# Parallel Solution Schemes for the Navier – Stokes Equations using the Crouzeix/Raviart–Element

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We consider a finite element discretization of the linearised incompressible Navier–Stokes equations using the non–conforming $P_1/P_0$ element of Crouzeix/Raviart and upwind techniques for stabilisation. This leads to a large linear system of the form

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}.$$

We present various algorithms to solve this system of linear equations, estimate their computational costs and discuss the costs of an implementation on a parallel computer. Finally we compare these algorithms using benchmark problems for the steady state and time dependent Navier–Stokes equations.

# The Incompressible Navier–Stokes Equations

$$\begin{aligned}
\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega\times(0,T] \\
\nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega\times(0,T] \\
\mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega\times(0,T] \\
\mathbf{u} &= \mathbf{u}_0 && \text{for } t=0
\end{aligned}$$

- steady state and time dependent
- linearisation (in each time step)

$$\begin{aligned}
\mathbf{u}_t^{n+1} - \nu\Delta\mathbf{u}^{n+1} + (\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} &= \mathbf{f} && \text{in } \Omega \\
\nabla\cdot\mathbf{u}^{n+1} &= 0 && \text{in } \Omega \\
\mathbf{u}^{n+1} &= \mathbf{g} && \text{on } \partial\Omega
\end{aligned}$$

- discretization in space with the non–conforming Crouzeix / Raviart–element ($P_1/P_0$–element)
- stabilisation with upwind techniques (Samarskij–upwinding)
- discretization in time with BDF(2)

$\Longrightarrow$ resulting saddle–point problem to solve

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$$

## Solution Methods

- experiences of benchmark computations (Schäfer and Turek 1996)

  **necessary** :

  - multi-grid techniques
  - higher order upwinding
  - implicit higher order discretization in time

  **no clear answer** :

  - <u>pressure correction schemes</u> or <u>fully coupled approaches</u> better for time dependent problems

- aims of our investigations :

  - compare one pressure correction scheme and one fully coupled approach for the Crouzeix / Raviart–element
  - estimate numerical and parallel costs
  - dependence of the results on parameters of the algorithms

# <u>Pressure Correction Schemes</u>

- operator splitting, discrete projection schemes
- Schur complement with respect to $p$

$$B^T A^{-1} B p = B^T A^{-1} r - s \tag{1}$$

- using fix point iteration for solving (1)

$$p^{m+1} = p^m - \alpha (B^T C^{-1} B)^{-1} (B^T A^{-1} B p^m - B^T A^{-1} r + s),$$

  $m = 0, 1, \ldots$

- linear system to solve in every step

$$B^T C^{-1} B x = -B^T A^{-1} B p^m + B^T A^{-1} r - s \tag{2}$$

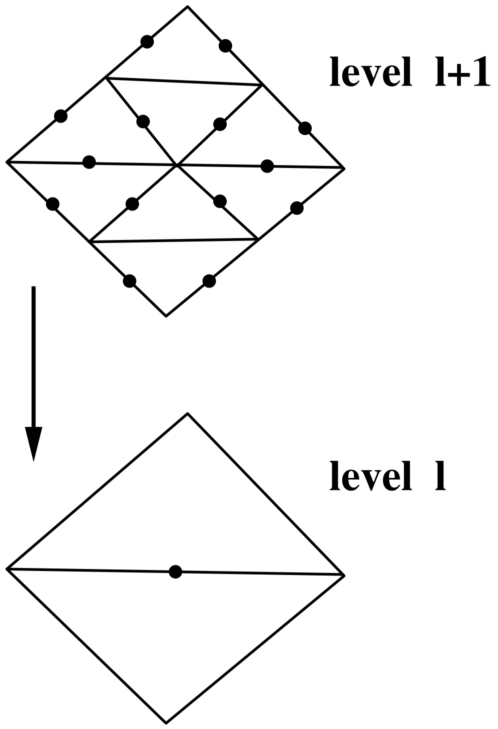
- choosing $C = \mathrm{diag}(A)$
- numerically : $A^{-1} \approx A_j^{-1}$ ($j$ **multi-grid** V–cycles for the unknowns of the **velocity**)
- solving (2) with **multi-grid** for the unknowns of the **pressure** (piecewise constant !)

# Multi-grid for the Unknowns of the Velocity

**components :**

- smoother
  - block Jacobi with inner ILU($\beta$)–iterations
  - block Jacobi with inner Gauss–Seidel iterations
- restriction : $L^2$–projection,
- prolongation : $L^2$–projection
- coarse grid solver : block Jacobi with inner ILU($\beta$)–iterations
- step–length control (on finest level only)

## restriction



- opms = multiplication (division) & addition (subtraction)

**costs (per unknown) :**

| operation | times/level | num. costs |
|---|---|---|
| smoothing | $\nu_1 + \nu_2$ | 6 (or 4) |
| residual | $\nu_1 + \nu_2$ | 6 |
| restriction | 1 | 17 |
| prolongation | 1 | 18 |
| step-length control | 1 | 7 |
| coarse grid | $It_0$ | 13 |

**costs (one V-cycle):**

- $N_u \approx 3N_{el}$
- costs for V-cycle $\approx 4N_u/3$ *opms*
- numerical costs without coarse grid solve

$$4[12(\nu_1 + \nu_2) + 42]N_{el} \; opms$$

-

$$\nu_1 = \nu_2 = 2 \quad : \quad 360N_{el} \; opms$$

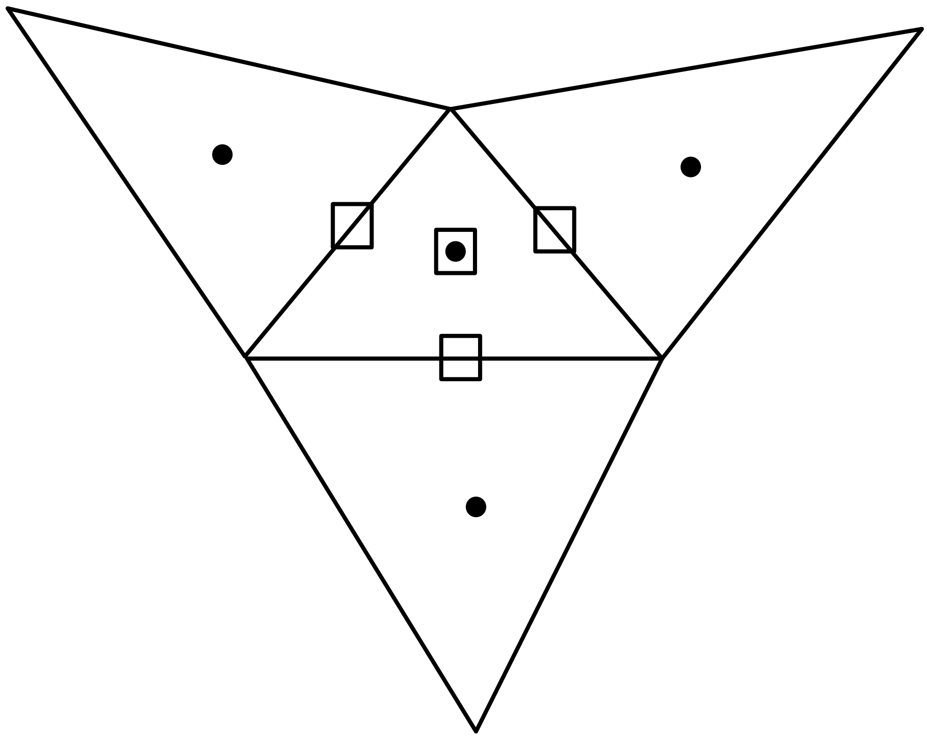
**parallel costs (one V-cycle), $L$ Levels:**

| operation | level/cyc | times/level | comm. |
|---|---|---|---|
| smoothing | L | $\nu_1 + \nu_2$ | 2 |
| residual | L | $\nu_1 + \nu_2$ | 0 |
| restriction | L | 1 | 2 |
| prolongation | L | 1 | 2 |
| step-length control | 1 | 1 | 2 |
| coarse grid | 1 | $It_0$ | 3 |

- communications on different levels is of different length
- table gives hints where communication is needed
- on the coarse grid norm of the residual as control of the iteration is computed (one communication)

## Multi-grid for the Unknowns of the Pressure

- impossible to assemble $\left(B^T \text{diag}(A)^{-1} B\right)$ before the iteration (piecewise constants)
- possible to compute and store $\left(B^T \text{diag}(A)^{-1} B\right)$ before the iteration



□ **matrix entries**

**components :**

- smoother : block Jacobi with inner Gauss–Seidel iterations
- restriction : $L^2$–projection,
- prolongation : $L^2$–projection
- coarse grid solver : block Jacobi with inner Gauss–Seidel iterations
- step–length control (on finest level only)

**costs (one V-cycle):**

| operation | times/level | num. costs |
|---|---|---|
| smoothing | $\mu_1 + \mu_2$ | 4 |
| residual | $\mu_1 + \mu_2$ | 5 |
| restriction | 1 | 4 |
| prolongation | 1 | 1 |
| step-length control | 1 | 6 |
| coarse grid | $It_0$ | 9 |

- costs for V-cycle $\approx 4N_{el}/3$ *opms*
- numerical costs without coarse grid solve

$$4[9(\mu_1 + \mu_2) + 11]/3N_{el} \; opms$$

-

$$\mu_1 = \mu_2 = 2 \quad : \quad 63N_{el} \; opms$$

**parallel costs (one V-cycle), $L$ Levels:**

| operation | level/cyc | times/level | comm. |
|---|---|---|---|
| smoothing | L | $\mu_1 + \mu_2$ | 0 |
| residual | L | $\mu_1 + \mu_2 + 1$ | 1 |
| restriction | L | 1 | 1 |
| prolongation | L | 1 | 1 |
| step-length control | 1 | 1 | 2 |
| coarse grid | 1 | $It_0$ | 2 |

# Coupled Multi-grid

**components :**

- smoother : Vanka smoother
- restriction : $L^2$–projection,
- prolongation : $L^2$–projection
- coarse grid solver : Vanka smoother
- step–length control (on finest level only)

**costs :** given per unknown of the pressure

| operation | times/level | num. costs |
|---|---|---|
| smoothing | $\lambda_1 + \lambda_2$ | 65 |
| residual | $\lambda_1 + \lambda_2$ | 31 |
| restriction | 1 | 55 |
| prolongation | 1 | 55 |
| step-length control | 1 | 35 |

**costs (one W-cycle):**

- without coarse grid solve

$$2[(\lambda_1 + \lambda_2)96 + 145]N_{el}\ opms$$

- 

$$\lambda_1 = \lambda_2 = 4 \quad : \quad 1826 N_{el}\ opms$$

**costs (one V-cycle):**

- $$\lambda_1 = \lambda_2 = 4 \quad : \quad 1218 N_{el}\ opms$$

**parallel costs (one W-cycle), $L$ Levels:**

| operation | level/cyc | times/level | comm. |
|---|---|---|---|
| pre–smoothing | $2^{L-1}$ | $\lambda_1$ | 2 |
| post–smoothing | $2^L - 1$ | $\lambda_2$ | 2 |
| residual | $2^L - 1$ | $\lambda_1 + \lambda_2$ | 1 |
| restriction | $2^L - 1$ | 1 | 2 |
| prolongation | $2^L - 1$ | 1 | 2 |
| step-length control | 1 | 1 | 2 |
| coarse grid | $2^{L-1}$ | $It_0$ | 4 |

**parallel costs (one V-cycle), $L$ Levels:**

| operation | level/cyc | times/level | comm. |
|---|---|---|---|
| smoothing | $L$ | $\lambda_1 + \lambda_2$ | 2 |
| residual | $L$ | $\lambda_1 + \lambda_2$ | 1 |
| restriction | $L$ | 1 | 2 |
| prolongation | $L$ | 1 | 2 |
| step-length control | 1 | 1 | 2 |
| coarse grid | $L$ | $It_0$ | 4 |

# Hard– & Software

- Parsytec GCPowerPlus (slow communication)
- moduls of *ugp*1.0 (e.g. load transfer) by P. Bastian

# Benchmark Problems

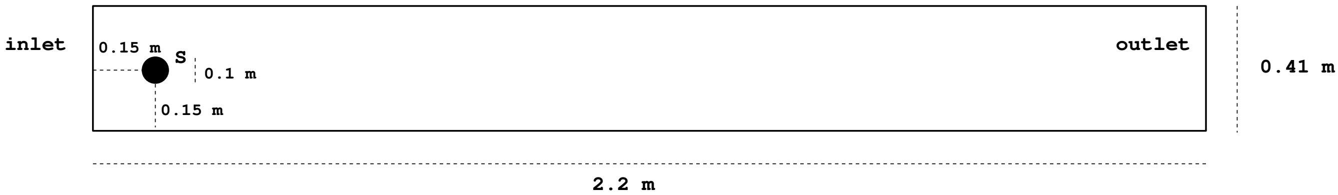



Figure 1: Domain for the benchmark problems

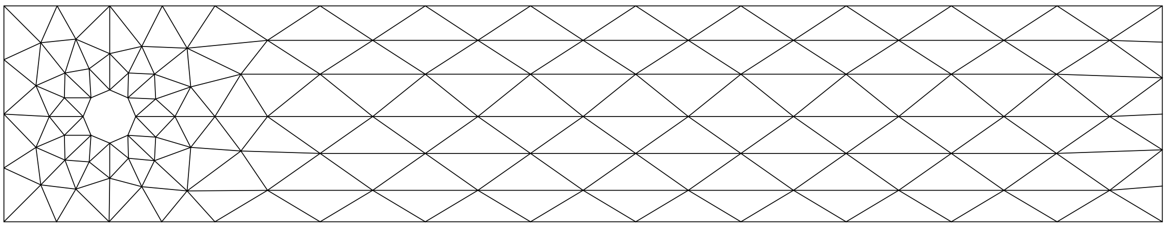


Figure 2: Grid 1

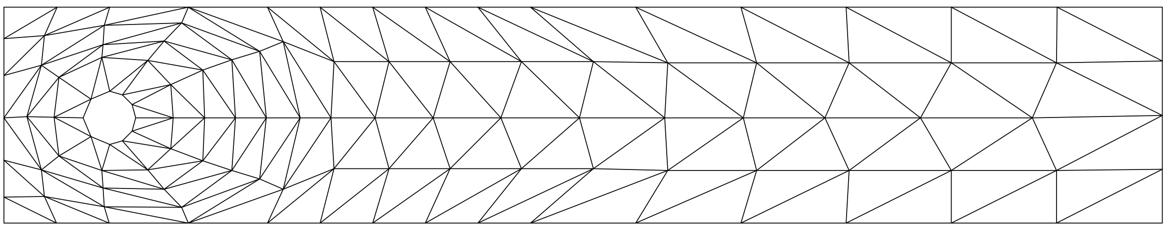


Figure 3: Grid 2

| level | unknowns (grid 1) |
|---|---|
| 3 | 49504 |
| 4 | 197312 |
| 5 | 787840 |

## Steady State Problem

- only fully coupled approach gives satisfactory results
- approximation $C = \text{diag}(A)$ to bad for pressure correction scheme

**convergence rates using the W–cycle :**

| level | grid 1 | grid 2 |
|---|---|---|
| 3 | 0.36 | 0.34 |
| 4 | 0.34 | 0.30 |
| 5 | 0.37 | 0.34 |

**convergence rates using the V–cycle :**

| level | grid 1 | grid 2 |
|---|---|---|
| 3 | 0.40 | 0.34 |
| 4 | 0.50 | 0.45 |
| 5 | 0.48 | 0.45 |

**communication time using the W–cycle (in %) :**

| node | 1 | 4 | 8 | 16 |
|---|---|---|---|---|
| 3 | 0 | 12 | 16 | 26 |
| 4 | 0 | 8 | 10 | 20 |
| 5 | | | | 12 |

**communication time using the V–cycle (in %) :**

| node | 1 | 4 | 8 | 16 |
|---|---|---|---|---|
| 3 | 0 | 8 | 10 | 20 |
| 4 | 0 | 3 | 4.5 | 9 |
| 5 | | | | 4 |

**time on level 0 using the W–cycle (in %) :**

| node | 1 | 4 | 8 | 16 |
|---|---|---|---|---|
| 3 | 6 | 30 | 29 | 40 |
| 4 | 4 | 24 | 18 | 29 |
| 5 | | | | 17 |

**time on level 0 using the V–cycle (in %) :**

| node | 1 | 4 | 8 | 16 |
|---|---|---|---|---|
| 3 | 3 | 21 | 13 | 22 |
| 4 | 1 | 6 | 3.5 | 8 |
| 5 | | | | 2 |

# Time Dependent Problem

- flow through the channel with steady state inflow (bench 2D-2)
- parabolic inflow profile
- periodic flow (Karman vortex street)

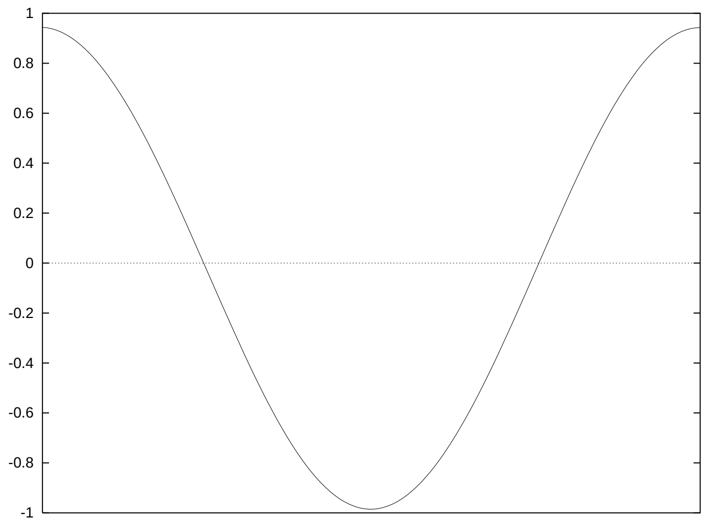



Figure 4: The lift coefficient for one period

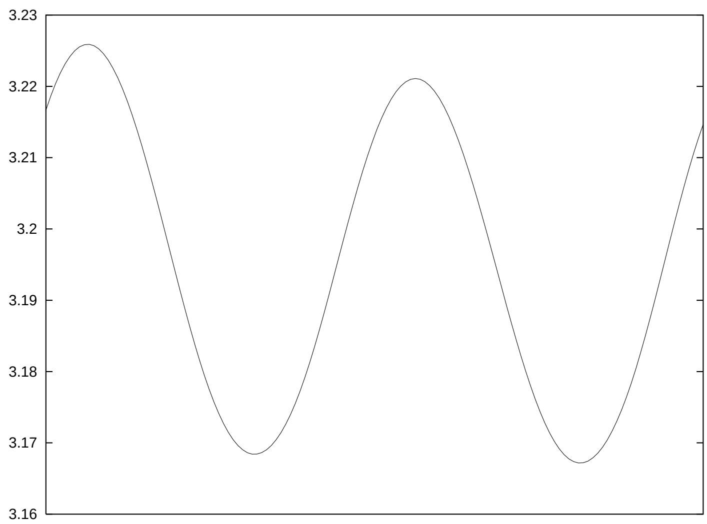



Figure 5: The drag coefficient for one period

**time step** 0.01

| method | coupled mg (W) | press. corr. |
|---|---|---|
| aver. no it | 6.44 | 6.53 |
| av. time/it. | 62 | 72 |
| time/timestep | 399 | 470 |
| comm. time (%) | 17 | 15 |
| level 0 (%) | 20 | 11 |

it = nonlinear iteration step

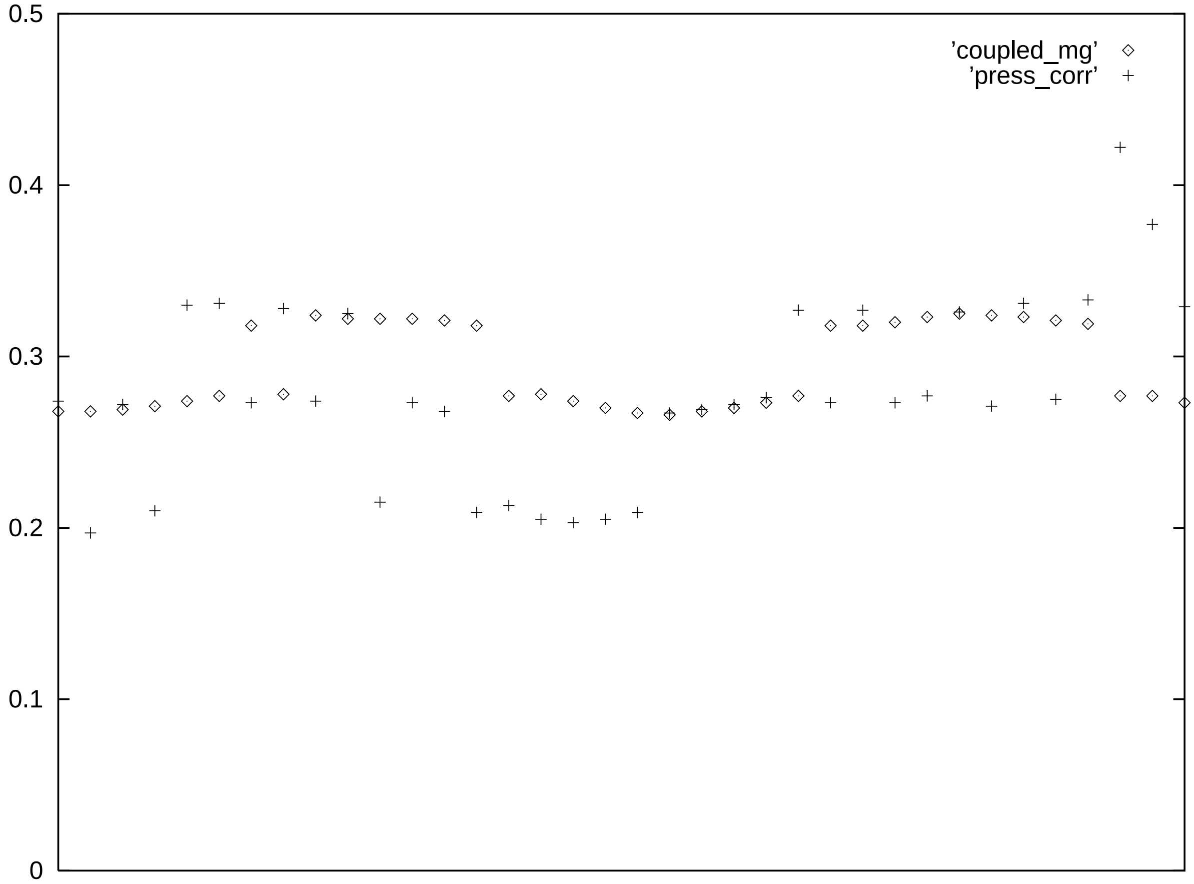



Figure 6: convergence rates for one period

**time step** 0.005

| method | coupled mg (V) | coupled mg (W) | press. corr. |
|---|---|---|---|
| aver. no it | 7.92 | 4.71 | 3.79 |
| av. time/it. | 35 | 59 | 70 |
| time/timestep | 277 | 278 | 265 |
| comm. time (%) | 5 | 15 | 10 |
| level 0 (%) | 4 | 17 | 6 |

it = nonlinear iteration step

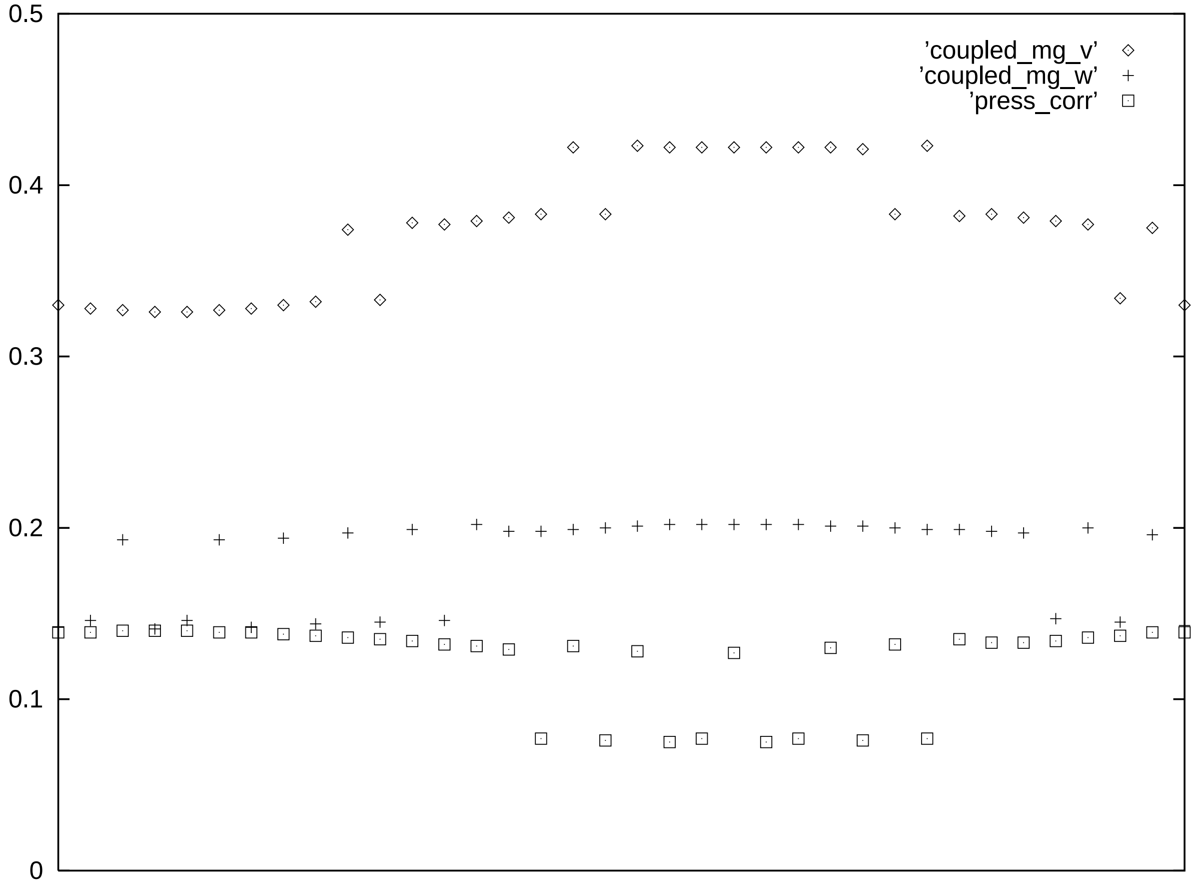



Figure 7: convergence rates for one period

# Concluding Remarks

- coupled multi-grid
  - not very sensitive on the parameters of the iteration
  - number of smoothing steps considerable influences the convergence rate
  - concerning the solution time both $V-$ and $W$ cycle are similar
  - coarse grid solve too expensive, must be improved
- pressure correction scheme
  - multi-grid for pressure unknowns is critical part of the algorithm
  - damping for smoothing and fix point iteration (Schur complement) very important
- comparison of all approaches (for time dependent problems)
  - computing times are similar
  - the smaller the time step the better are pressure correction schemes, $C = \mathrm{diag}(A)$ becomes a good approximation
  - dependence on parameters requires further investigations

**All Algorithms behave similar for time dependent problems. There are no substantial differences.**