Numerical Estimates of Inequalities in $H^{1/2}$

Ferdinand Kickinger, Sergei V. Nepomnyaschikh,
Ralf U. Pfau, Joachim Schöberl*

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Abstract

The Sobolev norm $H^{1/2}(\Gamma)$ plays a key role in domain decomposition (DD) techniques. For the efficiency of DD - preconditioners the quantitative values of several constants is important.

The goal of this paper is the numerical investigation of the constants in explicit extensions $H^{1/2}(\Gamma) \to H^1(\Omega)$ for the two and three dimensional case, the discrete imbedding of $H^{1/2}(\Gamma)$ in $L_\infty(\Gamma)$ and of the norm estimates between $H^{1/2}(\Gamma)$ and $H_{00}^{1/2}(\Gamma)$.

1 Introduction

Non-overlapping domain decomposition preconditioning is based on several operators between finite element subspaces of Sobolev spaces in the domain and on the boundary.

Theoretically, each operator can be performed to obtain optimal, i.e. mesh-size independent iteration numbers. The goal of this paper is to investigate numerically the quality of some optimal and nearly optimal components.

First, we give some definitions of norms(see, e.g. [1]). For measurable functions u on $\Omega \subset \mathbf{R}^d, \quad d = 2, 3$ we define the Lebesgue Norm and the Sobolev Norm

$$\begin{aligned} \|u\|_{L_2(\Omega)}^2 &:= \int_\Omega u^2 \,\mathrm{dx} \\ \|u\|_{H^1(\Omega)}^2 &:= \|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)} \end{aligned}$$

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and the according function spaces $L_2(\Omega)$ and $H^1(\Omega)$. Let $\Gamma := \partial\Omega$. The proper norm of boundary traces $\varphi = \operatorname{tr} u = u|_\Gamma$ is the $H^{1/2}(\Gamma)$-Norm defined as

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} := \int\limits_\Gamma \int\limits_\Gamma \frac{(\varphi(x)-\varphi(y))^2}{(x-y)^2} \mathrm{dxdy} + \int\limits_0^1 \varphi(x)^2 \mathrm{dx}.$$

Then the trace theorem is fulfilled (see [2], [3], [1]).

Theorem 1 (Trace theorem) *There exists constants c_1 and c_2 such that*
1. $\forall u \in H^1(\Omega)$ and $\varphi := \operatorname{tr} u$ there holds

$$\|\varphi\|_{H^{1/2}(\Gamma)} \leq c_1 \|u\|_{H^1(\Omega)}$$

2. $\forall \varphi \in H^{1/2}(\Gamma)\ \exists u \in H^1(\Omega), \operatorname{tr} u = \varphi$ such that

$$\|u\|_{H^1(\Omega)} \leq c_2 \|\varphi\|_{H^{1/2}(\Gamma)}$$

Further notation fixed throughout this paper is h for the finite element mesh size, $n = 1/h$, N is the dimension of the finite element space. $H_h(\Omega_h)$ or H_h is the finite element space on the domain and $V_h(\Gamma_h)$ or V_h is the trace space. Finite element functions on the domain will be named u_h or v_h, trace functions φ_h. Vector representations in $\mathbf{R}^N$ will be denoted by $\underline{u}$ or $\underline{\varphi}$.

The trace theorem holds also for certain finite element spaces [5]:

Theorem 2 (Finite Element Trace theorem) *Assume, the minimal angle of the underlying mesh is bounded uniformly from below. Then there exists constants c_1 and c_2 such that*
1. $\forall u_h \in H_h(\Omega)$ and $\varphi_h := \operatorname{tr} u_h \in V_h(\Gamma)$ there holds

$$\|\varphi_h\|_{V_h(\Gamma)} \leq c_1 \|u_h\|_{H_h(\Omega)}$$

2. $\forall \varphi_h \in V_h(\Gamma)\ \exists u_h \in H_h(\Omega), \operatorname{tr} u_h = \varphi_h$ such that

$$\|u_h\|_{H_h(\Omega)} \leq c_2 \|\varphi_h\|_{V_h(\Gamma)}$$

The outline of the paper is as follows. In Section 2 optimal explicit extension operators for 2D and 3D are investigated. Section 3 the norms for the discrete imbeddings from $(V_h(\Gamma_h), \|.\|_{H^{1/2}})$ into $(V_h(\Gamma_h), \|.\|_{L_\infty})$ are calculated. Section 4 deals with the connection of boundary condition and the space $(V_h, \|.\|_{H^{1/2}})$.

2 Explicit Extension Operators

First, we want to investigate the constants of explicit extension operators. Therefore, we consider the following extension problem on the domain $\Omega = (0,4) \times (0,1)$ of Fig. 1:

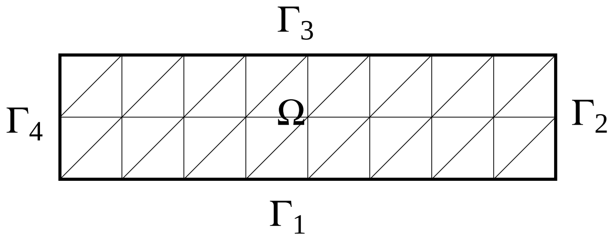

Figure 1: Problem Geometry and mesh

Given $\varphi \in H^{1/2}(\Gamma_1)$
Find $u \in H_0 := \{v \in H^1(\Omega) : v|_{\Gamma_3} = 0,\ v|_{\Gamma_2} = v|_{\Gamma_4}\}$
such that
$u|_{\Gamma_1} = \varphi, \quad \|u\|_{H_0} \leq c\|\varphi\|_{H^{1/2}(\Gamma_1)}$

In [5], the explicit extension operator t

$$u(x,y) = \mathrm{t}\,\varphi := \frac{1-y}{2y}\int_{x-y}^{x+y} \varphi(x)\mathrm{dx} \tag{1}$$

and the discrete analogue t_h are suggested and continuity is proven.

We want to determine values for the constants c such that the inequality

$$\|\,\mathrm{t}_h\,\varphi_h\|_{H_h} \leq c \inf_{\substack{v_h \in H_h \cap H_0 \\ v_h|_{\Gamma_1} = \varphi_h}} \|v_h\|_{H_h}$$

is sharp. The right-hand side is the extension with minimal energy, which corresponds to the solution of a Dirichlet-problem. It can be expressed by

$$\inf_{\substack{v_h \in H_h \cap H_0 \\ v_h|_{\Gamma_1} = \varphi_h}} \|v_h\|_{H_h} = \|\varphi_h\|_{S_C}$$

with the Schur complement Matrix

$$S_C = A_C - A_{CI}A_I^{-1}A_{CI}$$

with respect to the unknowns on the boundary Γ_1. We used a splitting of the H^1 matrix A

$$A = \begin{pmatrix} A_C & A_{CI} \\ A_{IC} & A_I \end{pmatrix}$$

into Γ_1 components (C) and other unknowns (I).

h	N	c
.1	400	1.2663
.05	1600	1.3205
.025	6400	1.3382
.0125	25600	1.3559
.00625	102400	1.3668

Table 1: Extension constants, 2D

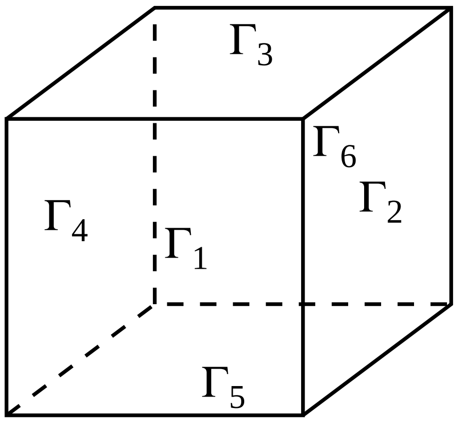

Figure 2: 3D problem geometry

To constant c^2 can be computed as largest eigen-value of the generalized eigen-value problem

$$(t_h^T A t_h)\varphi_h = \lambda S_c \varphi_h.$$

We calculated the constant c on meshes of mesh-size between $h = 1/10$ and $h = 1/160$ and got the results of Table 1. We observe that this explicit extension is fairly good and it can be implemented in a very efficient manner.

Next, we tested the corresponding extension for three dimensional problems, and calculated the constant. We consider the problem

$$\begin{aligned}
&\text{given } \varphi \in H^{1/2}(\Gamma_1) \\
&\text{find } u \in H_0 := \{v \in H^1(\Omega) : v|_{\Gamma_3} = 0,\ v|_{\Gamma_2} = v|_{\Gamma_4},\ v|_{\Gamma_3} = v|_{\Gamma_5}\} \\
&\text{such that} \\
&u|_{\Gamma_1} = \varphi, \quad \|u\|_{H_0} \le c\|\varphi\|_{H^{1/2}(\Gamma_1)},
\end{aligned}$$

on the domain $\Omega = (0,1)^3$, see Fig. 2. We discretized Ω by a uniform tetrahedral mesh of mesh-size h and chose a piece-wise linear finite element space. We tested the extension operator t

h	N	non GS	1 GS	4 GS	8 GS
.25	64	1.04415	1.00016	$\lvert . - 1\rvert < 1e-6$	$\lvert . - 1\rvert < 1e-8$
.125	512	1.15665	1.00226	$\lvert . - 1\rvert < 1e-6$	$\lvert . - 1\rvert < 1e-6$
.0625	4096	1.24827	1.05252	1.00556	1.00051
.03125	32768	1.30517	1.13796	1.06891	1.03033

Table 2: Extension constants, 3D

$$u(x, y, z) = \mathrm{t}\,\varphi := \frac{1-z}{4z^2} \int_{x-z}^{x+z} \int_{y-z}^{y+z} \varphi(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

To improve the constant, we added several steps of Gaus-Seidel iteration to the extension operator. The results are shown in Table 2.

3 About an Inequality Between L_∞ and $H^{1/2}$

It is known that for $u^h \in H_h^1(\Omega), \Omega \subset I\!R^2$, the inequality

$$\| u^h \|_{L_\infty(\Omega)} \leq c \cdot \left(\log\ h^{-1}\right)^{1/2} \| u^h \|_{H^1(\Omega)}$$

is valid with a constant c independent of the mesh size h. (see [4], [6])

This result can easily extended to an inequality on the one-dimensional interface Γ between the spaces $L_\infty(\Gamma)$ and $H^{1/2}(\Gamma)$:

Lemma

$$\| \varphi^h \|_{L_\infty(\Gamma)} \leq c \cdot \left(\log\ h^{-1}\right)^{1/2} \| \varphi^h \|_{H^{1/2}(\Gamma)}$$

Proof

Without loosing the generality Γ is identified with the interval $[0, 1]$. The interval is extended to the unit square $\Omega = [0, 1] \times [0, 1]$ where Γ is one edge of the square.

Let

$$u^h = \underset{v^h |_\Gamma = \varphi^h}{argmin} \| v^h \|_{H^1(\Omega)}$$

Then Theorem states that

$$\|u^h\|_{H^1(\Omega)} \leq c_1 \|\varphi^h\|_{H^{1/2}(\Gamma)}$$

with a constant c_1 independent of the mesh size h.

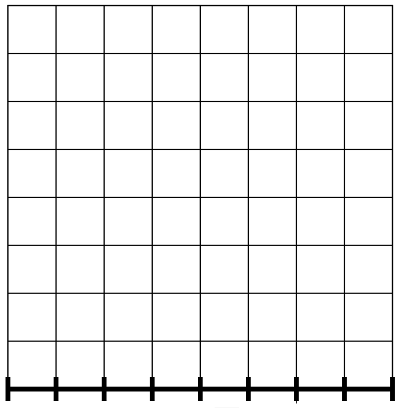

Figure 3: Unit square with Γ as one edge

Using the upper inequality for u^h we obtain

$$\|\varphi^h\|_{L_\infty(\Gamma)} \leq \|u^h\|_{L_\infty(\Omega)} \leq c_2 \left(\log\ h^{-1}\right)^{1/2} \|u^h\|_{H^1(\Omega)} \leq c_3 \left(\log\ h^{-1}\right)^{1/2} \|\varphi^h\|_{H^{1/2}(\Gamma)}$$

Now the aim of this study is to compute

$$u^h = \underset{v^h(0,0)=1}{argmin} \ \| v^h \|_{H^1(\Omega)}$$

with $\Omega = [0,1]^2$ with a uniform triangular mesh for different mesh sizes h to get an understanding

- whether the inequality is sharp or if the factor $\sqrt{\log\ h^{-1}}$ appears just for the reason of the proof
- and of the constant c.

It is clear that to find the minimal argument the norm may be squared and that

$$\| u^h \|_{H^1}^2 = \int_\Omega u^h \cdot u^h + \nabla u^h \cdot \nabla u^h \ dxdy \ =: a(u^h, u^h).$$

This leads to a linear system with the right hand side $u^h(0,0) = 1$ which has to be solved.

The calculations were done on a DEC3000 with the program MATLAB.

The following table shows the computed H^1-norm for different mesh sizes h and the right hand side of the inequality (without the constant c).

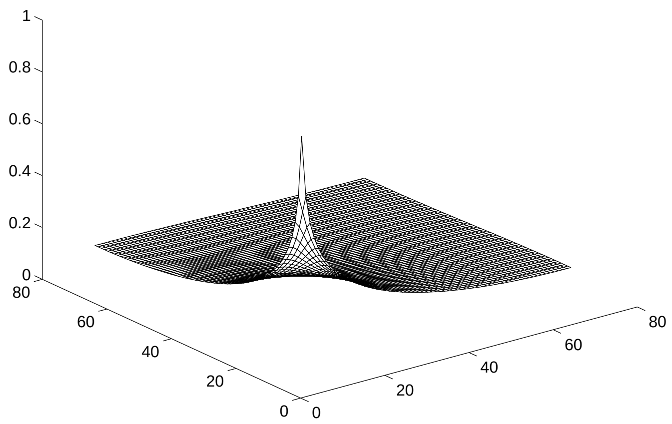

Figure 4: minimum norm solution for n = 64

J	$n = 2^J = h^{-1}$	$\|u^h\|_{H^1(\Omega)}$	$\|u^h\|_{H^1(\Omega)} \times \sqrt{\log(n)}$
1	2	0.7114	0.5923
2	4	0.6400	0.7535
3	8	0.5877	0.8475
4	16	0.5471	0.9109
5	32	0.5140	0.9570
6	64	0.4864	0.9920
7	128	0.4629	1.0196
8	256	0.4424	1.0418
9	512	0.4245	1.0602

It can be seen that the the factor $\|u^h\| \cdot \sqrt{\log(n)}$ is constant. For smaller n an increase is visible which is due to numerical errors. But the factors becomes constant for large n.

4 The gap between $H^{1/2}(\Gamma)$ and $H^{1/2}_{00}(\Gamma)$

Let us consider a decomposition of the domain $\Omega = (0,1) \times (-1,1)$ into two squares, see Fig. 6. For non-overlapping domain decomposition the function space of traces on the interface Γ is important. It is widely known, that this space is something like $H^{1/2}(\Gamma)$, but often no care is spent to the boundary conditions. In the left picture, we set Dirichlet b.c. at $\{0\} \times (-1,1)$, while in the right one we set Neumann b.c. at the according part.

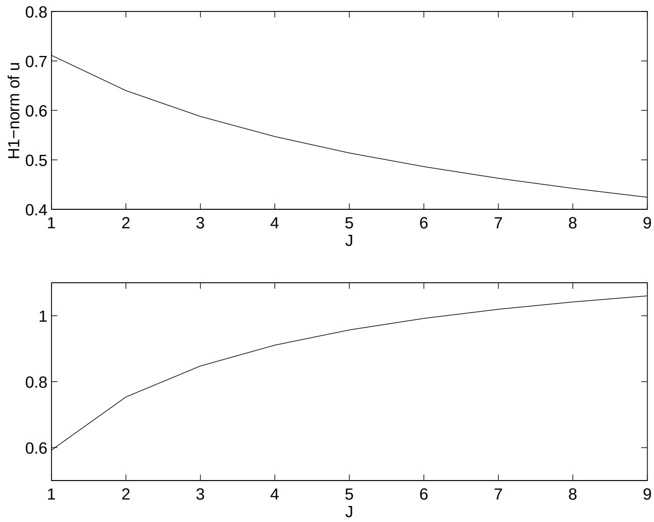

Figure 5: norm of the solution for different $n = 2^J$ (upper picture) and the right hand side of the inequality (lower picture)

For Neumann b.c. the trace theorem holds for the space $H^{1/2}(\Gamma)$ with norm

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} := \int_0^1 \int_0^1 \frac{(\varphi(x)-\varphi(y))^2}{(x-y)^2} \mathrm{dxdy} + \int_0^1 \varphi(x)^2 \mathrm{dx} \tag{3}$$

already given in Section 1, while for Dirichlet b.c. the according space is $H^{1/2}_{00}(\Gamma)$ with norm

$$\|\varphi\|^2_{H^{1/2}_{00}(\Gamma)} := \|\varphi\|^2_{H^{1/2}(\Gamma)} + \int_0^1 \frac{\varphi(x)^2}{x} \mathrm{dx}, \tag{4}$$

see [1]. Both norms are not equivalent.

We are interested in finite element subspaces, where all norms are equivalent. But of course, there must be a dependency on the mesh size h. The interface $(0,1)$ is dissected into $n = 2^J$ equidistant intervals of length $h = 1/n$, the nodes are denoted by $x_i = ih, i = 0, \ldots n$. The finite element space V_h consists of piecewise linear functions vanishing in x_0. We identify a finite element function φ_h with its counterpart $\underline{\varphi} \in \mathbf{R}^n$.

We define the Gramian matrices

$$(A\underline{\varphi}, \underline{\varphi}) \;:=\; \|\varphi_h\|^2_{H^{1/2}(\Gamma)}, \tag{5}$$

$$(B\underline{\varphi}, \underline{\varphi}) \;:=\; \int_0^1 \frac{\varphi_h(x)^2}{x} \mathrm{dx}, \tag{6}$$

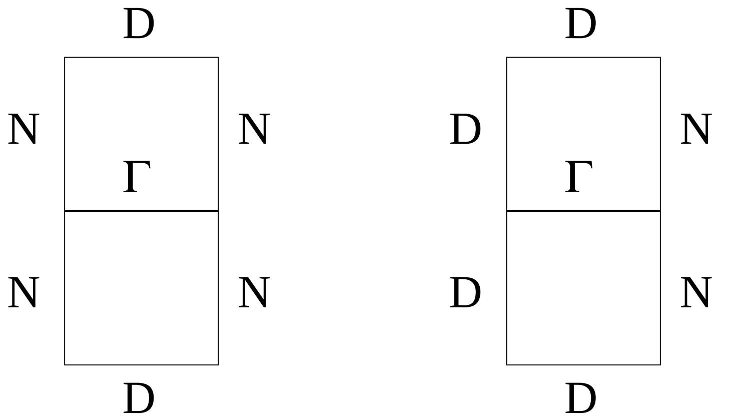

Figure 6: Internal boundaries

such that $\|\varphi_h\|^2_{H^{1/2}_{00}(\Gamma)}$ is given by $\|\underline{\varphi}\|^2_{A+B}$.

The goal is to find the function $c(h)$ such that the spectral inequality

$$B \leq c(h)A \tag{7}$$

is sharp. From Section 3 it is easy to derive the upper bound $c(h) \leq c_1(\log h^{-1})^2$, while the trivial example $\varphi_h(x) = 1$, $x \in (h, 1)$ provides only $c(h) \geq c_2 \log h^{-1}$. Both constants c_1 and c_2 do not depend on the mesh size h.

We assembled matrices $\tilde{A} \sim A$ and $\tilde{B} \sim B$ given by the discrete, spectrally equivalent forms to (5) and (6):

$$(\tilde{A}\underline{\varphi}, \underline{\varphi}) := \sum_{i=1}^{n}\sum_{j=1}^{i-1} \frac{(\varphi_h(x_i) - \varphi_h(x_j))^2}{(x_i - x_j)^2} h^2 + \sum_{i=1}^{n} \varphi_h(x_i)^2 h \tag{8}$$

$$(\tilde{B}\underline{\varphi}, \underline{\varphi}) := \sum_{i=1}^{n} \frac{\varphi_h(x_i)^2}{x_i} h \tag{9}$$

For several values of the mesh size $h = 2^{-J}$, the largest eigenvalue $\lambda_{\max}$ of the generalized eigenvalue problem

$$\tilde{A}x = \lambda \tilde{B}x \tag{10}$$

has been computed and compared to the mesh level J and J^2.

J	N	$\lambda_{\max}$	$\lambda_{\max}/J$	$\lambda_{\max}/J^2$
2	4	2.208	1.104	0.552
3	8	2.992	0.997	0.332
4	16	3.887	0.972	0.243
5	32	4.893	0.979	0.196
6	64	6.012	1.002	0.167
7	128	7.245	1.035	0.148
8	256	8.595	1.074	0.134
9	512	10.061	1.118	0.124
10	1024	11.646	1.165	0.117
11	2048	13.348	1.213	0.110

From these numbers it is not clear, whether $\lambda_{\max}$ behaves like J or J^2. But the picture of the eigen-function x_h gives a good idea about the worst case function for (7):

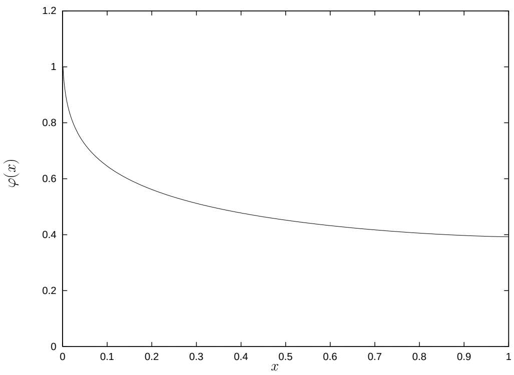

Figure 7: Eigenvectors to $\lambda_{\max}$

Figure 7 motivates a locally refined mesh to the left boundary. We tested meshes with $h_{\min} = 2^{-J}$ and $x_i = h_{\min}2^i, i = 0, \dots J$. On this mesh we can approximate the matrices by

$$(\tilde{A}\underline{\varphi}, \underline{\varphi}) \;:=\; \sum_{i=0}^{J}\sum_{j=0}^{i-1} \frac{(\varphi_h(x_i) - \varphi_h(x_j))^2}{(x_i - x_j)^2} h_i h_j + \sum_{i=0}^{J} \varphi_h(x_i)^2 h_i \tag{11}$$

$$(\tilde{B}\underline{\varphi}, \underline{\varphi}) \;:=\; \sum_{i=1}^{N} \frac{\varphi_h(x_i)^2}{x_i} h_i \tag{12}$$

with the local mesh size $h_i = x_{i+1} - x_i$. Now we could compute enough levels to see the trend

J	$\lambda_{\max}$	$\lambda_{\max}/J$	$\lambda_{\max}/J^2$
10	14.1	1.41	0.1410
20	45.5	2.27	0.1138
40	163	4.09	0.1022
60	355	5.92	0.09860
80	620	7.75	0.09683
100	958	9.58	0.09579
120	1369	11.4	0.09510
140	1854	13.3	0.09460
160	2413	15.1	0.09424
180	3044	16.9	0.09396
200	3749	18.7	0.09373

The estimate $c(h) \leq c_1(\log h^{-1})^2$ seems to be sharp. With the picture of the eigenfunction it was possible to construct asymptotically sharp functions explicitly:

$$\underline{\varphi}_i = \varphi_h(x_i) = 1 - \frac{i}{J}$$

This gives:

$$\begin{aligned}
(\tilde{B}\underline{\varphi}, \underline{\varphi}) &= \sum_{i=0}^{J} \frac{\left(1 - \frac{i}{J}\right)^2}{h_{\min} 2^i} h_{\min} 2^i = J^{-2} \sum_{i=0}^{J} i^2 \geq J/6 \\
(\tilde{A}\underline{\varphi}, \underline{\varphi}) &= \sum_{i=0}^{J} \sum_{j=0}^{i-1} \frac{\left(1 - \frac{i}{J} - 1 + \frac{j}{J}\right)^2}{(h_{\min} 2^i - h_{\min} 2^j)^2} h_{\min}^2 2^i 2^j + \sum_{i=0}^{J} \left(1 - \frac{i}{J}\right)^2 h_{\min} 2^i \\
&= \frac{1}{J^2} \sum_{i=0}^{J} \sum_{j=0}^{i-1} (i-j)^2 \frac{1}{1 - 2^{j-i}} \frac{1}{2^{i-j} - 1} + \frac{1}{J^2} \sum_{i=0}^{J} (J-i)^2 2^{i-l} \\
&= \frac{1}{J^2} \sum_{i=0}^{J} \sum_{k=1}^{i} k^2 \underbrace{\frac{1}{1 - 2^{-k}}}_{\leq 2} \underbrace{\frac{1}{2^k - 1}}_{\leq \frac{2}{2^k}} + \frac{1}{J^2} \sum_{k=0}^{J} \frac{k^2}{2^k} \\
&\leq \frac{5}{J^2} \sum_{i=0}^{J} \sum_{k=1}^{i} \frac{k^2}{2^k} \leq \frac{5}{J} \sum_{k=1}^{\infty} \frac{k^2}{2^k} \leq \frac{c}{J}
\end{aligned}$$

Now we can conclude

$$\lambda_{max} \geq \frac{(B\underline{\varphi}, \underline{\varphi})}{(A\underline{\varphi}, \underline{\varphi})} \geq c \frac{(\tilde{B}\underline{\varphi}, \underline{\varphi})}{(\tilde{A}\underline{\varphi}, \underline{\varphi})} \geq cJ^2$$

We sum up the final theorem:

Theorem 3 *The inverse inequality*

$$\|\varphi_h\|_{H_{00}^{1/2}(\Gamma)} \leq c(\log h^{-1})^2 \|\varphi_h\|_{H^{1/2}(\Gamma)}$$

is asymptotically sharp.

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