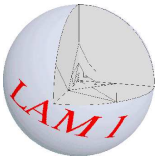


Universität Augsburg

Lehrstuhl für Angewandte Mathematik I

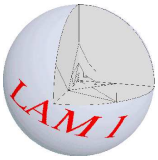
Time-parallel Multigrid Methods for Two-Phase Stefan Problems

Frank Wagner

This is a joint work with

Ronald H.W. Hoppe

9th Int. GAMM-Workshop on Parallel Multigrid Methods May 13, 1996



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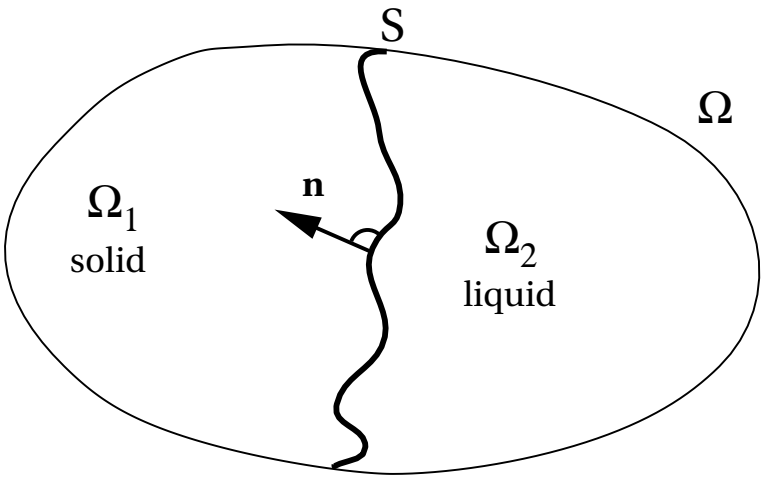
Time-parallel Multigrid Methods for Two-Phase Stefan Problems

Contents

- Introduction
- Enthalpy formulation
- Linearization
- Idea of time-parallel multigrid
- Implementational details
- Numerical results and efficiency rates

Two-Phase Stefan Problems



The equations for the *solid phase* (i=1) and *liquid phase* (i=2) are

$$\rho c_i \frac{\partial \vartheta_i}{\partial t} - \nabla \cdot [k_i \nabla \vartheta_i] = f(\vartheta_i)$$

in $Q_i := \Omega_i \times [0, T]$, with initial and boundary conditions.

On the *free boundary*

$$S := \left\{ (x,t) \in Q \;\middle|\; \vartheta_1(x,t) = \vartheta_2(x,t) = 0 \right\}$$

holds the *Stefan condition*

$$\rho\lambda v \cdot \mathbf{n} = k_2 \nabla \vartheta_2 \cdot \mathbf{n} - k_1 \nabla \vartheta_1 \cdot \mathbf{n},$$

where

ϑ_i	temperature in Q_i, i=1,2
$\mathbf{n}$	normal to S (into Ω_1)
v	velocity of the free boundary
$\rho\lambda,\ k_i$	material constants

Enthalpy formulation

Fixed domain: $Q := Q_1 \cup Q_2$.

generalized temperature

$$\vartheta(x) := \begin{cases} k_1\vartheta_1 & \text{if} \quad x \in Q_1, \\ k_2\vartheta_2 & \text{if} \quad x \in Q_2 \end{cases}$$

enthalpy

$$u(x) := \begin{cases} \rho c_1 u_1 & \text{if} \quad x \in Q_1, \\ \rho c_2 u_2 + \rho\lambda & \text{if} \quad x \in Q_2 \end{cases}$$

Defining $\beta : \mathbb{R} \to \mathbb{R}$ as

$$\beta(s) := \begin{cases} \frac{k_1}{\rho c_1} s & \text{if} \quad 0 < s, \\ 0 & \text{if} \quad 0 \le s \le \rho\lambda, \\ \frac{k_2}{\rho c_2}(s - \rho\lambda) & \text{if} \quad \rho\lambda < s, \end{cases}$$

we have $\beta(u) = \vartheta$.

The enthalpy formulation of the problem is

$$\frac{\partial u}{\partial t} - \Delta\beta(u) = f\Big(\beta(u)\Big) \quad \text{in} \quad Q.$$

and the free boundary is given as

$$S = \Big\{(x,t) \in Q \;\Big|\; u(x,t) \in [0, \rho\lambda]\Big\}.$$

(*mushy region*: $\operatorname{meas} S > 0$)

Introduction to linear Semigroups

Banach space X.

Linear semigroup $S(t) : X \to X$, $t > 0$, with

1. $S(0) = I_X$
2. $S(t)S(s) = S(t+s)$
3. $\lim_{\sigma \to 0} S(\sigma)x = x$ for all $x \in X$

Abstract Cauchy problem

for $u \in L^2([0,T]; X)$

$$\frac{d}{dt}u + Au = 0, \qquad u(0) = u_0.$$

The solution is given by $u(t) = S(t)u_0$, where $S(t)$ is *generated by* $-A$, i.e.

$$\lim_{\sigma \to 0} \frac{S(\sigma) - I_X}{\sigma} = -A.$$

(Therefore one writes $S(t) = \exp(-tA)$.)

Enthalpy update

In the nonlinear enthalpy formulation we approximate

$$-A \approx \frac{S(\sigma) - I_X}{\sigma}$$

and obtain the *enthalpy update* with respect to step size τ from

$$u(t+\tau) = u(t) + \frac{\tau}{\sigma}\Big[S(\sigma)\beta\Big(u(t)\Big) - \beta\Big(u(t)\Big)\Big],$$

for the time-discretized enthalpy.

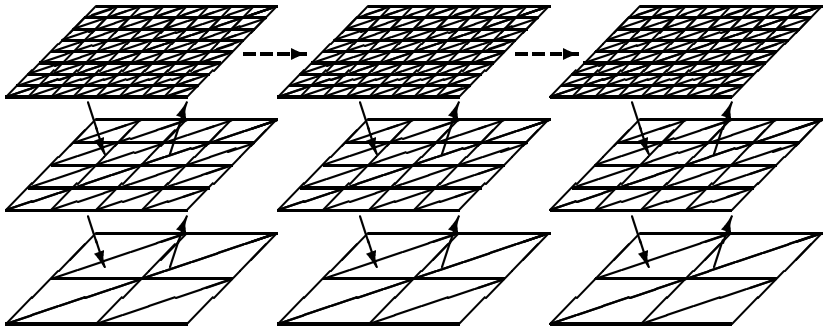
Here $\vartheta(\tau) := S(\sigma)\beta\Big(u(t)\Big)$ is the solution of a linear Cauchy problem

$$\frac{\tau}{\sigma}\frac{d}{dt}\vartheta + A\vartheta = \tilde{f}, \qquad \vartheta(0) = \beta\Big(u(t)\Big).$$

The linearization is characterized by the parameter $\mu := \frac{\tau}{\sigma}$.

[cf. Berger/Brezis/Rogers and Nochetto/Verdi]

Time-parallel multigrid



Discretizing the abstract Cauchy problem

$$\frac{d}{dt}u + Au = f, \qquad u(0) = u_0,$$

to the step size τ

$$u^{k+1} - u^k + \tau A u^{k+1} = \tau f^{k+1}, \quad 0 \le k \le p,$$

we solve p time steps *simultaneously* as

$$\mathbf{u} := \left(u^0, \ldots, u^p\right)$$

$$\mathbf{A}\mathbf{u} = \mathbf{f}.$$

where

$$\mathbf{A} := \begin{pmatrix} I_X & & & \\ -I_X & I_X + \tau A & & \\ & \ddots & \ddots & \\ & & -I_X & I_X + \tau A \end{pmatrix}$$

[cf. Hackbusch, Burmeister and Horton]

Two Grid Method

The *parallel two grid method* is constructed from a sequential two grid method for

$$I_X + \tau A v = g.$$

with smoother $S_l^\kappa(v_l, g_l)$.
We use a *Block-Jacobi smoother* $\mathbf{S}_l^\kappa(\mathbf{u}_l, \mathbf{f}_l)$:

PNTGM$(l, \mathbf{u}_l^\nu, \mathbf{f}_l)$

pre-smoothing:

$$\mathbf{u}_l^1 := \mathbf{S}_l^{\kappa 1}(\mathbf{u}_l^\nu, \mathbf{f}_l)$$

coarse grid correction:

$$\mathbf{d}_l := \mathbf{f}_l - \mathbf{A}_l(\mathbf{u}_l^1)$$

$$\text{solve} \quad \mathbf{A}_{l-1}\mathbf{v}_{l-1} = \mathbf{A}_{l-1}(\mathbf{r}\mathbf{u}_l^1) + \mathbf{r}\mathbf{d}_l$$

$$\mathbf{u}_l^2 := \mathbf{u}_l^1 + \mathbf{p}(\mathbf{v}_{l-1} - \mathbf{r}\mathbf{u}_l^1)$$

post-smoothing:

$$\mathbf{u}_l^{\nu+1} := \mathbf{S}_l^{\kappa 2}(\mathbf{u}_l^2, \mathbf{f}_l)$$

In the linearized case the enthalpy update has to be considered.

Parallel Multigrid Method

The parallel multigrid method for the k-th time step:

PNMGM$_k(l, \mathbf{u}_l^\nu, \mathbf{f}_l)$

if $(l == 0)$
  receive $u_0^{k-1,\nu}$
  solve $(I_0 + \tau A_0)(u_0^{k,\nu+1}) = u_0^{k-1,\nu} + \tau f_0^k$
  send $u_0^{k,\nu+1}$
else
  receive $u_l^{k-1,\nu}$
  send $u_l^{k,\nu}$
  pre-smoothing:

$$u_l^1 := S_l^{\kappa_1}(u_l^{k,\nu}, u_l^{k-1,\nu} + \tau f_l^k)$$

  coarse grid correction:

$$\begin{aligned} v_{l-1}^k &:= r u^1 \\ d_l^k &:= u_l^{k-1,\nu} + \tau f_l^k - (I_l + \tau A_l)(u_l^1) \\ g_{l-1}^k &:= (I_l + \tau A_{l-1})(v_{l-1}^k) + r d_l^k \\ i = 1..\gamma: &\quad \mathbf{PNMGM}_k(l-1, \mathbf{v}_{l-1}, \mathbf{g}_{l-1}) \\ u_l^2 &:= u_l^1 + p(v_{l-1}^k - r u_l^1) \end{aligned}$$

  post-smoothing:

$$u_l^{k,\nu+1} := S_l^{\kappa_2}(u_l^2, u_l^{k-1,\nu} + \tau f_l^k)$$

Experiments

Properties:

Spatial approximation: Linear linear finite elements on uniform grids.

Hierarchy: Coarsest level 1 with 32 elements and 9 nodes. Finest level 4 with 8192 elements and 3969 nodes.

Smoother: symmetric Gauss-Seidel (nonlinear or linear, resp.), $\kappa_1 = \kappa_2 = 3$ pre- and post-smoothing steps, in a W-cycle.

Relative accuracy: 10^{-12} for each time step.

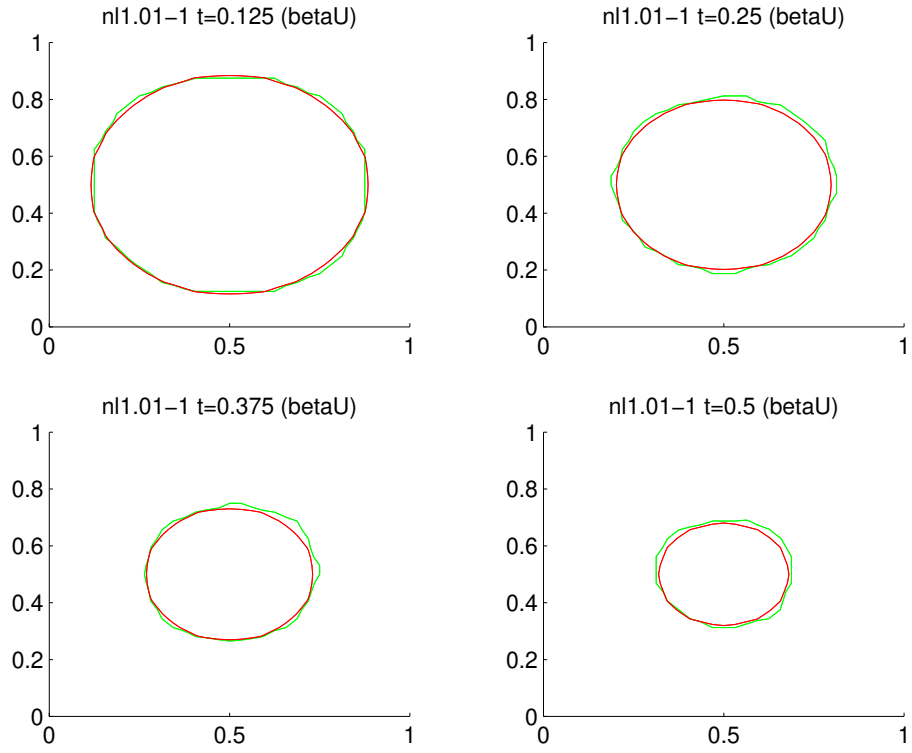
Implementation: In C language on an IBM SP2 machine using MPI message passing standard.

Numerical efficiency rate: $E_N := \dfrac{n_1}{n_p}$

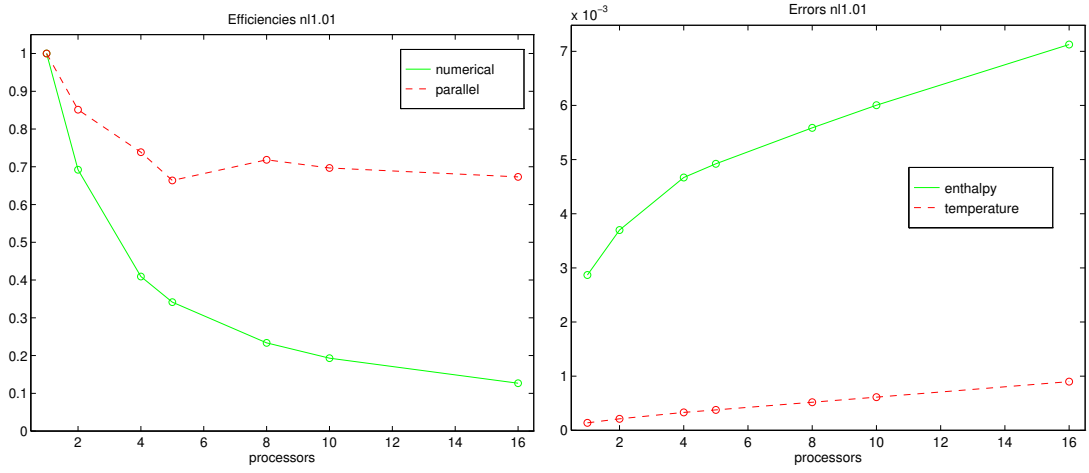
Parallel efficiency rate: $E_P := \dfrac{t_1}{t_p} \cdot E_N^{-1}$

Model Problem 1

Nonlinear calculation:

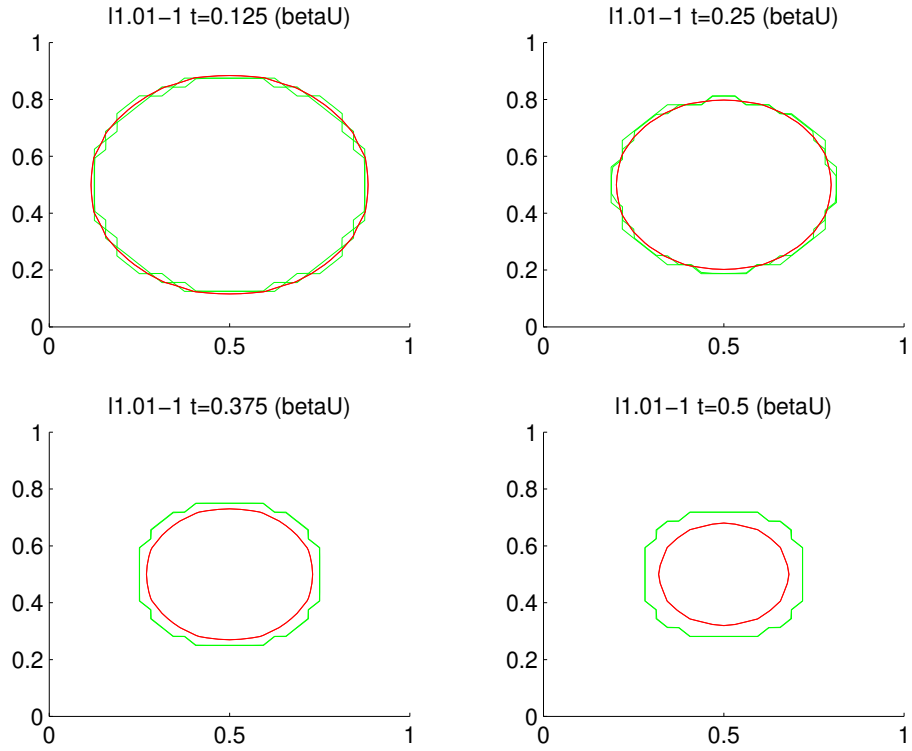


Efficiencies and errors:

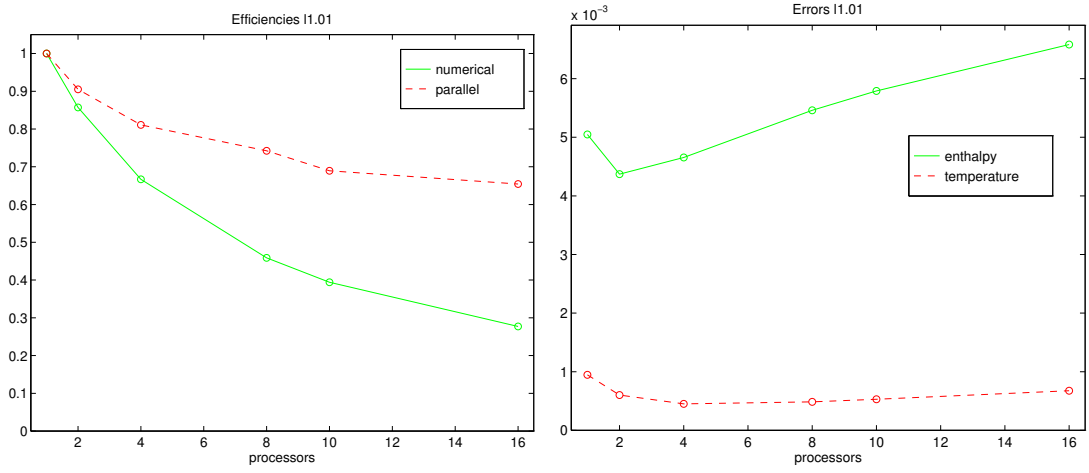


Model Problem 1

Linearized calculation:

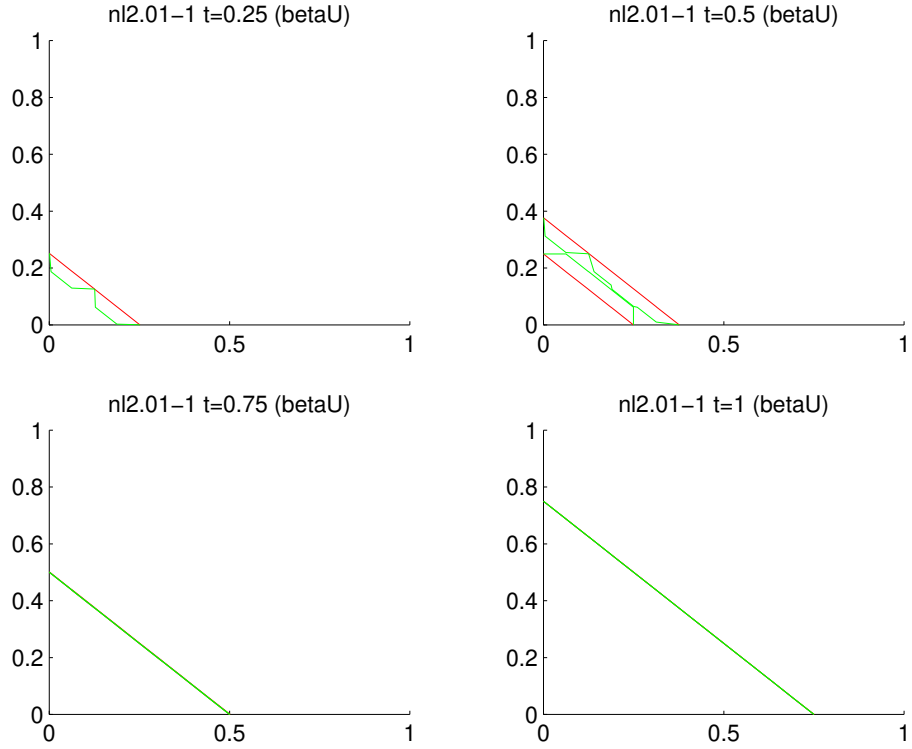


Efficiencies and errors:

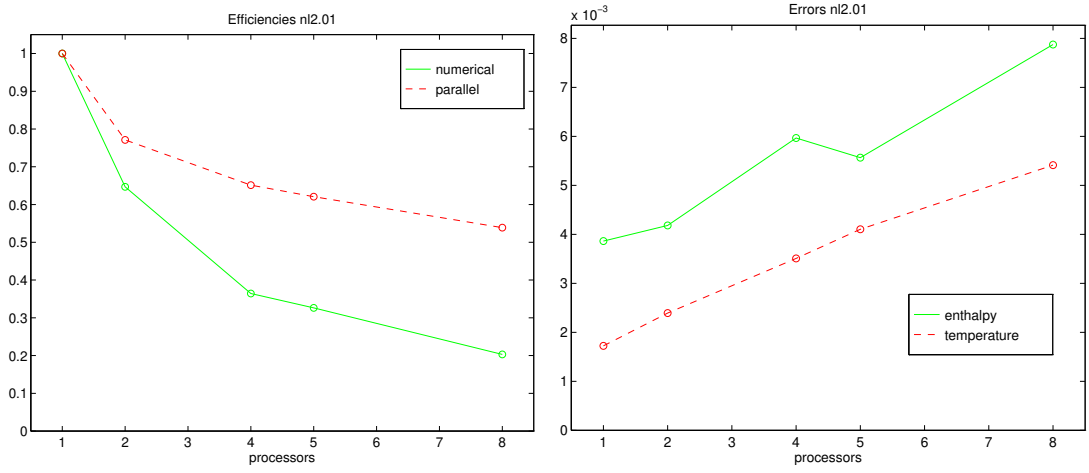


Model Problem 2

Nonlinear calculation:

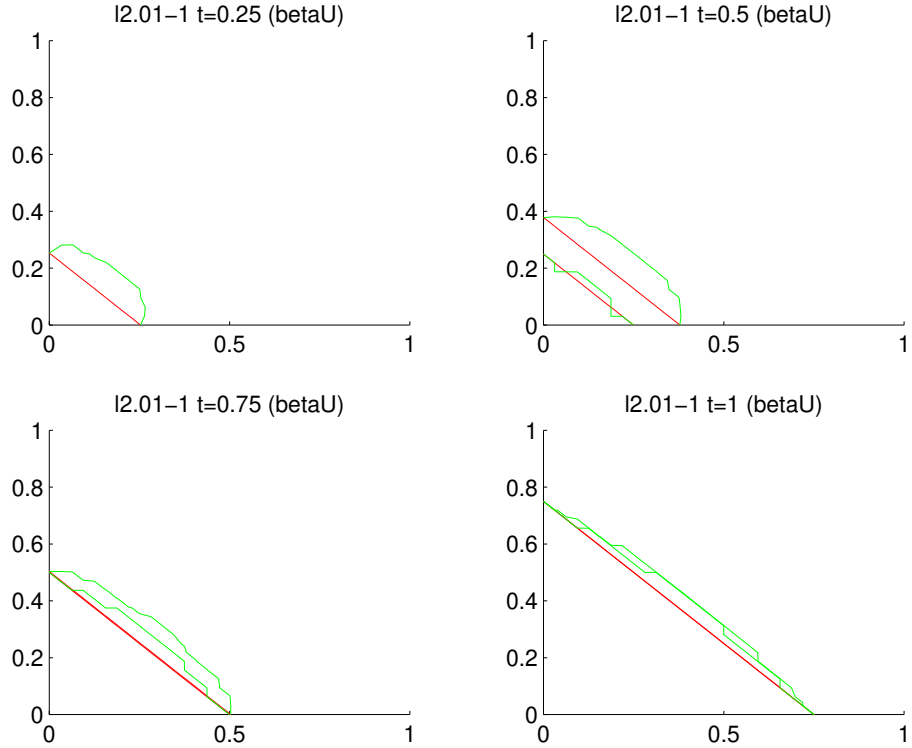


Efficiencies and errors:

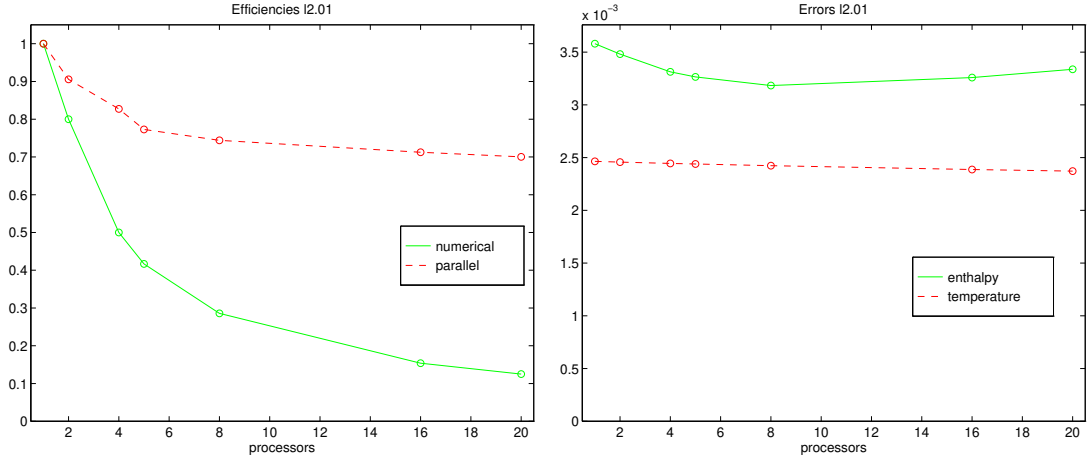


Model Problem 2

Linearized calculation:

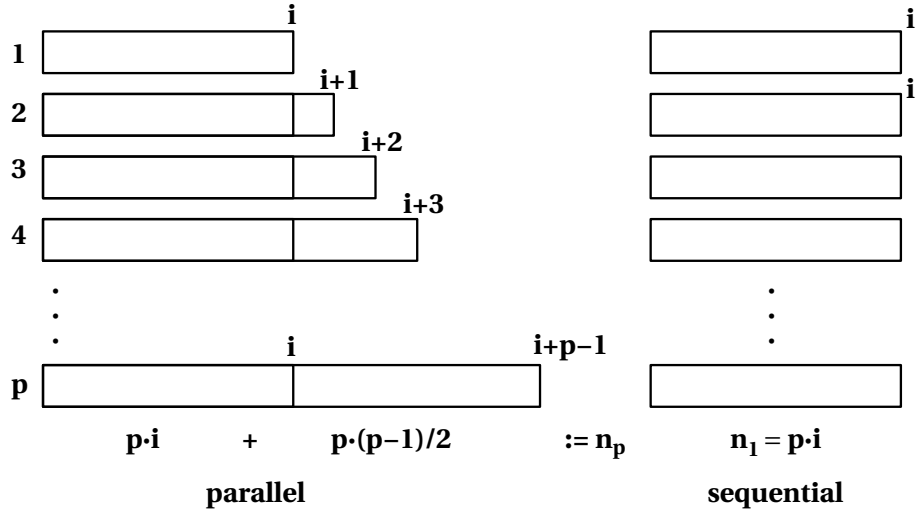


Efficiencies and errors:



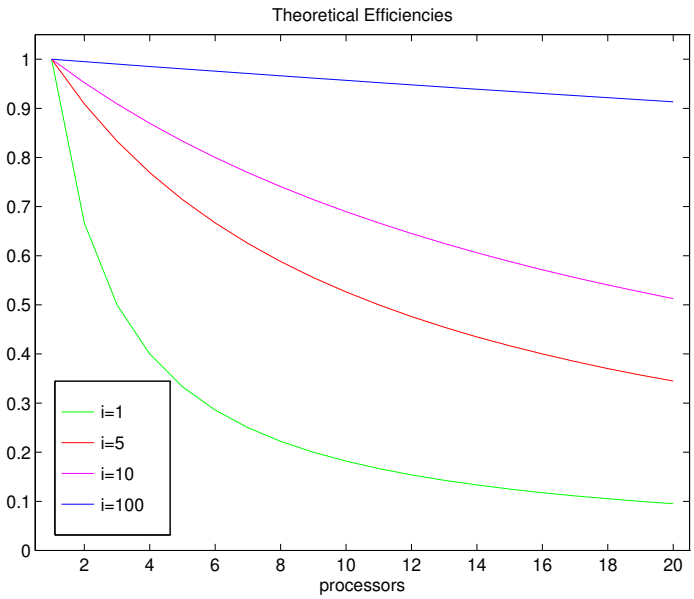
Numerical Efficiencies

Situation for the iteration of p time steps:



Numerical efficiency:

$$\frac{n_1}{n_p} = \frac{p \cdot i}{p \cdot i + \frac{p(p-1)}{2}} = \frac{1}{1 + \frac{p-1}{2i}}$$



Conclusions

We presented

- a fixed domain method for the free boundary problem
- the nonlinear enthaply formulation
- a linearization technique based on the theory of semigroups
- time-parallel multigrid methods for the nonlinear and linear problems
- an explanation for the observed rates of the numerical efficiency