Parallel preconditioners for plate and shell problems

- The Kirchhoff plate bending problem
- Finite element discretization and non-overlapping DD
- The ASM–DD–preconditioner
- A multilevel Schur-complement preconditioner
- A fast solver for the subdomain problems
- Results
- A DD–preconditioner for the shell problem

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References

[1] H. Matthes. Parallel Preconditioners for Plate Problems.
Preprint SPC95_31, TU Chemnitz–Zwickau, 1995.

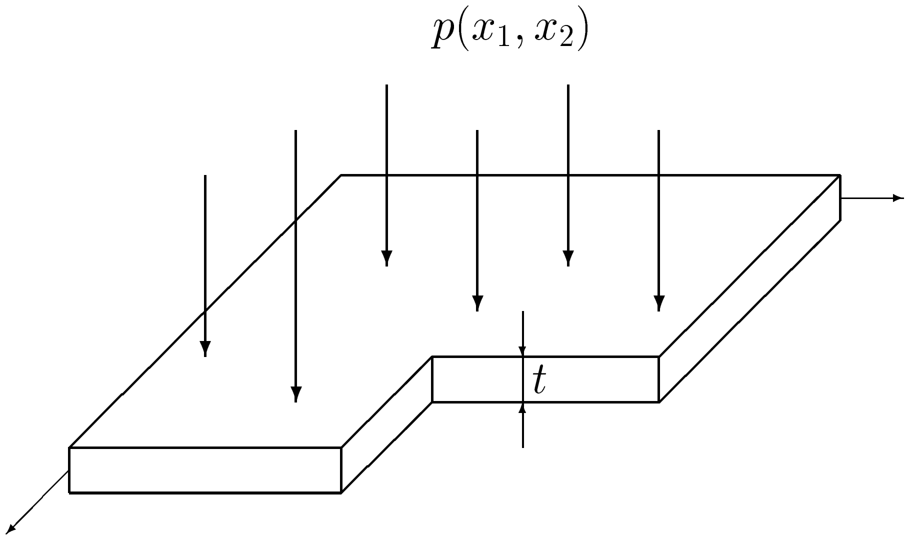
Kirchhoff plates

Assumptions

- Originally plane, thin plate of constant thickness t,
- homogeneous, isotropic material,
- transverse force of density

$$p = \frac{Et^3}{12(1-\sigma^2)} f,$$

  $E \ldots$ Young's modulus, $\sigma \ldots$ Poissons's coefficient,
- vertical displacement $u = u(x_1, x_2)$,
- small deformations $|u_{max}| < t/2$,
- small angles $\partial_k u = \tan\varphi \approx \varphi$.



The Kirchhoff plate bending problem

- Bending energy of a Kirchhoff plate:

$$J(v) = \frac{1}{2}\, a(v,v) - f(v),$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \{\Delta u\, \Delta v + \\ &\quad (1-\sigma)\,(2\,\partial_{12} u\, \partial_{12} v - \partial_{11} u\, \partial_{22} v - \partial_{22} u\, \partial_{11} v)\}\, dx, \\ &\quad \sigma \in (0, 1/2)\,, \\ f(v) &= \int_\Omega f\, v\, dx. \end{aligned}$$

- Consider the following boundary conditions:

  – clamped boundary

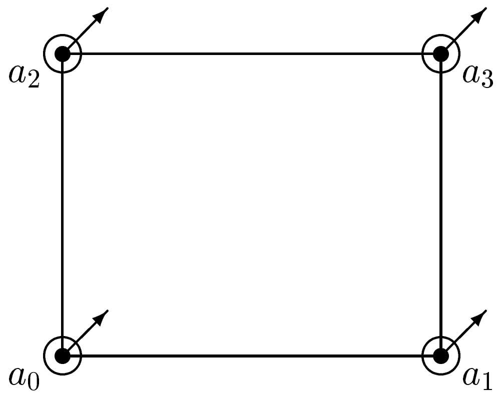
$$\begin{aligned} u \in V_{00} : J(u) &= \min_{v \in V_{00}} J(v), \\ V_{00} &= H_0^2(\Omega) \\ &= \left\{ v \in H^2(\Omega) : v = \partial_\nu v = 0 \text{ on } \Gamma = \partial\Omega \right\}, \end{aligned}$$

  – simply supported boundary

$$\begin{aligned} \bar{u} \in V_0 : J(\bar{u}) &= \min_{v \in V_0} J(v), \\ V_0 &= H^2(\Omega) \cap H_0^1(\Omega) \\ &= \left\{ v \in H^2(\Omega) : v = 0 \text{ on } \Gamma = \partial\Omega \right\}, \end{aligned}$$

  – free boundary.

Bogner-Fox-Schmidt rectangle

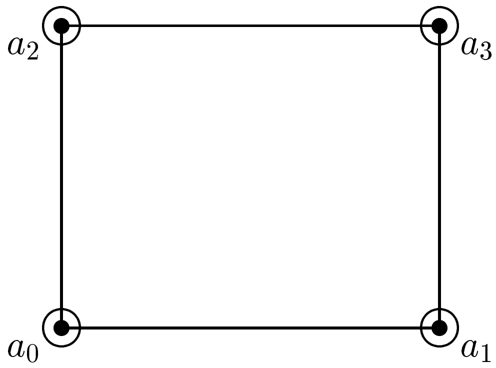


$$
\begin{aligned}
P_R &= P_3(R) \\
&= \left\{ p(x) = \sum_{k_1,\,k_2=0}^{3} \gamma_{k_1 k_2}\, \xi^{k_1}\, \eta^{k_2} \right\}, \\
&\quad \dim\, P_R = 16, \\
\Sigma_R &= \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), \partial_{12} p(a_i), i = 0,1,2,3\}.
\end{aligned}
$$

- Finite element of class $\mathcal{C}^1$,
- results in a conforming f.e. discretization, i.e.

$$V_h \subset H^2(\Omega).$$

Adini's rectangle



$$
\begin{aligned}
P_R &= \tilde{P}_3(R) \\
&= \left\{ p(x) = \sum_{k_1+k_2\le 3} \gamma_{k_1 k_2}\, \xi^{k_1}\, \eta^{k_2} + \gamma_{13}\, \xi\, \eta^3 + \gamma_{31}\, \xi^3\, \eta \right\}, \\
&\quad \dim\, P_R = 12, \\
\Sigma_R &= \{p(a_i), \partial_1 p(a_i), \partial_2 p(a_i), i = 0,1,2,3\}.
\end{aligned}
$$

- Finite element of class $\mathcal{C}^0$,
- results in a non-conforming f.e. discretization, i.e.

$$
\begin{aligned}
V_h &\subset H^1(\Omega) \cap C^0(\bar{\Omega}), \\
v_h|_R &\in H^2(R),\ \forall R \in \mathcal{R}_h, \\
&\Rightarrow \tilde{a}(.,.) = \sum_R \int_R \{\ldots\} dx, \text{etc.}
\end{aligned}
$$

The ASM-DD-preconditioner

- Consider a parallel iterative method for $K_h \underline{u}_h = \underline{f}_h$

$$\begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix},$$

- and construct a so called ASM-DD-preconditioner

$$C_h = \begin{pmatrix} I_C & K_{CI} B_I^{-T} \\ 0 & I_I \end{pmatrix} \begin{pmatrix} C_C & 0 \\ 0 & C_I \end{pmatrix} \begin{pmatrix} I_C & 0 \\ B_I^{-1} K_{IC} & I_I \end{pmatrix}.$$

- Components of C_h:
  - a preconditioner for the local subproblems $C_I = C_I^T$, p.d.:

$$\exists \overline{\gamma}_I, \underline{\gamma}_I > 0 : \quad \underline{\gamma}_I \, C_I \le K_I \le \overline{\gamma}_I \, C_I,$$

  - the basis transformation matrix B_I:

$$\begin{aligned} \mu &= \rho\left(S_C^{-1} T_C\right), \\ S_C &= K_C - K_{CI}\, K_I^{-1}\, K_{IC}, \\ T_C &= K_{CI} \left(K_I^{-1} - B_I^{-T}\right) K_I \left(K_I^{-1} - B_I^{-1}\right) K_{IC}, \end{aligned}$$

  - a (modified) Schur-complement preconditioner $C_C = C_C^T$, p.d.:

$$\begin{aligned} \exists \overline{\gamma}_C, \underline{\gamma}_C > 0 : \quad \underline{\gamma}_C \, C_C &\le S_C + T_C \le \overline{\gamma}_C \, C_C, \\ S_C &\le S_C + T_C \le (1+\mu) S_C. \end{aligned}$$

- Spectral equivalence of C_h and K_h (Haase/Langer/Meyer):

$$\underline{\gamma}/\overline{\gamma} \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2 \le \kappa(C_h^{-1} K_h) \le \overline{\gamma}/\underline{\gamma} \left(\sqrt{\mu} + \sqrt{1+\mu}\right)^2$$

where $\overline{\gamma} = \max\{\overline{\gamma}_I, \overline{\gamma}_C\}$, $\quad \underline{\gamma} = \min\left\{\underline{\gamma}_I, \underline{\gamma}_C\right\}$.

Model domains

- Consider model domains composed of p rectangles:

$$\bar{\Omega} = \cup_{s=1}^{p} \bar{\Pi}_s.$$

- subdomain pattern $\hat{=}$ coarse triangulation:

$$\mathcal{R}_{j_0} = \{\Pi_1, \ldots, \Pi_p\},$$

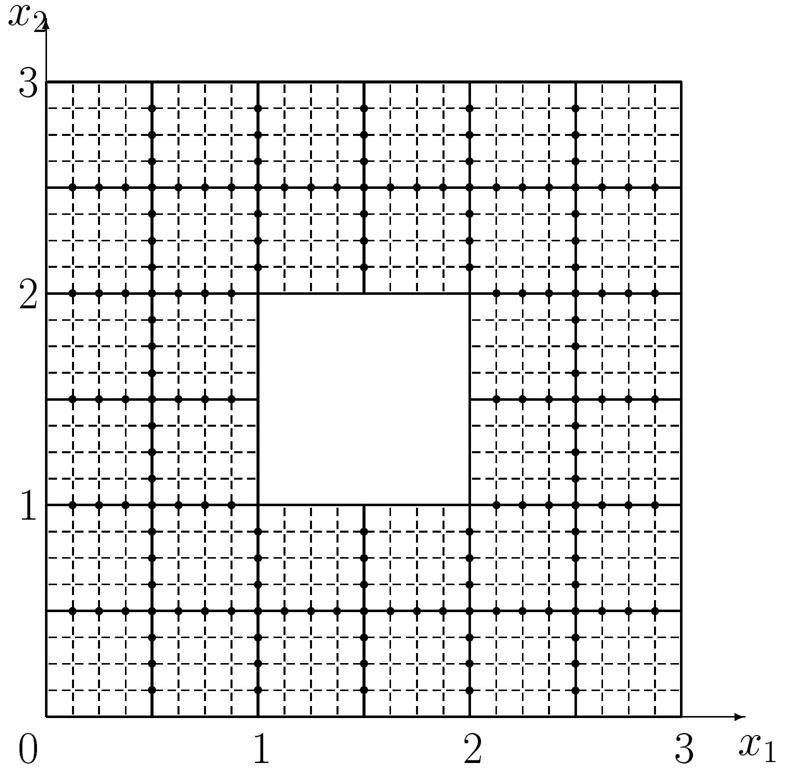
- dyadic refinement:

$$\to \mathcal{R}_l,\ \mathcal{N}_l,\ V_l = \mathrm{span}\Phi_l, \quad l = j_0 + 1, \ldots, j,$$

- $V_h = V_j$, $\mathcal{N}_h = \mathcal{N}_j$, ...

$$K_h \underline{u}_h = \underline{f}_h$$

- Example $p = 32$, $j_0 = 1$, $j = 3$.



Schur-complement preconditioner

- Bogner–Fox–Schmidt elements: Nested sequence of f.e. spaces

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j = V_h.$$

- Multilevel nodal basis preconditioner

$$\begin{aligned} C_h^{-1} &= \sum_{k=j_0}^{j} G_k\, D_k^{-1}\, G_k^T, \\ \text{where } D_{j_0} &= K_{j_0}, \\ D_k &= \operatorname{diag}(K_k), k = j_0+1, \ldots, j, \\ \Phi_k &= \Phi_h\, G_k,\ k = j_0, \ldots, j. \\ \kappa\left(C_h^{-1}\, K_h\right) &\le c \ne c(j, j_0). \end{aligned}$$

- $G_{IC,k} = 0 \Rightarrow$ Multilevel Schur-complement preconditioner

$$\begin{aligned} C_C^{-1} &= \sum_{k=j_0}^{j} G_{C,k}\, D_{C,k}^{-1}\, G_{C,k}^T, \\ \kappa\left(C_C^{-1}\, S_C\right) &\le c \ne c(j, j_0). \end{aligned}$$

- Adini case: No nested sequence of spaces $\tilde{V}_k$!
- Define mapping $\Pi_C : V_C \to \tilde{V}_C$

$$\begin{aligned} \Pi_C v_C|_P &= v_C|_P\,, \\ \nabla(\Pi_C v_C)|_P &= \nabla v_C|_P\,, \forall P \in \mathcal{N}_j \cap \Gamma_C. \end{aligned}$$

- Use V_C as an auxiliary space for $\tilde{V}_C$:

$$\begin{aligned} \tilde{C}_C^{-1} &= \Pi_C\, C_C^{-1}\, \Pi_C^T \\ \kappa(\tilde{C}_C^{-1}\, \tilde{S}_C) &\le \tilde{c} \ne \tilde{c}(j, j_0). \end{aligned}$$

Schur-complement preconditioner

- Clamped plate $\Omega = (0,1) \times (0,1)$,
- $p = 4^{j_0}$ subdomains.
- Spectral bounds and condition number of $C_C^{-1}\, S_C$:

j_0	1			2			3		
j	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ
2	0.71	1.8	2.5	—	—	—	—	—	—
3	0.83	2.4	2.9	0.63	1.9	3.0	—	—	—
4	0.87	2.8	3.2	0.85	2.5	2.9	0.69	2.0	2.9
5	0.87	3.1	3.6	0.87	2.9	3.3	0.85	2.5	2.9
6	0.88	3.4	3.8	0.87	3.2	3.6	0.87	2.9	3.3
7	—	—	—	0.87	3.4	3.9	0.87	3.2	3.7
8	—	—	—	—	—	—	0.87	3.4	3.9

- Spectral bounds and condition number of $\tilde{C}_C^{-1}\, \tilde{S}_C$:

j_0	1			2			3		
j	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ
2	0.52	1.5	2.9	—	—	—	—	—	—
3	0.51	2.2	4.3	0.49	1.8	3.6	—	—	—
4	0.58	2.7	4.7	0.50	2.4	4.7	0.49	1.8	3.8
5	0.69	3.1	4.4	0.61	2.8	4.6	0.51	2.4	4.7
6	0.71	3.3	4.6	0.70	3.1	4.4	0.65	2.8	4.3
7	—	—	—	0.72	3.4	4.7	0.71	3.1	4.4
8	—	—	—	—	—	—	0.72	3.4	4.7

- Note: $\underline{\gamma}_C < 1$ and $\overline{\gamma}_C > 1$!

Fast solution in rectangular subdomains

- $\Pi = (0, l_1) \times (0, l_2)$, uniform rectangular mesh $\bar{\omega}_h = \omega_h \cup \gamma_h$,
- degrees of freedom: ω_I, $\gamma_C = \gamma_D \cup \gamma_N$, $\omega_F = \omega_I \cup \gamma_N$.
- Non-homogeneous clamped plate problem:

$$\begin{pmatrix} \tilde{K}_I & \tilde{K}_{IN} \\ 0 & I_N \end{pmatrix} \begin{pmatrix} \underline{u}_I \\ \underline{g}_N \end{pmatrix} = \begin{pmatrix} \underline{f}_I - \tilde{K}_{ID}\underline{g}_D \\ \underline{g}_N \end{pmatrix}.$$

- Associated simply supported plate problem:

$$\bar{K}_F \underline{\bar{u}}_F = \begin{pmatrix} \tilde{K}_I & \tilde{K}_{IN} \\ \tilde{K}_{NI} & \tilde{K}_N \end{pmatrix} \begin{pmatrix} \underline{\bar{u}}_I \\ \underline{\bar{u}}_N \end{pmatrix} = \begin{pmatrix} \underline{f}_I - \tilde{K}_{ID}\underline{g}_D \\ -\tilde{K}_{ND}\underline{g}_D \end{pmatrix}.$$

- Basic idea (Bahlmann/Korneev):
  - Find $\underline{\bar{u}}_F$ via the Fourier method.
  - Find the correction $\underline{v}_F = \underline{u}_F - \underline{\bar{u}}_F$ by solving an auxiliary system of linear equations corresponding to γ_N.
- Fourier method:

$$\begin{aligned}
\bar{K}_F^{-1} &= M\,\Lambda^{-1}\,M^T = M\,\mathrm{blockdiag}(\Lambda_j^{-1})\,M^T, \\
\Lambda_j &\in \begin{cases} I\!R^{3\times 3} & : j \in \omega_h, \\ I\!R & : j \in \gamma_h. \end{cases} \\
M &= \left(\delta_{\alpha\beta} \cdot \xi_{i_1}(\alpha_1, j_1) \cdot \xi_{i_2}(\alpha_2, j_2)\right)_{\omega_F \times \omega_F}, \\
\xi_{i_k}(0, j_k) &= \sqrt{\frac{2}{N_k}} \sin \frac{i_k j_k \pi}{N_k}, \\
\xi_{i_k}(1, j_k) &= \sqrt{\frac{2}{N_k} \kappa_k(i_k)} \cos \frac{i_k j_k \pi}{N_k}, \\
\Rightarrow & \quad O(N_1\,N_2 \ln(N_1\,N_2)) \text{ arithmetical operations.}
\end{aligned}$$

Fast solution in rectangular subdomains

- The unknown correction $\underline{v} = \underline{u} - \underline{\bar{u}}$ satisfies:

$$\begin{aligned} \bar{K}_F \underline{v}_F &= \begin{pmatrix} \underline{0}_I \\ \underline{b}_N \end{pmatrix}, \\ \underline{v}_D &= \underline{0}_D, \\ \underline{v}_N &= \underline{g}_N - \underline{\bar{u}}_N. \end{aligned}$$

- Represent $\underline{b}_N$ in terms of the basis N:

$$N = \left(\pm\delta_{\alpha\beta} \cdot \xi_{i_\tau}(0, j_\tau)\right)_{\gamma_N \times \gamma_N},$$

$$\underline{b}_N = N \underline{b}_\nu \text{ bzw. } \begin{pmatrix} \underline{0}_I \\ \underline{b}_N \end{pmatrix} = \hat{N} \underline{b}_\nu.$$

- $\underline{b}_\nu$ is the solution of: $B \underline{b}_\nu = \underline{d}$,

$$\begin{aligned} B &= \hat{N}^T \bar{K}_F^{-1} \hat{N}, \\ \underline{d} &= N^T \underline{g}_N - \hat{N}^T \underline{\bar{u}}_F. \end{aligned}$$

- Solution of the non-homogeneous Dirichlet problem:

$$\begin{aligned} \underline{\bar{f}}_F &= \begin{pmatrix} \underline{f}_I - \tilde{K}_{ID} \underline{g}_D \\ -\tilde{K}_{ND} \underline{g}_D \end{pmatrix}, \\ \underline{u}_I &= \left[\bar{K}_F^{-1} \underline{\bar{f}}_F + \bar{K}_F^{-1} \hat{N} B^{-1} \left(N^T \underline{g}_N - \hat{N}^T \bar{K}_F^{-1} \underline{\bar{f}}_F \right) \right]_I \\ &= \left[M \left(\Lambda^{-1} M^T \underline{\bar{f}}_F + \Lambda^{-1} N_\mu B^{-1} \left(N^T \underline{g}_N - N_\mu^T \Lambda^{-1} M^T \underline{\bar{f}}_F \right) \right) \right]_I \end{aligned}$$

- additional arithmetical cost: $O(N_1 \, N_2 \, \ln \varepsilon^{-1})$
  - Jacobi method: $S = (I - D^{-1} B)$,

$$\|S\|_\infty \leq c_0 \neq c_0(h) \Longrightarrow I(\varepsilon) = c \neq c(h).$$

  - Nonzero elements in B and $N_\mu = M^T \hat{N}$: $O(N_1 \, N_2)$.

Using the boundary potential method within DD-preconditioning

- Subdomain preconditioner:

$$\begin{aligned}
\tilde{C}_I &= \text{blockdiag}(\tilde{C}_{I,s}), \\
\tilde{C}_{I,s}^{-1}\underline{r}_{I,s} &= \left[\left(\bar{K}_{F,s}^{-1} - \bar{K}_{F,s}^{-1}\hat{N}_s\,(I - S_s^{n_C})\,B_s^{-1}\hat{N}_s^T\bar{K}_{F,s}^{-1}\right)\begin{pmatrix}\underline{r}_{I,s}\\ 0_{N,s}\end{pmatrix}\right]_{I,s}, \\
\Longrightarrow &\qquad \underline{\gamma}_I = 1 - c_0^{n_C}, \quad \overline{\gamma}_I = 1 + c_0^{n_C}.
\end{aligned}$$

- Basis transformation:
  - Exact biharmonic basis: $\mu = 0$,

$$\begin{aligned}
V_* &= \begin{pmatrix} I_C & 0 \\ -\tilde{K}_I^{-1}\tilde{K}_{IC} & I_I \end{pmatrix}, \\
\tilde{C}_*^{-1} &= V_*\begin{pmatrix}\tilde{C}_C^{-1} & 0 \\ 0 & \tilde{C}_I^{-1}\end{pmatrix}V_*^T.
\end{aligned}$$

  - Boundary potential method: V_{n_B}: $\mu = \mu(n_B)$?

$$\tilde{C}_{n_B}^{-1} = V_{n_B}\begin{pmatrix}\tilde{C}_C^{-1} & 0 \\ 0 & \tilde{C}_I^{-1}\end{pmatrix}V_{n_B}^T.$$

  - The existence of positive, h-independent constants μ_1, μ_2:

$$\mu_1\tilde{C}_* \le \tilde{C}_{n_B} \le \mu_2\tilde{C}_*$$

    requires not more than $n_B = O(\ln h^{-1})$ inner iterations.
  - practically sufficient: $n_B \approx 15\ldots 20$.

Numerical results

- Square clamped plate, $p = 4^{j_0}$ subdomains,
- Adini elements,
- DD–preconditioner $\tilde{C}_{n_B}$: $n_C = n_B = 20$,

j_0	0			1			2			3		
j	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ	λ_{min}	λ_{max}	κ
2	1.00	1.13	1.13	0.53	1.51	2.88	0.59	1.00	1.69	—	—	—
3	1.01	1.13	1.12	0.59	2.37	4.00	0.50	1.78	3.58	0.52	1.05	2.03
4	1.03	1.10	1.07	0.71	2.83	4.01	0.48	2.66	5.55	0.49	1.83	3.76
5	1.00	1.06	1.05	0.72	3.13	4.36	0.50	3.02	6.00	0.48	2.73	5.72
6	1.00	1.03	1.03	0.72	3.35	4.65	0.58	3.24	5.63	0.49	3.05	6.17
7	1.00	1.01	1.01	0.72	3.53	4.89	0.63	3.42	5.39	0.56	3.27	5.88
8	1.00	1.02	1.02	0.72	3.67	5.08	0.70	3.57	5.13	0.63	3.43	5.43
9	—	—	—	0.72	3.79	5.24	0.71	3.70	5.18	0.69	3.57	5.15
10	—	—	—	—	—	—	0.72	3.81	5.28	0.71	3.70	5.18
11	—	—	—	—	—	—	—	—	—	0.72	3.81	5.28

- ParPCG: $\left\| \underline{u} - \underline{u}^{I(\varepsilon)} \right\|_{\tilde{K}\tilde{C}^{-1}\tilde{K}} \leq \varepsilon \left\| \underline{u} - \underline{u}^{0} \right\|_{\tilde{K}\tilde{C}^{-1}\tilde{K}}$.

	j_0	0	1	2	3	4	0	1	2	3	4
j	N_*	$I(10^{-5})$					T_T[s]				
1	3	1	1	—	—	—	0.03	0.04	—	—	—
2	27	1	4	4	—	—	0.05	0.16	0.17	—	—
3	147	2	7	8	5	—	0.13	0.26	0.41	0.62	—
4	675	2	8	11	8	4	0.39	0.47	0.65	1.21	7.15*
5	2883	2	8	11	11	6	1.55	1.26	0.92	1.99	11.47*
6	11907	2	8	11	12	10	6.55	4.71	2.01	2.69	20.18*
7	48387	2	8	10	11	11	28.25	19.58	6.16	3.86	24.81*
8	195075	2	8	10	10	10	122.59	83.89	24.26	7.99	27.05*
9	783363	—	8	10	9	8	—	364.23	104.18	24.02	31.23*
10	3139587	—	—	10	9	8	—	—	446.87	96.04	65.98*
11	12570627	—	—	—	9	8	—	—	—	409.80	212.00*

- Use $\tilde{V}_h$ as an auxiliary space for V_h.
- $\Pi : \tilde{V}_h \to V_h$

$$\begin{aligned} \Pi \tilde{v}_h|_P &= \tilde{v}_h|_P \,, \\ \nabla(\Pi \tilde{v}_h)|_P &= \nabla \tilde{v}_h|_P \,, \\ \partial_{12}(\Pi \tilde{v}_h)|_P &= \frac{1}{m_P} \sum_{R:P\in R} \partial_{12,R} \tilde{v}_h|_P \,. \end{aligned}$$

- Smoother $\mathcal{S} : V_h \to V_h$.

$$\mathcal{S} v_h = \sum_{P\in\mathcal{N}_h} \frac{(v_h, \varphi_{(\mathbf{1},P)})}{a(\varphi_{(\mathbf{1},P)}, \varphi_{(\mathbf{1},P)})} \varphi_{(\mathbf{1},P)}.$$

- Parallel preconditioner for BFS elements

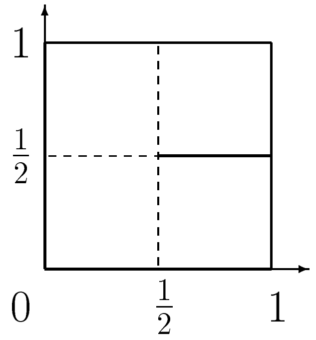
$$\begin{aligned} C_h^{-1} &= S_h + \Pi\, \tilde{C}_h^{-1}\, \Pi^T \\ \kappa(C_h^{-1} K_h) &\leq c \neq c(j, j_0). \end{aligned}$$

- Square clamped plate, $p = 4^{j_0}$ subdomains.

	j_0	0	1	2	3	4	0	1	2	3	4
j	N_*	$I(10^{-5})$					T_T[s]				
1	4	1	1	—	—	—	0.04	0.05	—	—	—
2	36	6	6	3	—	—	0.17	0.28	0.17	—	—
3	196	7	12	9	6	—	0.37	0.52	0.61	3.97	—
4	900	6	12	12	7	4	0.98	0.81	0.96	1.37	7.85*
5	3844	4	11	12	12	6	2.76	1.92	1.31	2.93	12.53*
6	15876	3	11	12	12	11	9.37	6.88	2.61	4.28	23.73*
7	64516	3	10	11	11	11	40.26	25.93	7.42	4.81	26.51*
8	260100	3	9	10	10	10	174.79	100.91	26.60	9.52	28.89*
9	1044484	—	9	10	9	9	—	437.55	111.86	26.51	37.19*
10	4186116	—	—	10	9	8	—	—	483.69	104.36	70.26*
11	16760836	—	—	—	9	8	—	—	—	442.98	227.99*

Results for the slit domain

- Slit domain, clamped boundary,
- $p = 4^{j_0+1}$ subdomains,
- $n_I = n_C = 20$,
- Adini elements



	j_0	0	1	2	3	0	1	2	3
j	N_*	$I(10^{-5})$				T_T[s]			
1	21	5	6	—	—	0.21	0.18	—	—
2	135	9	10	7	—	0.32	0.49	0.81	—
3	651	11	14	10	7	0.64	0.78	1.49	11.41*
4	2835	12	15	14	10	1.83	1.19	2.47	17.96*
5	11811	13	16	15	13	7.31	2.81	3.30	25.68*
6	48195	14	16	16	15	32.62	9.39	5.44	33.02*
7	194691	15	17	16	15	149.18	39.75	12.36	39.31*
8	782595	16	17	17	16	689.68	168.60	43.20	58.94*
9	3138051	—	18	17	16	—	770.98	172.31	123.99*
10	12567555	—	—	18	17	—	—	775.72	422.88*

- BFS-elements

	j_0	0	1	2	3	0	1	2	3
j	N_*	$I(10^{-5})$				T_T[s]			
1	28	8	6	—	—	0.36	0.29	—	—
2	180	12	11	9	—	0.52	0.78	1.28	—
3	868	13	14	11	9	0.89	1.05	2.03	15.73*
4	3780	13	15	14	11	2.23	1.56	4.20	21.44*
5	15748	14	16	15	14	8.60	3.38	3.99	29.64*
6	64260	14	16	15	14	35.16	10.42	6.02	33.37*
7	259588	15	17	16	15	160.75	43.22	14.92	41.97*
8	1043460	15	17	17	16	697.91	182.51	47.68	63.16*
9	4184068	—	18	17	16	—	832.01	187.18	132.45*
10	16756740	—	—	17	16	—	—	805.14	426.64*

Thin elastic shells

- Consider Kirchhoff type shell models: Koiter, Novozhilov.
- Let $\Omega \subset I\!R^2$ the reference domain of a circular cylinder $S = \varphi(\Omega)$

$$\begin{aligned} S &= \Big\{(x,y,z) : x^2 + z^2 = R^2, y \in (0,L)\Big\}, \\ \Omega &= (0,2\pi) \times (0,L). \end{aligned}$$

- Find the displacement $\mathbf{u} = (u_1, u_2, u_3)$ that minimizes the energy

$$J(\mathbf{u}) = \frac{1}{2}\, a(\mathbf{u},\mathbf{u}) - f(\mathbf{u})$$

$$\mathbf{u} \in \mathbf{V} \subset H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega).$$

$$\begin{aligned} a(\mathbf{u},\mathbf{u}) &= \frac{E\,t}{1-\sigma^2} \int_\Omega (\varepsilon_{11} + \varepsilon_{22})^2 - 2\,(1-\sigma)(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}) dx \\ &+ \frac{E\,t^3}{12(1-\sigma^2)} \int_\Omega (\kappa_{11} + \kappa_{22})^2 - 2\,(1-\sigma)(\kappa_{11}\kappa_{22} - \kappa_{12}) dx \end{aligned}$$

where

$$\begin{aligned} \varepsilon_{11} &= \partial_1 u_1 + \tfrac{1}{R}\, u_3, & \kappa_{11} &= -\partial_{11} u_3 + \tfrac{1}{R}\, \partial_1 u_1, \\ \varepsilon_{22} &= \partial_2 u_2, & \kappa_{22} &= -\partial_{22} u_2, \\ \varepsilon_{12} &= \tfrac{1}{2}\left(\partial_2 u_1 + \partial_1 u_2\right), & \kappa_{12} &= -\partial_{12} u_3 + \tfrac{1}{R}\, \partial_2 u_1. \end{aligned}$$

- Fourth order system of p.d.e.'s

$$\mathcal{L} = \begin{pmatrix} -\partial_{11} - \frac{1-\sigma}{2}\,\partial_{22} + \frac{t^2}{12\,R^2}\delta & -\frac{1+\sigma}{2}\,\partial_{12} & -\frac{1}{R}\,\partial_1 + \frac{t^2}{12\,R}\, d \\ -\frac{1+\sigma}{2}\,\partial_{12} & -\frac{1-\sigma}{2}\,\partial_{11} - \ \partial_{22} & -\frac{\sigma}{R}\,\partial_2 \\ \frac{1}{R}\,\partial_1 - \frac{t^2}{12\,R} d & \frac{\sigma}{R}\,\partial_2 & \frac{1}{R^2} + \frac{t^2}{12}\Delta^2 \end{pmatrix}$$

$$\begin{aligned} \delta &= -\partial_{11} - 2(1-\sigma)\,\partial_{22} \\ d &= -\partial_{111} + (2-\sigma)\,\partial_{122} \end{aligned}$$

A DD preconditioner for the shell problem

- Non–overlapping rectangular subdomains

$$\bar{\Omega} = \cup_{s=1}^{p} \bar{\Pi}_s .$$

- Conforming f.e. discretization:

$$\mathbf{V_h} = V_h^{BFS} \times V_h^{BFS} \times V_h^{BFS} \subset \mathbf{V},$$

- Non–conforming auxiliary f.e. space

$$\tilde{\mathbf{V}}_\mathbf{h} = V_h^{A} \times V_h^{A} \times V_h^{A} \not\subset \mathbf{V}.$$

- DD-preconditioner $\tilde{C}_h^{-1}$ for the Adini–discretization
  - $\tilde{C}_C$:
    multilevel nodal basis preconditioner for $\tilde{S}_C$,
  - $\tilde{C}_{I,n_C}$ and V_{n_B}:
    Solve decoupled auxilliary subdomain problems

$$\begin{pmatrix} -\partial_{11} - \frac{1-\sigma}{2}\,\partial_{22} & 0 & 0 \\ 0 & -\frac{1-\sigma}{2}\,\partial_{11} - \partial_{22} & 0 \\ 0 & 0 & \frac{t^2}{12}\Delta^2 \end{pmatrix},$$

    using the boundary potential method.

- Parallel preconditioner for the conforming BFS–discretization

$$C_h^{-1} = S + \Pi\,\tilde{C}_h^{-1}\,\Pi^T .$$

Examples

- Clamped shell,

$$
\begin{aligned}
S &= \left\{(x,y,z) : x^2 + y^2 = 1, y \geq 0, z \in (0,1)\right\}, \\
\Omega &= (0,\pi) \times (0,1).
\end{aligned}
$$

$j - j_0$	4×1	8×2	16×4	4×2	8×4	16×8
1	56	29	23	44	27	25
2	80	35	22	54	26	22
3	86	39	22	58	28	21
4	90	40	20	64	29	19
5	93	41	21	62	29	19
6	95	42	21	64	29	19

- Clamped / simply supported cylinder

$$
\begin{aligned}
S &= \left\{(x,y,z) : x^2 + y^2 = 1, z \in (0,1)\right\}, \\
\Omega &= (0, 2\,\pi) \times (0,1).
\end{aligned}
$$

$j - j_0$	4×1	8×2	16×4	4×2	8×4
0	—	44	22	—	22
1	157	53	25	92	48
2	141	54	23	79	60
3	144	59	22	84	64
4	147	63	23	89	63
5	148	65	23	91	63
6	153	65	24	92	63
7	145	65	24	89	63