harmonic polynomial p in the basis (1),

$$p = p_0\phi_0 + \sum_{k=1}^{n} (p_{k,1}\phi_{k,1} + p_{k,2}\phi_{k,2}), \tag{6}$$

we obtain a linear system $A\underline{p} = \underline{\gamma}$ equivalent to (5) with the matrix

$$A = \begin{pmatrix} \int_{I_1} 1 & \int_{I_1} \phi_{1,1} & \cdots & \int_{I_1} \phi_{n,1} & \int_{I_1} \phi_{n,2} \\ \int_{I_2} 1 & \int_{I_2} \phi_{1,1} & \cdots & \int_{I_2} \phi_{n,1} & \int_{I_1} \phi_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int_{I_{2n+1}} 1 & \int_{I_{2n+1}} \phi_{1,1} & \cdots & \int_{I_{2n+1}} \phi_{n,1} & \int_{I_{2n+1}} \phi_{n,2} \end{pmatrix}$$

and the vector $\underline{p} = \{p_0, p_{1,1}, p_{1,2}, \dots, p_{n,1}, p_{n,2}\}^\top \in \mathbb{R}^{2n+1}$.

We call $\mathcal{I}$ *regular* if the interpolation problem (5) has a unique solution for all given values $\{\gamma_I\}$. Clearly, the scheme $\mathcal{I}$ is regular if and only if A is regular. We cite here a result on regularity of a family of schemes $\mathcal{I}$ with constant distances of the chords.

Theorem 3 (Existence and uniqueness [6]). *The interpolation problem* (5) *always has a unique solution for the choice* $\mathcal{I} = \{I(\theta_j, t_j) : j = 1, \ldots, 2n+1\}$ *with*

$$0 < \theta_1 < \theta_2 < \ldots < \theta_{2n+1} \le 2\pi,$$

while the distances $t_j = t \in (-1, 1)$ *are constant and* t *is not a zero of any Chebyshev polynomial of the second kind* $U_1, \ldots, U_n$.

In the following, we use the notations

$$\begin{aligned}
c(k) &:= (\cos(k\theta_1), \ldots, \cos(k\theta_{2n+1}))^\top, \\
s(k) &:= (\sin(k\theta_1), \ldots, \sin(k\theta_{2n+1}))^\top, \\
G &:= \begin{pmatrix} | & | & | & \cdots & | & | \\ 1 & c(1) & s(1) & \ldots & c(n) & s(n) \\ | & | & | & \cdots & | & | \end{pmatrix}, \\
\alpha_k &:= \frac{2}{k+1}\sqrt{1-t^2}U_k(t), \\
\beta_k &:= \begin{cases} \frac{1}{2(2n+1)}(\sqrt{1-t^2})^{-1} = \frac{1}{2n+1}\alpha_k^{-1}, & k = 0, \\ \\ \frac{k+1}{2n+1}(\sqrt{1-t^2}U_k(t))^{-1} = \frac{2}{2n+1}\alpha_k^{-1}, & k \ge 1, \end{cases} \\
F &:= \mathrm{diag}(\alpha_0, \alpha_1, \alpha_1, \ldots, \alpha_n, \alpha_n), \\
E &:= \mathrm{diag}(\beta_0, \beta_1, \beta_1, \ldots, \beta_n, \beta_n).
\end{aligned}$$

For equally spaced angles, the columns of G are orthogonal. This fact allows a simple representation of the inverse of the system matrix A in this case.

Theorem 4 ([6]). *Assume chords with equally spaced angles* $\theta_j = \frac{2\pi j}{2n+1}$ *and fixed distance* $t_j = t \in (0, 1)$, $j = 1, \ldots, 2n+1$, *such that* $U_k(t) \neq 0$ *for all* $k \in \{0, \ldots, n\}$. *Then the inverse of* A *is given by*

$$A^{-1} = EG^\top.$$

4.2 Interpolatory cubature

Consider a regular scheme of chords $\mathcal{I}$ with arbitrary distinct angles

$$0 < \theta_1 < \theta_2 < \cdots < \theta_{2n+1} \le 2\pi$$

and fixed distance $t_j = t \in (0, 1)$, $j = 1, \ldots, 2n+1$, such that $U_k(t) \neq 0$ for all $k \in \{0, \ldots, n\}$. According to Theorem 3, for any given harmonic function u on D, there exists a unique harmonic polynomial $p_u \in \mathcal{H}_n$ with $\mathcal{R}_{\theta_j}(p_u, t) =$

k	0	1	2	3	4	5
$Q^*[\phi_{k,1}]$	3.141	0	0	0	−0.159	0.016
$Q^*[\phi_{k,2}]$		0	0	0	0.0181	0.127

Table 1: Result of cubature with non-equispaced angles for the harmonic basis functions

$\mathcal{R}_{\theta_j}(u,t)$ for all $j = 1, \dots, 2n+1$. Thus we can define the interpolatory cubature formula

$$Q^*[u] := I[p_u] = \pi p_0,$$

where p_0 is the coefficient of the constant part of p_u as in (6). The above formula holds since $I[\phi_0] = \pi$ and the integrals of all other basis functions vanish.

Theorem 5. *The interpolatory cubature rule Q^* with constant distances t such that $U_k(t) \neq 0$, $k = 0, \dots, n$, is precise at least for harmonic polynomials of degree up to n.*

If the angles are equispaced, we have $Q^* \equiv Q$ as in (4).

If in addition t is a root of U_{2n+1}, then again Q^* is exact for all harmonic polynomials of total degree up to $4n+1$.

Proof. The fact that the formula is precise up to degree n follows immediately from the existence of a unique interpolant, Theorem 3. For equispaced angles, we see from Theorem 4 that

$$p_0 = [EG^\top \underline{\gamma}]_1 = \beta_0 \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u,t).$$

From the definitions, it follows that

$$Q^*[u] = \frac{\pi}{(4n+2)\sqrt{1-t^2}} \sum_{j=1}^{2n+1} \mathcal{R}_{\theta_j}(u,t),$$

which is (4). The last statement then follows with Theorem 2. □

Remark. A simple numerical example for interpolatory cubature with non-equispaced angles confirms that in general these rules are precise only up to degree n. For Table 1, we chose chords with angles $\theta_j = \frac{2j\pi}{7} - 0.4\sin(1.3j)$, $j = 1, \dots, 7$, and $t = 0.4$. We use interpolatory cubature over these chords to approximate the integrals of the harmonic basis functions $\phi_{k,1}$ and $\phi_{k,2}$. As expected, the cubature yields exact values up to degree $n = 3$, since $2n+1 = 7$ chords were used. (Values with magnitude smaller than 10^{-14} which occurred due to limited precision were rounded to 0.)

5 Error estimate

In this section, we derive an error estimate for the special case of a cubature rule with equispaced angles and a distance t which is sufficiently large in the sense that it is larger by some bound than the largest root of U_n. For this, we make use of our previous results from [6] on the error in the coefficients of the interpolating polynomial. In the process, we generalize a statement from this previous paper to hold in a more general setting.

We start by proving a more general version of a technical lemma from [6].

Lemma 6. *Let*

$$a(k,n) := \frac{k+1}{U_k(\cos x(n))}, \quad 0 < x(n) \le \frac{\pi-\varepsilon}{n+1},$$

with $\varepsilon > 0$. *For* $n \in \mathbb{N}$, *we have*

$$1 = a(0,n) \le a(1,n) \le \ldots \le a(n-1,n) \le a(n,n) < \frac{\pi-\varepsilon}{\sin(\pi-\varepsilon)}.$$

Proof. By the definition of $U_k(t)$, we have that

$$a(k,n) = \frac{(k+1)\sin x(n)}{\sin\left[(k+1)x(n)\right]} = \frac{\frac{\sin(x(n))}{x(n)}}{\frac{\sin(y(k,n))}{y(k,n)}}, \tag{7}$$

with $y(k,n) = (k+1)x(n)$.

All arguments to sin are in the range $(0,\pi)$, and thus the sines are all positive. Thus, $U_k(\cos x(n)) > 0$, and we use the well-known fact $|U_k(t)| \le k+1$ to conclude that $1 \le a(k,n)$.

In the following, we use the well-known properties of the sinc function $y \mapsto \frac{\sin y}{y}$ that in $(0,\pi)$ it is positive, bounded by 1 and monotonically decreasing. By assumption on $x(n)$ and because $k \le n$, we have $y(k,n) \in (0,\pi-\varepsilon]$. Since clearly $y(k,n)$ is monotonically increasing in k, we have that $\frac{\sin(y(k,n))}{y(k,n)}$ is monotonically decreasing in k, and from (7) we conclude that $a(k,n)$ is monotonically increasing in k.

We now bound $a(n,n)$ from above. Because $x(n) \in (0,\pi)$, we have $\frac{\sin(x(n))}{x(n)} < 1$. We use that $y(n,n) \le \pi-\varepsilon$ and the monotonicity of the sinc function to bound

$$\frac{\sin(y(n,n))}{y(n,n)} \ge \frac{\sin(\pi-\varepsilon)}{\pi-\varepsilon},$$

from which the desired upper bound for $a(n,n)$ follows. ☐

The following lemma is a straightforward generalization of an analogous result in [6].

Lemma 7. *Assume that* $f = u|_{\partial D}$ *has a uniformly convergent Fourier series*

$$f(\theta) = f_0 + \sum_{k=1}^{\infty}(f_k\cos(k\theta) + f_{-k}\sin(k\theta)). \tag{8}$$

and its Fourier coefficients $(f_k)_{k\in\mathbb{Z}}$ *decay like* $|f_k| \leq M|k|^{-s}$ *with* $M > 0$*,* $s > 1$*. Let*

$$p^{(n)} = p_0^{(n)}\phi_0 + \sum_{k=1}^{n}(p_k^{(n)}\phi_k + p_{-k}^{(n)}\phi_{-k}) \in \mathcal{H}_n$$

be the interpolating polynomial of degree n *according to Theorem 3, where the angles* θ_j *are chosen equispaced as in* (3) *and* $t = \cos(x(n))$ *with* $0 < x(n) \leq \frac{\pi-\varepsilon}{n+1}$ *and* $\varepsilon > 0$*.*

Then the error in the coefficients of the interpolating polynomial $p^{(n)}$ *satisfies*

$$|f_k - p_k^{(n)}| \leq MC_{s,\varepsilon}n^{-s} \qquad \forall |k| \leq n,$$

where $C_{s,\varepsilon} > 0$ *is a constant which depends only on* s *and the* ε*.*

Proof. The lemma was proved in [6] for the special choice $t = \cos\frac{\pi}{2n+1}$, i.e., $x(n) = \frac{\pi}{2n+1}$. Lemma 6 provides a generalization of [6, Lemma 5] for more general choices of $x(n)$ and thus of t. The proof can then be followed step by step, replacing the estimates for $a(k,n)$ by the more general version in Lemma 6. □

With this result, an error estimate for the cubature rule Q, which in this setting coincides with the interpolatory cubature rule Q^*, follows immediately.

Theorem 8. *Let the assumptions of Lemma 7 be satisfied. Then, for the cubature rule* Q *using* $2n+1$ *Radon projections, we have the cubature error estimate*

$$|Q[u] - I[u]| = \pi|f_0 - p_0^{(n)}| \leq \pi MC_{s,\varepsilon}n^{-s},$$

Proof. The identity

$$|Q[u] - I[u]| = \pi|f_0 - p_0^{(n)}|$$

follows immediately from (2), and the statement from Lemma 7. □

6 Numerical examples

6.1 Example 1

We test our cubature rule (4) on the harmonic function

$$u(x,y) = \log\sqrt{(x-1)^2 + (y-1)^2}.$$

In Figure 2 we plot the cubature errors for varying degree n (x-axis) with the choice $t = \cos\frac{n\pi}{2n+2}$ (circles) and $t = 0$ (squares). Convergence is exponential for both choices of t, with the case $t = 0$ yielding considerably smaller errors.

In Figure 3 we plot the cubature errors for varying t (x-axis) with degree $n = 3$ (circles), $n = 5$ (squares) and $n = 7$ (rhombuses). We observe that choices of t which are optimal in the sense of Theorem 2 (t-s are zeros of U_{2n+1}) also tend to yield smaller errors in practice. This is seen from the dips in the error plots which coincide with the non-negative roots of U_7, U_{11}, and U_{13}, respectively.

For this particular example, it seems that $t = 0$ is always the most favorable choice. However, in the next example we will see that this is not always true.

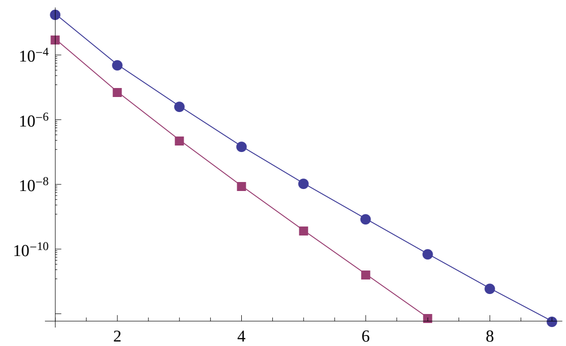

Figure 2: (Ex. 1) Cubature errors for varying degree n (x-axis) with the choice $t = \cos \frac{n\pi}{2n+2}$ (circles) and $t = 0$ (squares).

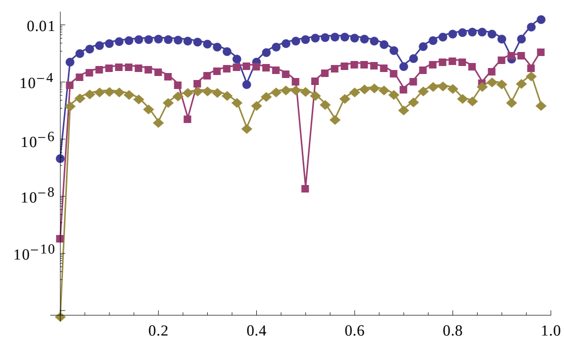

Figure 3: (Ex. 1) Cubature errors for varying t (x-axis) with degree 3 (circles), 5 (squares), 7(rhombuses)

6.2 Example 2

In order to test the cubature rule for functions with less smoothness, we construct the harmonic extension of the boundary function $f(\theta) = \theta^2$ on the unit circle in radial coordinates, where the argument θ is chosen in the interval $[-\pi, \pi]$. This function is only C^0 on the unit circle, but analytic within the unit disk. By expanding f into its Fourier series, it can be shown that the corresponding harmonic function has the representation

$$u(x, y) = \mathrm{Re}\left(\frac{\pi^2}{3} + 2(\mathrm{Li}_2(-x - \mathrm{i}y) + \mathrm{Li}_2(-x + \mathrm{i}y))\right),$$

where

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

is the dilogarithm or Spence's function. See Figure 4 for a plot of the harmonic function u.

The boundary function f satisfies the smoothness assumption from Lemma 7 with $s = 2$. In Figure 5, we plot the cubature errors for $t = 0$ with the degree n varying from 1 to 15. We find that the decay of the error is approximately $\mathcal{O}(n^{-2.7})$, which is slightly faster than the rate $\mathcal{O}(n^{-s})$ predicted by Theorem 8. In Figure 6, we plot the cubature errors for fixed degree $n = 3$ with t varying in $[0, 1)$. As in Example 1, we find that the smallest errors are attained when t is chosen as a root of $U_{2n+1} = U_7$. In contrast to the previous example, here not 0, but the largest root of U_7 yields the smallest overall error.

7 Conclusions

We have constructed cubature rules for harmonic functions on the unit disk using Radon projections as their given data. The below table categorizes the presented rules in terms of the angles $\{\theta_m\}$ and distances t of the used chords. The third column indicates the degree k of the space $\mathcal{H}_k$ on which a rule of the respective type using $2n + 1$ pieces of data is precise.

θ_m	t (constant)	deg. of precision	ref.
distinct	not a root of $U_1, \dots, U_n$	n	Theorem 5
equispaced	arbitrary	$2n$	Theorem 2
equispaced	root of U_{2n+1}	$4n + 1$	Theorem 2

Acknowledgments

The authors acknowledge the support by Bulgarian National Science Fund, Grant DMU 03/17 and Grant DFNI-T01/0001. The research of the second author was supported by the National Research Network "Geometry + Simulation" (NFN S117, 2012–2016), funded by the Austrian Science Fund (FWF).

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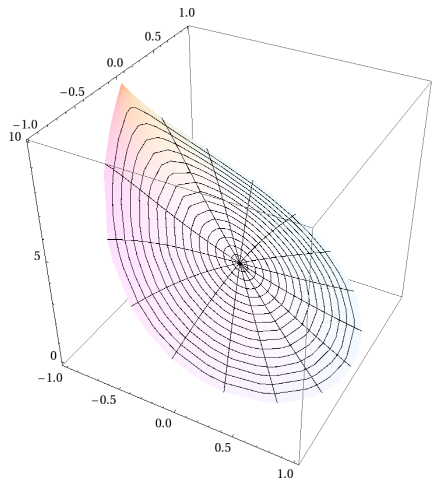

Figure 4: (Ex. 2) The harmonic function u with C^0 boundary data

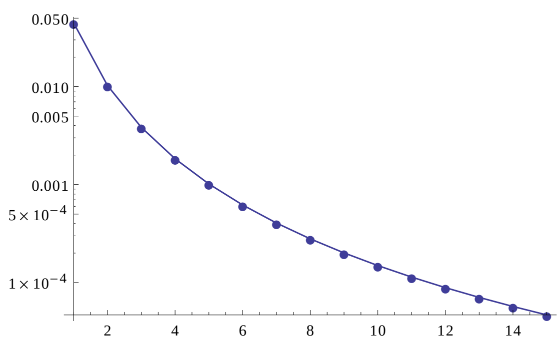

Figure 5: (Ex. 2) Cubature errors for varying degree $n = 1, \ldots, 15$ (x-axis)

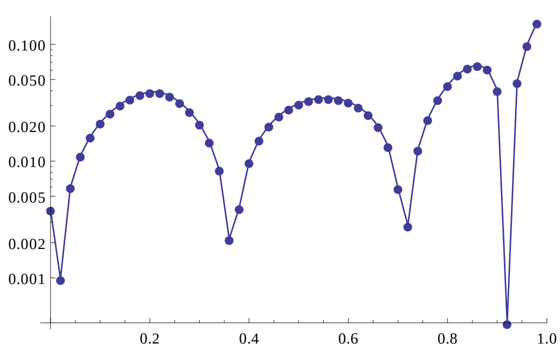

Figure 6: (Ex. 2) Cubature errors for varying t (x-axis) with degree 3

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