Explicit Extension Operators on Hierarchical Grids*

G. Haase† S.V. Nepomnyaschikh‡

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Abstract

Extension operators extend functions defined on the boundary of a domain into its interior. This paper presents explicit extension operators by means of multilevel decompositions on hierarchical grids. It is shown that the norm-preserving property of these operators holds for the 2D as well for the 3D case with constants independent on discretization and domain size. These constants can be further improved by an additional iteration scheme applied to the extension operator. Some implementation of these techniques is presented for a domain decomposition preconditioner and numerical experiments are given.

Keywords : Boundary value problems, trace theory, multilevel methods, domain decomposition, preconditioning, finite element method.

1 Introduction

The purpose of this paper is to discuss the construction of norm-preserving explicit extension operators of functions at a boundary into the interior of the domain. The theorems on traces of functions from Sobolev spaces play an important role in studying boundary value problems of mathematical physics [2, 3, 4]. These theorems are commonly used for deriving a priori estimates of the stability with respect to boundary conditions. For the case

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†Johannes Kepler University Linz, Inst. of Math., Altenberger Str. 69, A–4040 Linz, Austria, ghaase@numa.uni-linz.ac.at

‡Computing Center, Siberian Branch of Russian Academy of Sciences, Novosibirsk, 630090, Russia, svnep@oapmg.sscc.ru

of grid functions the first constructive analysis of this problem seems to be carried out in [1], where the case of rectangular grids was considered. For a numerical solution of non-homogeneous Dirichlet problems it is important to have a good extension of the Dirichlet conditions inside the domain. If the trivial extension, i.e. extension by zero at interior nodes of the grid, is used then instead of the numerical solution of a boundary value problem with a smooth solution, we have to compute a solution with a huge gradient in the vicinity of the boundary. The use of norm-preserving explicit extension operators gives a "good" initial guess for iteration processes and reduces the boundary value problems with non-homogeneous Dirichlet conditions to the boundary value problems with homogeneous Dirichlet conditions. The solution of the last is uniformly bounded with respect to the grid size.

Another important application of the explicit extension operators is connected with domain decomposition methods [18, 19, 21]. Using the explicit extension operators in domain decomposition methods, instead of exact solvers in the subdomains, local preconditioning operators can be utilized. Using this operators, optimal estimates of the convergence rate of the domain decomposition methods and optimal arithmetic cost are obtained.

The first construction of the norm-preserving explicit extension operators for unstructured grids seems to be suggested in [18] and used for the construction of preconditioning operators in [18, 19, 20, 21]. For the case of hierarchical grids, near norm-preserving explicit extension operators have been suggested in [14] which are easy to implement. Its iterative improving was discussed in [10, 11]. In this paper we follow [22]. To construct the explicit extension operators, functions at the boundary are decomposed into series of functions at hierarchical grids. Then, each component of this decomposition will be extended in a very simple way. Multilevel decomposition in trace spaces was also considered in [25]. To prove the trace theorem for hierarchical grids, a multilevel decomposition of the boundary has been used in [24]. In the present paper, we design the multilevel decomposition of functions on the boundary with very cheap arithmetic costs and the arithmetic costs of the resulting explicit extension operators (as well as adjoint operators) is proportional to the number of grid nodes.

The paper is organized as follows. In Section 2, the construction of explicit extension operators is presented. Realization aspects of these extensions are discussed in Section 3. The improvement of the extensions by an iterative scheme is proposed in Section 4. The application to domain decomposition methods is considered in Section 5 and some numerical experiments are presented in Section 6.

2 Construction of Extension Operators

Let Ω be a bounded, polygonal domain and Γ be its boundary. Consider a coarse grid triangulation of Ω

$$\Omega_0^h = \bigcup_{i=1}^{M_0} \tau_i^{(0)} \quad , \qquad \mathrm{diam}(\tau_i^{(0)}) \; = \; \mathcal{O}(1)$$

which will be successively refined an number of times. This results in a sequence of nested triangulations $\Omega_0^h,\ \Omega_1^h,\ \ldots\ ,\ \Omega_J^h$ such that

$$\Omega_k^h = \bigcup_{i=1}^{M_k} \tau_i^{(k)} \quad , \quad k = 0, 1, \ldots , J \ ,$$

where the triangles $\tau_i^{(k+1)}$ are generated by subdividing triangles $\tau_i^{(k)}$ into four congruent subtriangles by connecting the midpoints of the edges. Introduce the spaces $\mathbb{W}_k$ and $\mathbb{V}_k$ of FE (finite element) functions. The space $\mathbb{W}_k$ consists of real-valued functions which are continuous on Ω and linear on the triangles in Ω_k^h. The space $\mathbb{V}_k$ is the space of traces on Γ of functions from $\mathbb{W}_k$:

$$\mathbb{V}_k = \{\varphi^h | \varphi^h = u^h|_\Gamma, \ \text{ with } u^h \in \mathbb{W}_k\}$$

We consider $\mathbb{W}_k$ and $\mathbb{V}_k$ as the subspaces of the Sobolev spaces $H^1(\Omega)$ and $H^{\frac{1}{2}}(\Gamma)$, respectively, with corresponding norms [4]. The main goal is the construction of some norm–preserving explicit extension operator t from $\mathbb{V}_J$ to $\mathbb{W}_J$:

$$t : \ \mathbb{V}_J \to \mathbb{W}_J \ \ .$$

This construction is based on the idea from [14], but instead of Yserentant's hierarchical decomposition [27, 28] of the space $\mathbb{V}_J$ we use some analogous of the so–called BPX–decomposition of $\mathbb{V}_J$ [7, 26]. Denote by $\varphi_i^{(k)}, \quad i = 1, 2, \ldots , N_k$ the nodal basis of $\mathbb{V}_k$ and by $\Phi_i^{(k)}$ the one–dimensional subspace spanned by this function $\varphi_i^{(k)}$ with support $\sigma_i^{(k)}$. Define

$$Q_i^{(k)} : L_2(\Gamma) \to \Phi_i^{(k)}$$

the L_2–like projection from $L_2(\Gamma)$ onto $\Phi_i^{(k)}$:

$$Q_i^{(k)} \psi^h = d_i^{(k)}(\psi^h) \varphi_i^{(k)},$$

where

$$d_i^{(k)} = \frac{1}{(\varphi_i^{(k)}, 1)_{L_2(\Gamma)}} (\psi^h, \varphi_i^{(k)})_{L_2(\Gamma)} \ , \qquad \text{or} \tag{1a}$$

$$d_i^{(k)} = \frac{1}{\text{meas}\left(\sigma_i^{(k)}\right)} (\psi^h, 1)_{L_2(\sigma_i^{(k)}))} \ . \tag{1b}$$

Denote by

$$Q_k = \sum_{i=1}^{N_k} Q_i^{(k)} \ , \quad k = 0, 1, \dots, J-1 \ .$$

For $k = J$, Q_k is defined as the L_2–orthoprojection from $L_2(\Omega)$ onto $\mathbb{V}_J$.

LEMMA 1. *There exist positive constants c_1, c_2, independent of h, such that for any $\varphi^h \in \mathbb{V}_J$*

$$\begin{aligned} c_1 \, \|\varphi^h\|^2_{H^{\frac{1}{2}}(\Gamma)} &\leq \|Q_0 \varphi^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J} 2^k \|(Q_k - Q_{(k-1)})\varphi^h\|^2_{L_2(\Gamma)} \\ &\leq c_2 \, \|\varphi^h\|^2_{H^{\frac{1}{2}}(\Gamma)}. \end{aligned}$$

PROOF The result above is a simple consequence of well-known properties of Q_k and a technique from [5, 9, 23, 24, 28]. However, for completeness we give the proof. It is easy to see that Q_k is the linear projection onto $\mathbb{V}_k$ and there exists a constant c_3, independent of h, such that

$$\|Q_k \varphi\|_{L_2(\sigma_i^{(k)})} \leq c_3 \, \|\varphi\|_{L_2(\sigma_i^{(k)'})} \qquad \forall \, \varphi \in L_2(\Gamma) \tag{2}$$

holds, where $\sigma_i^{(k)'}$ is the union of all $\sigma_j^{(k)}$ which posses at least one common point with $\sigma_i^{(k)}$.
Also the following approximation property of Q_k holds :

$$\begin{aligned} \|\varphi - Q_k \varphi\|^2_{L_2(\Gamma)} &\leq \sum_{i=1}^{N_k} \|\varphi - Q_k \varphi\|^2_{L_2(\sigma_i^{(k)})} = \sum_{i=1}^{N_k} \|\varphi - \kappa_i + \kappa_i - Q_k \varphi\|^2_{L_2(\sigma_i^{(k)})} \\ &= \sum_{i=1}^{N_k} \|\varphi - \kappa_i + Q_k \left(\kappa_i - \varphi\right)\|^2_{L_2(\sigma_i^{(k)})} \\ &\leq 2(1 + c_3^2) \sum_{i=1}^{N_k} \|\varphi - \kappa_i\|^2_{L_2(\sigma_i^{(k)'})} \ . \end{aligned}$$

Here κ_i is an arbitrary constant function. Then the estimate

$$\|\varphi - Q_k\varphi\|^2_{L_2(\Gamma)} \;\le\; c_4\, 2^{-k}\, |\varphi|_{H^1(\Gamma)} \qquad \forall \varphi \in H^1(\Gamma)$$

is valid with some h-independent constant c_4. According to [24] there exists also an h-independent constant c_1 such that

$$\begin{aligned} c_1\, \|\varphi^h\|^2_{H^{1/2}(\Gamma)} \;&\le\; \inf_{\substack{\xi_k^h \in \mathbb{V}_k \\ \sum_{k=0}^{J} \xi_k^h = \varphi^h}} \sum_{k=0}^{J} 2^k\, \|\xi_k^h\|^2_{L_2(\Gamma)} \\ &\le\; \|Q_0\varphi^h\|^2_{L_2(\Gamma)} \;+\; \sum_{k=1}^{J} 2^k\, \|\,(Q_k - Q_{k-1})\,\varphi^h\|^2_{L_2(\Gamma)} \\ &\le\; \|Q_0\varphi^h\|^2_{L_2(\Gamma)} \;+\; 4\sum_{k=1}^{J} 2^k\, \|\varphi^h - Q_k\varphi^h\|^2_{L_2(\Gamma)} \;\;. \end{aligned}$$

Let us estimate $\|\varphi^h - Q_k\varphi^h\|_{L_2(\Gamma)}$. For any $\psi \in H^1(\Gamma)$ we have

$$\begin{aligned} \|\varphi^h - Q_k\varphi^h\|_{L_2(\Gamma)} \;&=\; \|\varphi^h - \psi + \psi - Q_k\psi + Q_k\psi - Q_k\varphi^h\|_{L_2(\Gamma)} \\ &\le\; \|\varphi^h - \psi\|_{L_2(\Gamma)} + \|\psi - Q_k\psi\|_{L_2(\Gamma)} + \|Q_k(\psi - \varphi^h)\|_{L_2(\Gamma)} \\ &\le\; (1 + c_3')\|\varphi^h - \psi\|_{L_2(\Gamma)} \;+\; c_4 2^{-k}|\psi|_{H^1(\Gamma)} \;\;, \end{aligned}$$

where the constant c_3' arises from (2) by summing of $\sigma_i^{(k)}$. Taking the infimum over all $\psi \in H^1(\Gamma)$, we obtain the K-functional

$$\|\varphi^h - Q_k\varphi^h\|_{L_2(\Gamma)} \;\le\; c_5 \inf_{\psi \in H^1(\Gamma)} \left\{\|\varphi^h - \psi\| + 2^{-k}|\psi|_{H^1(\Gamma)}\right\} \;=\; c_5\, K_1(2^{-k}, \varphi^h) \;\;,$$

where the constant c_5 is independent of h. Using the equivalence of norms (see, e.g., [9, 24])

$$\|\varphi^h\|^2_{H^{1/2}(\Gamma)} \;\approx\; \|\varphi^h\|^2_{L_2(\Gamma)} + \sum_{k=1}^{\infty} 2^k\, K_1^2(2^{-k}, \varphi^h) \;\;,$$

one obtains the statement of Lemma 1. ∎

Denote by $x_i^{(k)}$, i=1,2, ... ,L_k the nodes of the triangulation Ω_k^h (we assume that nodes $x_i^{(k)}$ are enumerated first on Γ and then inside Ω) and define the extension operator t in the following way. For any $\varphi^h \in \mathbb{V}_J$ set

$$\psi_0^h \;=\; Q_0\,\varphi^h \;\;, \tag{3a}$$

$$\psi_k^h \;=\; (Q_k - Q_{k-1})\,\varphi^h, \quad k = 1, 2, \ldots, J \;\;. \tag{3b}$$

Then

$$\varphi^h = \psi_0^h + \psi_1^h + \ldots + \psi_J^h \ .$$

Define the extension $u_k^h \in \mathbb{W}_k$ of the function ψ_k^h according to [14, 22]:

$$u_0^h(x_i^{(0)}) = \begin{cases} \psi_0^h(x_i^{(0)}) & , \ x_i^{(0)} \in \Gamma \\ \overline{\psi} & , \ x_i^{(0)} \notin \Gamma \ , \end{cases} \tag{4a}$$

$$u_k^h(x_i^{(k)}) = \begin{cases} \psi_k^h(x_i^{(k)}) & , \ x_i^{(k)} \in \Gamma \ , \\ 0 & , \ x_i^{(k)} \notin \Gamma \ , \end{cases} \tag{4b}$$

For $\overline{\psi}$ we choose either the mean value of the boundary function ψ_0^h or the solution of the proper PDE on the coarsest grid with Dirichlet boundary conditions ψ_0^h.

Now, we define the extension

$$t\ \varphi^h := u^h \equiv u_0^h + u_1^h + \ldots + u_J^h \ . \tag{5}$$

By setting $t\ \varphi^h := v_J$ the definition above can be written in a recursive way :

$$v_0 := u_0^h \tag{6a}$$

$$v_k := v_{k-1} + u_k^h \qquad k = 1, \ldots, J \ . \tag{6b}$$

Note, that $v_k \in \mathbb{W}_k$ is the extension of $Q_k\varphi^h$ on level $k = 0, \ldots, J$.

REMARK 1. *We can use the L_2–orthoprojection from $L_2(\Omega)$ onto $\mathbb{V}_k$ instead of Q_k, $k = 0, 1, \ldots, J-1$. But in this case the cost of the decomposition (3) is expensive (especially for three dimensional problems).*

LEMMA 2. *There exists a positive constant c_6, independent of h, such that*

$$\|u_k^h\|_{H^1(\Omega)} \le c_6\ 2^k\ \|\psi_k^h\|_{L_2(\Gamma)}, \ k = 0, 1, \ldots, J.$$

PROOF The proof of this lemma is obvious and was done in [14]. ■

THEOREM 1. *There exists a positive constant c_7, independent of h, such that*

$$\|t\varphi^h\|_{H^1(\Omega)} \le c_7\|\varphi^h\|_{H^{\frac{1}{2}}(\Gamma)} \quad \forall\varphi^h \in \mathbb{V}_J \ .$$

Here the operator t is defined in (5).

PROOF We have (see,e.g., [24])

$$\begin{aligned}\|t\varphi^h\|^2_{H^1(\Omega)} &= \|u^h\|^2_{H^1(\Omega)} \le c_8 \inf_{\substack{w_k^h\in\mathbb{W}_k\\ \sum_{k=0}^J w_k^h = u^h}} \sum_{k=0}^J 4^k\,\|w_k^h\|^2_{L_2(\Omega)}\\ &\le c_8 \sum_{k=0}^J 4^k\,\|u_k^h\|^2_{L_2(\Omega)}\ .\end{aligned}$$

Here c_8 is independent of h and the u_k^h, $k=0,\dots,J$ are defined in (4). Then it follows from Lemma 1, 2 and from the special structure of the functions u_k^h that

$$\sum_{k=0}^J 4^k\,\|u_k^h\|^2_{L_2(\Omega)} \le c_9 \sum_{k=0}^J 2^k\,\|\psi_k^h\|^2_{L_2(\Gamma)} \le c_{10}\|\varphi^h\|^2_{H^{1/2}(\Gamma)}$$

holds, where c_9, c_{10} are independent of h. Note that $c_7 = \sqrt{c_8\cdot c_{10}}$. ∎

REMARK 2. *The construction of the extension operator t for three dimensional problems can be done in the same way. The Theorem 1 is valid too.*

If the original domain is split into many subdomains in domain decomposition methods [19], then the diameters of the subdomains depend on some small parameter ε and we need the extension operator t such that the constant c_7 from the Theorem 1 is independent of ε. To do this, let us assume that by making the change of variables

$$x = \varepsilon\cdot s, \quad x\in\Omega \tag{7}$$

the domain Ω is transformed into the domain Ω' with the boundary Γ' and that the properties of Ω' are independent of ε. From [19, 20] we have the following.

LEMMA 3. *There exists a positive constant c_{11}, independent of h and ε, such that*

$$c_{11}\|\varphi^h\|_{H_\varepsilon^{\frac12}(\Gamma)} \le \|u^h\|_{H^1(\Omega)}$$

for any function $u^h\in\mathbb{W}_J$, where $\varphi^h\in\mathbb{V}_J$ is the trace of u^h at the boundary Γ. And there exists a positive constant c_{12}, independent of h and ε, such that for any $\varphi^h\in\mathbb{V}_J$ there exists $u^h\in\mathbb{W}_J$:

$$\begin{aligned} u^h(x) &= \varphi^h(x), \quad x\in\Gamma,\\ \|u^h\|_{H^1(\Omega)} &\le c_{12}\|\varphi^h\|_{H_\varepsilon^{\frac12}(\Gamma)}\ .\end{aligned}$$

Here

$$\|\varphi^h\|^2_{H^{\frac{1}{2}}_{\varepsilon}(\Gamma)} = \varepsilon\|\varphi^h\|^2_{L_2(\Gamma)} + |\varphi^h|^2_{H^{\frac{1}{2}}(\Gamma)} \ ,$$

$$\|\varphi^h\|^2_{L_2(\Gamma)} = \int\limits_{\Gamma}(\varphi^h(x))^2 dx \ ,$$

$$|\varphi^h|^2_{H^{\frac{1}{2}}(\Gamma)} = \int\limits_{\Gamma}\int\limits_{\Gamma}\frac{(\varphi^h(x)-\varphi^h)y))^2}{|x-y|^2}dx\ dy \ .$$

LEMMA 4. *There exists a positive constant c_{13}, independent of h and ε, such that for any $\varphi^h \in \mathbb{V}_J$*

$$\|\varphi^h_0\|^2_{H^{\frac{1}{2}}_{\epsilon}(\Gamma)} + \frac{1}{\varepsilon}\|\varphi^h_1\|^2_{L_2(\Gamma)} + |\varphi^h_1|^2_{H^{\frac{1}{2}}(\Gamma)} \le c_{13}\|\varphi^h\|^2_{H^{\frac{1}{2}}_{\varepsilon}(\Gamma)}. \tag{8}$$

Here

$$\varphi^h_0 = Q_0\varphi^h, \quad \varphi^h_1 = \varphi^h - \varphi^h_0.$$

The following lemma is valid.

LEMMA 5. *There exists a positive constant c_{14}, independent of h and ε, such that*

$$\|\varphi^h_0\|^2_{H^{\frac{1}{2}}_{\varepsilon}(\Gamma)} + \frac{1}{\varepsilon}\left(\|Q_0\varphi^h_1\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J}2^k\|(Q_k - Q_{k-1})\varphi^h_1\|^2_{L_2(\Gamma)}\right) \le c_{14}\|\varphi^h\|^2_{H^{\frac{1}{2}}_{\epsilon}(\Gamma)}$$

Here φ^h_0, φ^h_1, are from (8).

PROOF Using (7) and Lemma 1, we have

$$\begin{aligned}
\tfrac{1}{\varepsilon}\|\varphi^h_1\|^2_{L_2(\Gamma)} + |\varphi^h_1|^2_{H^{\frac{1}{2}}(\Gamma)} &= \|\varphi^h_1\|^2_{L_2(\Gamma')} + |\varphi^h_1|^2_{H^{\frac{1}{2}}(\Gamma')} \\
&\le \tfrac{1}{c_1}\left(\|Q'_0\varphi^h_1\|^2_{L_2(\Gamma')} + \sum_{k=1}^{J}2^k\|(Q'_k - Q'_{k-1})\varphi^h_1\|^2_{L_2(\Gamma')}\right) \\
&= \tfrac{1}{\varepsilon c_1}\left(\|Q_0\varphi^h_1\|^2_{L_2(\Gamma)} + \sum_{k=1}^{J}2^k\|(Q_k - Q_{k-1})\varphi^h_1\|^2_{L_2(\Gamma)}\right) .
\end{aligned}$$

Here Q'_k is the projection which corresponds to Q_k with the change of variables. ■

THEOREM 2. *There exists a positive constant c_{15}, independent of h and ε, such that*

$$\|t\varphi^h\|_{H^1(\Omega)} \leq c_{15}\|\varphi^h\|_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} \quad \forall\varphi^h \in \mathbb{V}_J \ .$$

Here the operator t is defined in (5).

PROOF For φ_0^h, φ_1^h from (8) we have

$$\begin{aligned}
&\|Q_0\varphi^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} + \sum_{k=1}^{J} 2^k\|(Q_k - Q_{k-1})\varphi^h\|^2_{L_2(\Gamma)} \\
\leq\ & \|Q_0\varphi^h\|^2_{H_\varepsilon^{\frac{1}{2}}(\Gamma)} + \sum_{k=1}^{J} 2^k\|(Q_k - Q_{k-1})\varphi_1^h\|^2_{L_2(\Gamma)} \\
& + \sum_{k=1}^{J} 2^k\|(Q_k - Q_{k-1})\varphi_0^h\|^2_{L_2(\Gamma)} \ .
\end{aligned}$$

For the function φ_0^h consider the following decomposition:

$$\begin{aligned}
\varphi_0^h &= \varphi_{0,0}^h + \varphi_{0,1}^h, \\
\varphi_{0,0}^h &= \ \text{const} \ = \frac{1}{\text{meas}(\Gamma)} \int\limits_{\Gamma} \varphi_0^h(x)dx \\
\varphi_{0,1}^h &= \varphi_0^h - \varphi_{0,0}^h.
\end{aligned}$$

It is easy to see that $\quad (Q_k - Q_{k-1})\varphi_{0,0}^h = 0, \quad k = 1, 2, \cdots, J$.
Then we can use the evident trick from [14] with the Poincare inequality in $H^{\frac{1}{2}}(\Gamma')$:

$$\begin{aligned}
\sum_{k=1}^{J} 2^k\|(Q_k - Q_{k-1})\varphi_0^h\|^2_{L_2(\Gamma)} \ &= \ \sum_{k=1}^{J} 2^k\|(Q_k - Q_{k-1})\varphi_{0,1}^h\|^2_{L_2(\Gamma)} \\
&= \ \varepsilon\sum_{k=1}^{J} 2^k\|(Q'_k - Q'_{k-1})\varphi_{0,1}^h\|^2_{L_2(\Gamma')} \ \leq \ c_2\varepsilon\|\varphi_{0,1}^h\|^2_{H^{\frac{1}{2}}(\Gamma')} \\
&\leq \ c_{16}\,\varepsilon|\varphi_{0,1}^h|^2_{H^{\frac{1}{2}}(\Gamma^1)} \ = \ c_{16}\,\varepsilon|\varphi_{0,1}^h|^2_{H^{\frac{1}{2}}(\Gamma)} \ = \ c_{16}\,\varepsilon|\varphi_0^h|^2_{H^{\frac{1}{2}}(\Gamma)} \ .
\end{aligned}$$

Here c_{16} is from the Poincare inequality. It is easy to see that there exists a positive constant c_{17}, independent of h and ε, such that

$$\|u_0^h\|_{H^1(\Omega)} \leq c_{17}\|\psi_0^h\|_{H_\varepsilon^{\frac{1}{2}}(\Gamma)},$$

where $\psi_0^h = \varphi_0^h = Q_0\varphi^h$, and $u_0^h \in \mathbb{W}_0$ is from (4). The rest of the estimates for Theorem 2 and Theorem 1 is the same. ■

3 Realization of the Extension

Consider the symmetric, $\overset{\circ}{H}{}^1$–elliptic and $\overset{\circ}{H}{}^1$–bounded variational problem

$$\text{find } u \in \overset{\circ}{H}{}^1(\Omega) \; : \qquad \int_\Omega \lambda(x)\, \nabla^T u(x)\, \nabla v(x)\, dx \;=\; \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in \overset{\circ}{H}{}^1(\Omega)\,, \tag{9}$$

arising from the weak formulation of a scalar second–order, symmetric and uniformly bounded elliptic boundary value problem given in a plane bounded domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\Gamma = \partial\Omega$.

Define the usual finite elements (FE) nodal basis

$$\Phi = [\Phi_C, \Phi_I] = \left[\vartheta^J_1, \cdots, \vartheta^J_{N_C^{(J)}}, \vartheta^J_{N_C^{(J)}+1}, \cdots, \vartheta^J_{N_C^{(J)}+N_I^{(J)}}\right], \tag{10}$$

where the first $N_C^{(J)} = N_J$ basis functions on the finest level J correspond to nodes on Γ, the remaining basis functions are in the interior. The proper nodes will be denoted by "C" and "I". Similarly, $N_C^{(k)}$ and $N_I^{(k)}$ represent the number of basis functions on the boundary and in the interior on level k. Then the FE isomorphism leads to the symmetric and positive definite system of equations on each level k

$$K_k\, \underline{u}_k \;:=\; \begin{pmatrix} K_{C,k} & K_{CI,k} \\ K_{IC,k} & K_{I,k} \end{pmatrix} \begin{pmatrix} \underline{u}_{C,k} \\ \underline{u}_{I,k} \end{pmatrix} \;=\; \begin{pmatrix} \underline{f}_{C,k} \\ \underline{f}_{I,k} \end{pmatrix} \;=:\; \underline{f}_k \;, \tag{11}$$

where $K_{I,k}$ is symmetric, positive definite.

In the following, the matrices $I_{C,k}$, $I_{I,k}$ denote the proper identities on level k and the matrices $P^{k+1}_{C,k}$, $P^{k+1}_{I,k}$, $P^{k+1}_{IC,k}$ represent the usual linear FE interpolation matrices (Fig. 1) on the proper subsets of nodes.
The multilevel extension of a function $\psi^h = \sum_i^{N_C^{(J)}} \psi_i\, \varphi_i^{(J)} \in \mathbb{V}_J$ represented by the vector $\underline{\psi} \in \mathbb{R}^{N_C^{(J)}}$ into a function $u^h = \sum_i^{\overline{N_C^{(J)}+N_I^{(J)}}} u_i\, \vartheta_i^{(J)} \in \mathbb{W}_J$ represented by the vector $\underline{u} \in \mathbb{R}^{N_I^{(J)}}$ consists of three steps :

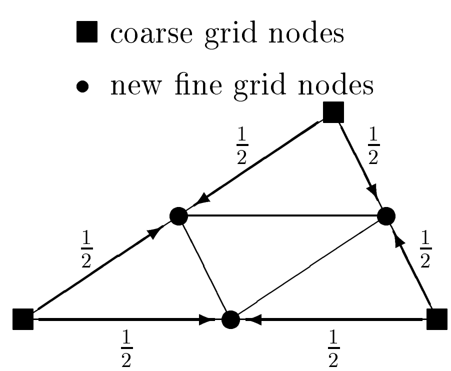

Figure 1: Linear FE interpolation

1. Determine the rectangular $N_C^{(k)} \times N_C^{(J)}$ projection matrix $\mathcal{Q}_k$ and define the coefficients of the projection $Q_k \psi^h$ in the FE nodal basis of level k

$$\underline{\beta}_k := \mathcal{Q}_k \underline{\psi} \qquad k = 0, \ldots, J \ . \tag{12}$$

2. According to (3) split the vectors $\underline{\beta}_k$ into the coefficients of the multi-level nodal basis presentation of $\psi^h = \sum_{k=0}^{J} \sum_{i}^{N_C^{(k)}} \alpha_i^{(k)} \varphi_i^{(k)}$ and determine the coefficient vectors $\underline{\alpha}_k$

$$\underline{\alpha}_0 \ := \ \underline{\beta}_0 \tag{13a}$$

$$\underline{\alpha}_k \ := \ \begin{pmatrix} -P_{C,k-1}^k & I_{C,k} \end{pmatrix} \begin{pmatrix} \underline{\beta}_{k-1} \\ \underline{\beta}_k \end{pmatrix} \qquad k = 1, \ldots, J \ . \tag{13b}$$

3. The coefficients $\underline{v}_k$ of the extensions $v_k = \sum_{i=1}^{N_C^{(k)}+N_I^{(k)}} v_i^{(k)} \vartheta_i^{(k)}$ are determined by

$$\underline{v}_0 = \begin{pmatrix} \underline{v}_{C,0} \\ \underline{v}_{I,0} \end{pmatrix} \ := \ \begin{pmatrix} I_{C,0} \\ B_{IC,0} \end{pmatrix} \underline{\alpha}_0 \tag{14a}$$

$$\underline{v}_k = \begin{pmatrix} \underline{v}_{C,k} \\ \underline{v}_{I,k} \end{pmatrix} \ := \ \begin{pmatrix} I_{C,k} & P_{C,k-1}^k & 0 \\ 0 & P_{IC,k-1}^k & P_{I,k-1}^k \end{pmatrix} \begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} \ . \tag{14b}$$

Denote by E the matrix representation of the extension t (5), then we set $E\underline{\varphi} := \underline{v}_J$.
The matrix $B_{IC,0}$ can be chosen as $\frac{1}{N_C^{(0)}} \left(\begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{smallmatrix}\right)_{N_I^{(0)} \times N_C^{(0)}}$, mapping the mean value of the boundary data into the interior. Another approach is the discrete harmonic extension on the coarsest grid with respect to the PDE, i.e., $B_{IC,0} = -K_{I,0}^{-1} K_{IC,0}$.

4 Improving the Extension by an Iteration Scheme

Denote by $\underline{z}_{I,k}$ some extension of boundary data $\underline{\alpha}_{C,k}$, then the functional $\mathcal{J}(\underline{z}_{I,k}) = \left(K_k \begin{pmatrix} \underline{\alpha}_{C,k} \\ \underline{z}_{I,k} \end{pmatrix}, \begin{pmatrix} \underline{\alpha}_{C,k} \\ \underline{z}_{I,k} \end{pmatrix} \right)$ represents the square of the energy norm of that extension. The extensions given in (6) are just an approximation of the following minimization problem

$$\underline{v}_{I,k} \ = \ \arg\min_{\underline{z}_{I,k}} \mathcal{J}(\underline{z}_{I,k}) \ , \tag{15}$$

which is equivalent to the system of equations

$$\begin{array}{rcll} \underline{v}_{C,k} & = & \underline{\alpha}_{C,k} & \\ K_{I,k}\underline{v}_{I,k} & = & -K_{IC,k}\underline{\alpha}_{C,k} & \qquad k=0,\ldots,J \ . \end{array} \tag{16}$$

To improve the quality of those extensions given in (6), i.e., decrease the constants in Theorems 1 and 2, we apply some iteration scheme on (16)

$$\underline{v}^{j}_{I,k} \ := \ \underline{v}^{j-1}_{I,k} - \tau^j_k B_{I,k} \cdot \left(K_{I,k}\underline{v}^{j-1}_{I,k} + K_{IC,k}\underline{\alpha}_{C,k} \right) \ . \tag{17}$$

By defining the iteration matrix $M^{\nu_k}_{I,k} := \prod_{j=1}^{\nu_k} \left(I_{I,k} - \tau^j_k B_{I,k} K_{I,k} \right)$ the ν_k iterations can be rewritten to

$$\underline{v}^{\nu_k}_{I,k} \ := \ M^{\nu_k}_{I,k}\underline{v}^0_{I,k} - \left(I_{I,k} - M^{\nu_k}_{I,k} \right) K^{-1}_{I,k} K_{IC,k}\underline{\alpha}_{C,k} \tag{18}$$

with some initial guess $\underline{v}^0_{I,k}$. To define the preconditioner $B_{I,k}$, for instance, we can split $K_{I,k}$ into the strictly upper, diagonal and lower part, i.e., $K_{I,k} = L_{I,k} + D_{I,k} + U_{I,k}$, then the Gauß-Seidel iteration matrix is defined via $M_{I,k} := (D_{I,k} + L_{I,k})^{-1} U_{I,k}$.

There are two opportunities to choose the initial guess $\underline{v}^0_{I,k}$ and the boundary data $\underline{\alpha}_{C,k}$.
First, we start the iteration with the zero initial guess and the discrete representation of ψ^h_k, i.e., $\underline{\alpha}_k$, on the boundary. This changes (14b) into

$$\begin{pmatrix} \underline{v}_{C,k} \\ \underline{v}_{I,k} \end{pmatrix} := \begin{pmatrix} I_{C,k} & P^k_{C,k-1} & 0 \\ -(I_{I,k} - M^{\nu_k}_{I,k})K^{-1}_{I,k}K_{IC,k} & P^k_{IC,k-1} & P^k_{I,k-1} \end{pmatrix} \begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} \ . \tag{19}$$

The second opportunity consists in using the actual extension on level k as initial guess and the proper boundary data $\underline{v}_{C,k}$. The relation $\underline{v}_{C,k} = \underline{\alpha}_k + P^k_{C,k-1}\underline{v}_{C,k-1}$ leads directly to the second reformulation of (14b)

$$\begin{pmatrix} \underline{v}_{C,k} \\ \underline{v}_{I,k} \end{pmatrix} := \begin{pmatrix} I_{C,k} & 0 \\ -(I_{I,k} - M^{\nu_k}_{I,k})K^{-1}_{I,k}K_{IC,k} & M^{\nu_k}_{I,k} \end{pmatrix} \begin{pmatrix} I_{C,k} & P^k_{C,k-1} & 0 \\ 0 & P^k_{IC,k-1} & P^k_{I,k-1} \end{pmatrix} \begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} \ . \tag{20}$$

In the following, we omit the level index k. Recalling the functional $\mathcal{J}(\underline{z}_I)$, the proper error functional is $\mathcal{E}(\underline{z}_I) = \| \underline{z}_I - \underline{w}_I \|^2_{K_I}$ with $\underline{w}_I = -K^{-1}_I K_{IC}\underline{\alpha}_C$ as the exact discrete extension.

LEMMA 6. *Denote by* $\underline{w}_I^j$ *the* j*-th iterate in iteration scheme (17), which converges in the* K_I*-energy norm, i.e., there exists a positive constant* $q < 1$ *such that*

$$\| \underline{w}_I^j - \underline{w}_I \|_{K_I} \leq q \| \underline{w}_I^{j-1} - \underline{w}_I \|_{K_I} \tag{21}$$

holds for all $j > 0$*. Then the extension* $\begin{pmatrix} \underline{\alpha}_C \\ \underline{w}_I^j \end{pmatrix}$ *fulfills*

$$\mathcal{J}(\underline{w}_I^j) = \mathcal{J}(\underline{w}_I) + \mathcal{E}(\underline{w}_I^j) \leq \mathcal{J}(\underline{w}_I) + q^{2j}\mathcal{E}(\underline{w}_I^0)$$

with some initial guess $\underline{w}_I^0$*.*

PROOF With the definitions for $\mathcal{J}$ and $\mathcal{E}$ we write

$$\begin{aligned}
\mathcal{J}(\underline{w}_I^j) &= \left(K \begin{pmatrix} \underline{\alpha}_C \\ \underline{w}_I^j \end{pmatrix}, \begin{pmatrix} \underline{\alpha}_C \\ \underline{w}_I^j \end{pmatrix} \right) \\
&= \left(K_I \underline{w}_I^j, \underline{w}_I^j\right) + \left(K_{IC}\underline{\alpha}_C, \underline{w}_I^j\right) + \underbrace{\left(K_{CI}\underline{w}_I^j, \underline{\alpha}_C\right)}_{=(\underline{w}_I^j, K_{IC}\underline{\alpha}_C)} + (K_C \underline{\alpha}_C, \underline{\alpha}_C) \\
&= \left(K_I \underline{w}_I^j, \underline{w}_I^j\right) + \Big(K_I \underbrace{K_I^{-1} K_{IC}\underline{\alpha}_C}_{=-\underline{w}_I}, \underline{w}_I^j\Big) + \Big(\underline{w}_I^j, K_I \underbrace{K_I^{-1} K_{IC}\underline{\alpha}_C}_{=-\underline{w}_I}\Big) \\
&\qquad + (K_I \underline{w}_I, \underline{w}_I) - (K_I \underline{w}_I, \underline{w}_I) + (K_C \underline{\alpha}_C, \underline{\alpha}_C) \\
&= \left(K_I(\underline{w}_I^j - \underline{w}_I), (\underline{w}_I^j - \underline{w}_I)\right) - (K_I \underline{w}_I, \underline{w}_I) + (K_C \underline{\alpha}_C, \underline{\alpha}_C) \\
&= \underbrace{\| \underline{w}_I^j - \underline{w}_I \|_{K_I}^2}_{\mathcal{E}(\underline{w}_I^j)} \underbrace{- \| \underline{w}_I \|_{K_I}^2 + \| \underline{\alpha}_C \|_{K_C}^2}_{\mathcal{J}(\underline{w}_I)} \\
&\overset{(21)}{\leq} q^2 \| \underline{w}_I^{j-1} - \underline{w}_I \|_{K_I}^2 + \mathcal{J}(\underline{w}_I) = q^2 \mathcal{E}(\underline{w}_I^{j-1}) + \mathcal{J}(\underline{w}_I) \\
&\overset{(21)}{\leq} q^{2j} \| \underline{w}_I^0 - \underline{w}_I \|_{K_I}^2 + \mathcal{J}(\underline{w}_I) = q^{2j}\mathcal{E}(\underline{w}_I^0) + \mathcal{J}(\underline{w}_I) \ .
\end{aligned}$$

■

5 Application to a DD Preconditioner

The domain Ω will be decomposed into p non-overlapping subdomains Ω_s $(s = 1, \ldots, p)$ such that $\overline{\Omega} = \bigcup_{s=1}^p \overline{\Omega}_s$. The grid triangulations Ω^h, will be distributed analogously for all level $k = 0, \ldots, J$.

Now, the submatrices in system (11) are changed into blockmatrices, especially $K_I = \operatorname{diag}(K_{I,i})_{i=1,\ldots,p}$. This new system will be solved by some parallelized preconditioned iterative method, e.g., CG-method. As a preconditioner we use the ASM–DD preconditioner

$$C = \begin{pmatrix} I_C & -B_{IC}^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ -B_{IC} & I_I \end{pmatrix} \quad . \tag{22}$$

This preconditioner contains the three components $C_I = \operatorname{diag}(C_{I,i})_{i=1,\dots,p}$, C_C and the block matrix B_{IC}, which can be chosen freely in order to adapt the preconditioner to the particulars of the problem under consideration. For the choice $B_{IC,i} = -B_{I,i}K_{IC,i}$ see [13]. As preconditioner C_C for the Schur complement $S_C = K_C - K_{CI}K_I^{-1}K_{IC}$ the BPS [6] is used.

The preconditioning step $\underline{w} = C^{-1}\underline{r}$ can be rewritten in the form

Algorithm 1 : The ASM-DD Preconditioner [13]
$\underline{\mathbf{w}}_C \;=\; C_C^{-1} \sum\limits_{i=1}^{p} A_{C,i}^T \left(\underline{r}_{C,i} + B_{IC,i}^T \underline{r}_{I,i}\right)$
$\underline{\mathbf{w}}_{I,i} \;=\; C_{I,i}^{-1}\underline{r}_{I,i} + B_{IC,i}\underline{\mathbf{w}}_{C,i} \qquad ; \; i = 1, 2, \dots, p$

where $A_i = \left(\begin{smallmatrix} A_{C,i} & A_{CI,i} \\ A_{IC,i} & A_{I,i} \end{smallmatrix}\right)$ denotes the subdomain connectivity matrix which is used for a convenient notation only. The subdomain FE assembly process which is connected with nearest neighbour communication stands behind this notation. For further investigations on DD preconditioners see [17, 13, 12, 16, 21, 20].

Assume positive, h-independent spectral equivalence constants $\underline{\gamma}_C, \overline{\gamma}_C, \underline{\gamma}_I, \overline{\gamma}_I$ fulfilling the spectral equivalence inequalities

$$\underline{\gamma}_C\, C_C \;\le\; S_C \;\le\; \overline{\gamma}_C\, C_C \qquad \text{and} \qquad \underline{\gamma}_I\, C_I \;\le\; K_I \;\le\; \overline{\gamma}_I\, C_I \quad .$$

If we have additionally a constant c_E so that

$$\left\| \begin{pmatrix} \underline{v}_C \\ B_{IC}\underline{v}_C \end{pmatrix} \right\|_K \;\le\; c_E \, \| \, \underline{v}_C \, \|_{S_C} \qquad \forall \underline{v}_C \in \mathbb{R}^{N_c} \tag{23}$$

holds then the upper and lower bounds of the condition number $\kappa(C^{-1}K)$ [13, 8] can be estimated as

$$\mathcal{O}(c_E^2) \;\le\; \kappa(C^{-1}K) \;\le\; \mathcal{O}(c_E^4) \quad . \tag{24}$$

Estimate (23) represents the result from Theorem 1 in a discrete sense, so that B_{IC} can be chosen as the discrete extension operator defined in (14), (19) or (20). Additionally, Algorithm 1 requires $B_{CI} = B_{IC}^T$ so that the transposed of that extension have to be applied. Whereas transposing (14) is quite simple one have to take care when smoothing is included.

If we denote by $\overset{*}{M}_{I,k}$ the adjoint operator to $M_{I,k}$ with respect to the $K_{I,k}$ inner product, then the transposed operation to (19) can be written as

$$\begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} = \begin{pmatrix} \underline{v}_{C,k} - K_{CI,k}\left(I_{I,k} - \left(\overset{*}{M}_{I,k}\right)^{\nu_k}\right) K_{I,k}^{-1}\,\underline{v}_{I,k} \\ \left(P_{C,k-1}^k\right)^T \underline{v}_{C,k} + \left(P_{IC,k-1}^k\right)^T \underline{v}_{I,k} \\ \left(P_{I,k-1}^k\right)^T \underline{v}_{I,k} \end{pmatrix}, \tag{25}$$

i.e., we have to perform ν_k sweeps of the iteration procedure defined by $\overset{*}{M}_{I,k}$ with the right hand side $\underline{v}_{I,k}$ and a zero initial guess. If $M_{I,k}$ represents a lexicographically forward Gauß-Seidel iteration, then the adjoint iteration is the lexicographically backward one.

In the transposed operation to (20), the action

$$\begin{aligned} \left(M_{I,k}^T\right)^{\nu_k} \underline{v}_{I,k} &= \underline{v}_{I,k} - K_{I,k}K_{I,k}^{-1}\underline{v}_{I,k} + \left(M_{I,k}^T\right)^{\nu_k} \underline{v}_{I,k} \\ &= \underline{v}_{I,k} - K_{I,k}\underbrace{\left(I_{I,k} - \left(\overset{*}{M}_{I,k}\right)^{\nu_k}\right) K_{I,k}^{-1}\underline{v}_{I,k}}_{=:\ \underline{w}_{I,k}} \end{aligned}$$

is just the defect calculation using the result $\underline{w}_{I,k}$ of ν_k iterations with the iteration operator $\overset{*}{M}_{I,k}$. So, this transposed operation can be written as

$$\begin{pmatrix} \underline{\alpha}_k \\ \underline{v}_{C,k-1} \\ \underline{v}_{I,k-1} \end{pmatrix} = \begin{pmatrix} \underline{v}_{C,k} - K_{CI,k}\underline{w}_{I,k} \\ \left(P_{C,k-1}^k\right)^T \left[\underline{v}_{C,k} - K_{CI,k}\underline{w}_{I,k}\right] + \left(P_{IC,k-1}^k\right)^T \left[\underline{v}_{I,k} - K_{I,k}\underline{w}_{I,k}\right] \\ \left(P_{I,k-1}^k\right)^T \left[\underline{v}_{I,k} - K_{I,k}\underline{w}_{I,k}\right] \end{pmatrix}. \tag{26}$$

When the operator C_I in Algorithm 1 is also defined as a multilevel operator, the action $C_I^{-1}\underline{r}_I$ can be combined with $B_{IC}^T\underline{r}_I$, see [11, 10].

6 Numerical Experiments

In the numerical experiments we used two simple and one challenging examples.

Example 1:
$$\begin{aligned} -\triangle u &= 1 \quad \text{in}\,\Omega = [0,1]\times[0,0.5] \\ u &= 0 \quad \text{on}\,\partial\Omega \end{aligned}$$

Example 2:
$$\begin{aligned} -\triangle u &= 1 \quad \text{in}\,\Omega = [0,1]^2 \\ u &= 0 \quad \text{on}\,\partial\Omega \end{aligned}$$

For Example 1, the domain Ω was subdivided into 2 squares, while the domain in Example 2 was partitioned into 16 squares.

Example 3 (Electrical machine): As a more challenging example we calculated the magnetic potential in an electrical machine with a rather complex geometry and large jumps in the coefficients (for more details see [15]), for the decomposition of the domain into 16 subdomains see Fig. 2.

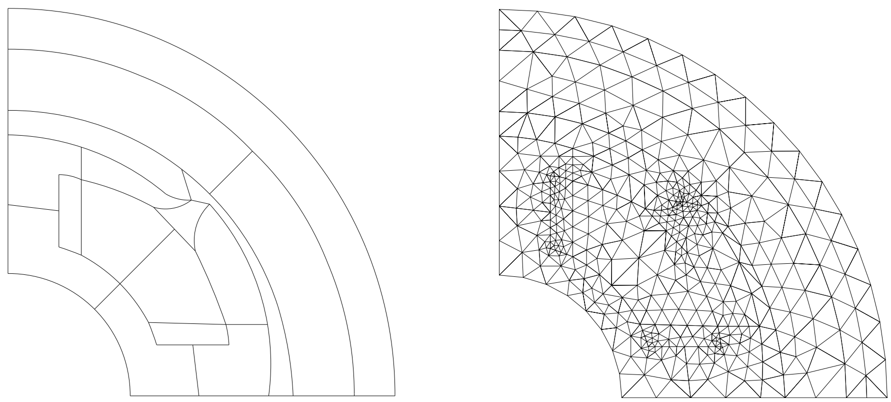

Figure 2: Material adapted decomposition and initial mesh of Example 3

All calculations were done on a 16 processor Parsytec POWER-XPLORER with 32 MByte memory per node. All examples were solved with the preconditioned parallelized CG using Algorithm 1 as preconditioning step until an accuracy, measured in the $KC^{-1}K$-energy norm, of 10^{-6} was achieved. As Schur complement preconditioner C_C the BPS [6] was used. Unless mentioned specifically, the inner problem was solved exactly, i.e., $C_I = K_I$. For comparison we used in example 3 also a multigrid V-cycle with one pre- and one post-smoothing sweep (V11) for defining C_I. The iteration procedure (18) implemented via (20) was applied at the most one time ($\nu \in \{0, 1\}$). In tables 1 - 3 the projection (1a) was tested; the tables 4 - 6 present the results using projection (1b). For measuring the quality of the extensions we calculate $C_{\text{aver}} = \| E\underline{\varphi} \|_K \Big/ \left\| \begin{pmatrix} \underline{\varphi} \\ -K_I^{-1} K_{IC} \underline{\varphi} \end{pmatrix} \right\|_K$, the ratio between an approximate extension $E\underline{\varphi}$ and the exact one in the energy norm.

proj. (1a)	ν	$J=0$	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
Iterations	0	2	7	8	11	12	13	13
C_{aver}	0	1.00	1.05	1.07	1.10	1.14	1.17	1.19
Iterations	1	2	6	8	8	8	9	9
C_{aver}	1	1.00	1.01	1.03	1.05	1.06	1.07	1.07

Table 1: # CG-iterations for Example 1 using 2 processors

proj. (1a)	ν	$J=0$	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
Iterations	0	9	13	15	17	19	21	22
C_{aver}	0	1.00	1.04	1.06	1.10	1.13	1.15	1.17
Iterations	1	9	13	15	15	16	18	20
C_{aver}	1	1.00	1.02	1.04	1.05	1.06	1.07	1.08

Table 2: # CG-iterations for Example 2 (459777 d.o.f.) using 16 processors

proj. (1a)	ν	$J=0$	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$
# unknowns		440	1.715	6.773	26.921	107.345	428.705
C_I exact: Iterations	0	25	33	41	50	59	69
Quality C_{aver}	0	1.00	1.05	1.08	1.11	1.13	1.14
C_I exact: Iterations	1	25	32	39	46	52	59
Quality C_{aver}	1	1.00	1.02	1.04	1.05	1.06	1.06
C_I - V11: Iterations	0	25	33	41	50	59	67
solver in sec.	0	1.5	2.2	3.5	7.1	21.5	80.5
C_I - V11: Iterations	1	25	32	39	45	51	58
solver in sec.	1	1.7	2.2	3.4	6.8	21.5	87.2

Table 3: # CG-iterations for Example 3 using 16 processors

For all three examples, the behavior of the iteration counts reflects the logarithmic grow of the condition number $\kappa(C^{-1}K)$ for the BPS Schur complement preconditioner, i.e., the constant c_E (23) seems to be h-independent. Also the quality ratio C_{aver} seems to be bounded with growing level number J, especially in example 3. Both observations are in agreement with the result of Theorem 1. The additional iteration ($\nu = 1$) for defining the extension decreases the iteration count and really improves the quality of the extension. The solver times in Table 3 indicate that the additional iteration does not speed up the solution process for the example presented.

proj. (1b)	ν	$J=0$	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
Iterations	0	2	7	10	12	13	14	14
C_{aver}	0	1.00	1.06	1.08	1.12	1.15	1.17	1.19
Iterations	1	2	6	8	9	9	9	9
C_{aver}	1	1.00	1.01	1.02	1.04	1.05	1.07	1.07

Table 4: # CG-iterations for Example 1 using 2 processors

proj. (1b)	ν	$J=0$	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
Iterations	0	9	13	16	19	20	22	23
C_{aver}	0	1.00	1.04	1.09	1.12	1.16	1.17	1.18
Iterations	1	9	13	14	16	17	18	19
C_{aver}	1	1.00	1.01	1.03	1.04	1.05	1.06	1.06

Table 5: # CG-iterations for Example 2 (459777 d.o.f.) using 16 processors

proj. (1b)	ν	$J=0$	$J=1$	$J=2$	$J=3$	$J=4$	$J=5$
# unknowns		440	1.715	6.773	26.921	107.345	428.705
Iterations	0	25	34	43	52	60	70
Quality C_{aver}	0	1.00	1.05	1.10	1.13	1.16	1.17
Iterations	1	25	32	39	46	53	59
Quality C_{aver}	1	1.00	1.02	1.04	1.05	1.05	1.05
C_I - V11: Iterations	0	25	34	43	53	60	68
solver in sec.	0	1.6	2.4	3.6	7.4	20.8	81.5
C_I - V11: Iterations	1	25	32	39	47	54	60
solver in sec.	1	1.6	2.1	3.5	7.5	22.8	90.0

Table 6: # CG-iterations for Example 3 using 16 processors

Although the iteration counts in tables 4 - 6 are slightly higher, we can draw the same conclusions for projection (1b) as for the projection according to (1a).

7 Conclusions

The extension technique (5) presented in Section 2 is a fast and qualitatively good approximation of the homogeneous extension of Laplacian-like differential operators when using the projections (1). The independence of the constants in Theorems 1 and 2 from the discretization parameter h and the diameter ε of the domain is still valid in the 3D-case but has to be tested in future. In combination with a proper iteration procedure (18), the extension procedure works also with respect to more complicated second order differential operators (see [11]). For the examples given, the additional iteration step per level did not result in an improved overall solution time. But this property may change in other examples.

When using the extensions in a DD preconditioner, the transposed of those extensions is needed. In combination with a proper preconditioner C_I a sophisticated implementation reduces significantly the time per iteration (for more details see [11, 10]). Here, again the 3D-case has to be investigated. Using an efficient h-independent preconditioner C_I, e.g. multigrid, the asymptotic behavior of the condition number $\kappa(C^{-1}K)$ depends only on the asymptotic behavior of the Schur complement preconditioner C_C. Therefore, the numerical effort of the whole parallel algorithm is nearly optimal (logarithmically optimal).

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