Improved Separators by Multigrid Methods

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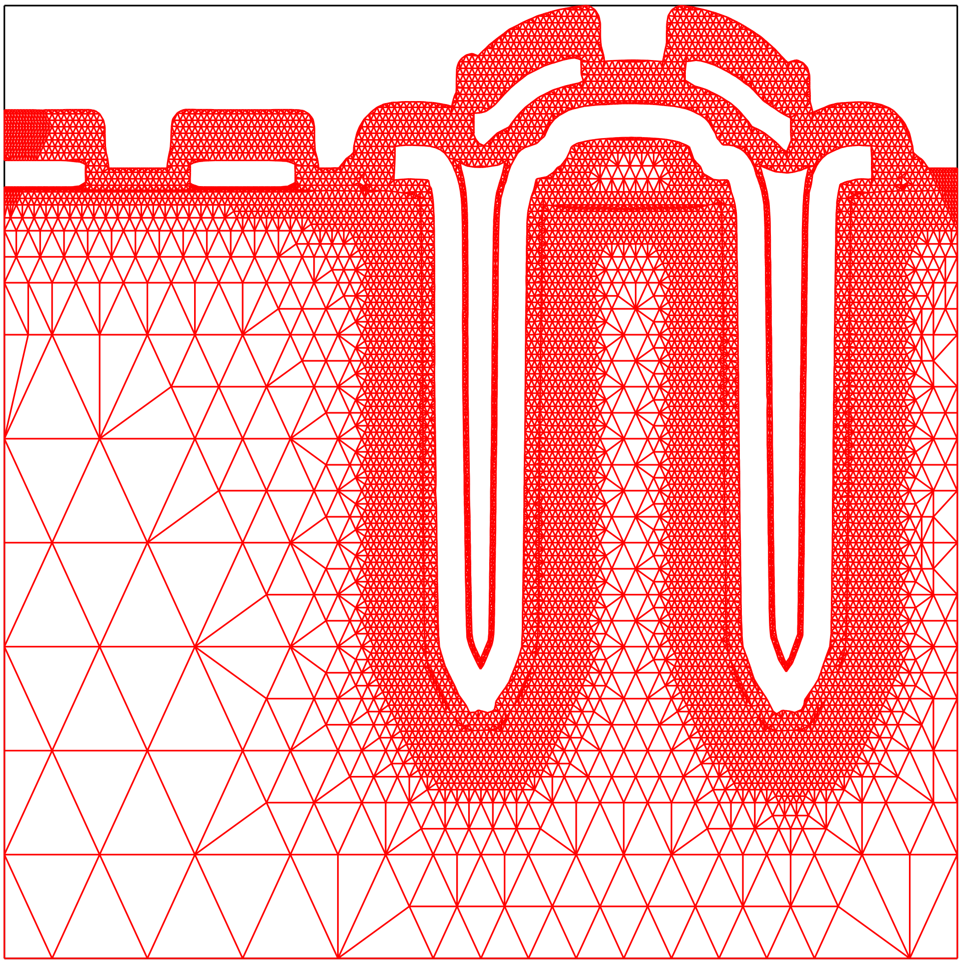


All grids are generated by Norbert Strecker's DIOS–code

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a graph, with the set of vertices $\mathcal{V}$, edges $\mathcal{E}$ and numbers of elements $v = |\mathcal{V}|$, $e = |\mathcal{E}|$.

One asks for two subsets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ with

a) the number of edges connecting $\mathcal{V}_1$, $\mathcal{V}_2$ minimal, and

b) $|v_1 - v_2| \to \min$.

The combinatorial problem is NP hard.

The simplest analytical analog is a Neumann eigenvalue problem:

the basis for spectral bisection algorithms.

Better approximate solutions due to better analytical analogs?

$$\tilde{\mathcal{L}}_\alpha(u,\lambda) := \frac{\|\nabla u\|^2}{\|h(u)\|^2} + \lambda_2 \frac{f_\alpha^2(u)}{\|h(u)\|^2} \to \min\,. \qquad (1)$$

Condition a) is replaced by $\|\nabla u\|^2$**,
a suitable norm of the gradient, hence**
$\|\nabla u\|^2/\|h(u)\|^2$ **is the Lagrange function
(with some weight function** $h(u)$**),** λ_2 **the Lagrange multiplier.**

b) is replaced by

$$f(u) = \int_{\mathrm{supp}\, u^+} dV - \int_{\mathrm{supp}\, u^-} dV = 0, \qquad (2)$$

where $\mathrm{supp}\, u^+$**,** $\mathrm{supp}\, u^-$ **means the support of the positive / negative part of** u**.**
$f_\alpha(u)$ **represents an approximation of this side condition.**

Using the L_2 norm of the gradient, relaxing (2) to

$$\tilde{f}(u) = \int u dV = 0, \quad h(s) = s$$

yields the Neumann eigenvalue problem.

Condition b) has to be approximated by an ordering and counting procedure or: finding a constant c that fulfills

$$\int_{\text{supp }(u+c)^+} dV - \int_{\text{supp }(u+c)^-} dV = 0.$$

Hence $u(x) + c = 0$ defines the separator $S_0(x)$.

a) would yield: $S_{max}(|\nabla u|)$.

The distance can be easily measured:

$$d(S_0, S_{max}) = |(\mathcal{V}_1(S_0) \cup \mathcal{V}_1(S_{max})) \setminus (\mathcal{V}_1(S_0) \cap \mathcal{V}_1(S_{max}))|.$$

Let (2) be approximated

$$f_\alpha(u) = \int g_\alpha(u) dV = 0,$$

$$\lim_{\alpha\to 1} g_\alpha(s) = \text{ sign } (s),$$

$$g_\alpha(-s) = -g_\alpha(s), \quad g'_\alpha(s) > 0,$$

$$g_\alpha(s) = |s|^{1-\alpha} \text{ sign } (s), \quad \sinh^{-1}(s/(1-\alpha)), \ldots.$$

Euler equation:

$$\begin{aligned} \frac{\partial}{\partial u}\mathcal{L}_\alpha(u,\lambda) &= 0 \\ \frac{\partial}{\partial \lambda}\mathcal{L}_\alpha(u,\lambda) &= 0. \end{aligned}$$

For the L_2 norm of $|\nabla u|$ follows:

$$\begin{aligned} -\nabla\cdot\nabla u - \lambda_1 h(u)h'(u) + \lambda_2 \bar{g}_\alpha g'_\alpha(u) &= 0, \\ \int h(u)^2 dV - 1 &= 0, \\ (\int g_\alpha(u) dV)^2 = \bar{g}_\alpha^2 &= 0, \end{aligned}$$

Linearization at
(U,Λ) **(**$u = U + \delta u$**,** $\lambda = \Lambda + \delta\lambda$**):**

$$
\begin{aligned}
&(-\Delta - \Lambda_1(hh'' + h'^2) + \\
&\qquad \Lambda_2(\bar{g}'_\alpha g'_\alpha + \bar{g}_\alpha g''_\alpha))\delta u + hh'\delta\lambda_1 + \bar{g}_\alpha g'_\alpha \delta\lambda_2 = r_1, \\
&\qquad\qquad (hh', \delta u) = r_2, \\
&\qquad\qquad (\bar{g}_\alpha g'_\alpha, \delta u) = r_3,
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= -(-\Delta U - \Lambda_1 h(U)h'(U)) - \Lambda_2 \bar{g}_\alpha(U) g'_\alpha(U) \\
r_2 &= (1 - (h(U), h(U)))/2 \\
r_3 &= -f_\alpha^2(U)/2
\end{aligned}
$$

The expressions for

$$\nabla\epsilon(1/|\nabla u|)\nabla u,$$

($\epsilon(s)$ **the regularization function for the** L_1 **norm) are not given here.**

Simplifications:

$$\begin{aligned} -\nabla\cdot\nabla u - \lambda_1 h(u)h'(u) + \lambda_2 \bar{g}_\alpha g'_\alpha(u) &= 0,\\ \int h(u)^2 dV - 1 &= 0,\\ (\int g_\alpha(u)dV)^2 = \bar{g}^2_\alpha &= 0, \end{aligned}$$

a) $h(s) = s$ **and approximation of the integrals:**
$-\nabla\cdot\nabla \rightarrow A$**,** $\bar{g}_\alpha =$ $\mathbf{g}_\alpha^T\,\mathbf{1}$**,** $\mathbf{g}_\alpha^T\,\mathbf{1} = 1/(1-\alpha)\,\mathbf{g}_\alpha'^T\,\mathbf{u}$**:**

$$\begin{aligned} A\,\mathbf{u} + \bar{g}_\alpha\,\mathbf{g}'_\alpha(u)\lambda_2 &= \lambda_1\,\mathbf{u},\\ \bar{g}'_\alpha/c\,\mathbf{g}_\alpha'^T\,\mathbf{u} + \lambda_1\lambda_2 &= \lambda_1\lambda_2. \end{aligned}$$

Freezing $\bar{g}_\alpha g'_\alpha(u) = \bar{g}_\alpha(U)g'_\alpha(U)$ **and** $\lambda_1 = \Lambda_1$ **yields:**

$$\begin{aligned} A\,\mathbf{u} + \bar{c}\,\mathbf{g}'_\alpha(U)\lambda_2 &= \lambda_1\,\mathbf{u},\\ \bar{c}\,\mathbf{g}_\alpha'^T(U)\,\mathbf{u} + \Lambda_1\lambda_2 &= \lambda_1\lambda_2 \end{aligned}$$

Eigenvalue problem for the eigenvalue λ_1
and the eigenvector $(\,\mathbf{u}^T, \lambda_2)$**.**
Define (to minimize $\lambda_{1,i}$**):**

$$U_{i+1} = \sum_{j=0}^{i} \beta_j u(U_j),$$

$$\begin{aligned} (A - \tilde{\Lambda}_{1,i}I)\,\mathbf{u}_i + \bar{c}\,\mathbf{g}'_\alpha(U_i)\lambda_{2,i} &= \lambda_{1,i}\,\mathbf{u}_i,\\ \bar{c}\,\mathbf{g}_\alpha'^T(U_i)\,\mathbf{u}_i + (\Lambda_{1,i} - \tilde{\Lambda}_{1,i})\lambda_{2,i} &= \lambda_{1,i}\lambda_{2,i}. \end{aligned}$$

b) $h(s) = g_\alpha(s)$ **and** $w = g_\alpha(u)$ **yields:**

$$-\nabla \cdot g_\alpha^{-1\prime} \nabla w = \lambda_1 w h'(u).$$

Freezing the coefficients $g_\alpha^{-1\prime}(W_i)$**,** $h'(U_i)$ **yields a sequence of eigenvalue problems again.**
c) Computation of some eigenfunctions (including the constant one) and minimizing (1) with respect to that small basis.
From such an initial guess Newtons method can be applied until 'valid gradient information' (the distance of the two differently defined separators is small) or a sufficient reduction of the separator size is reached.

If the distances are not measured in the metric of the graph, one gets a partitioning of the domain.

Example: 11738 nodes, 4 multigrid levels;

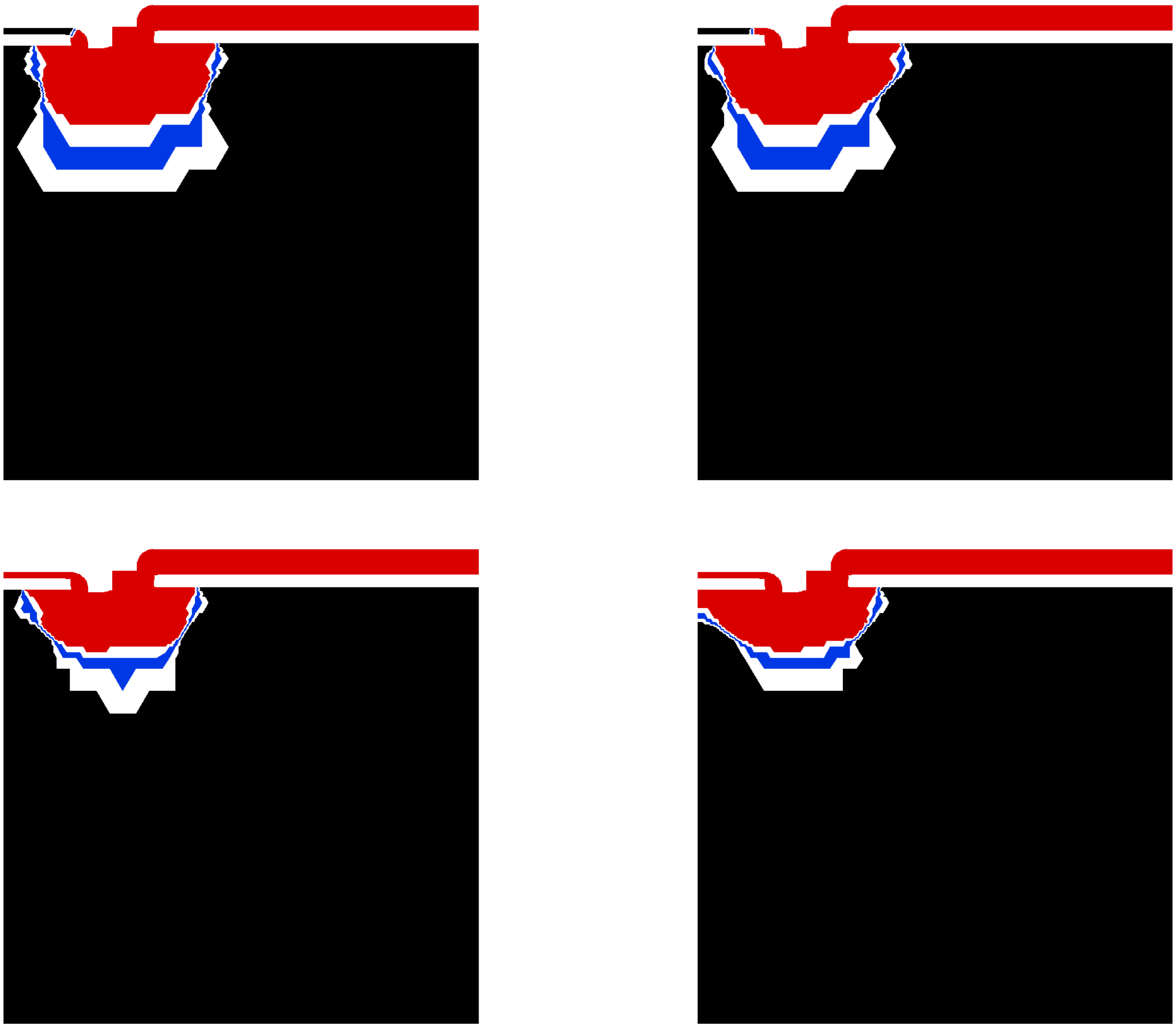
level	# nodes	# matrix elem.	# added (ref.) nodes	max. next neighbours
1	11738	79566	0	10
2	3703	22913	773	13
3	1527	8951	489	14
4	741	4215	284	17

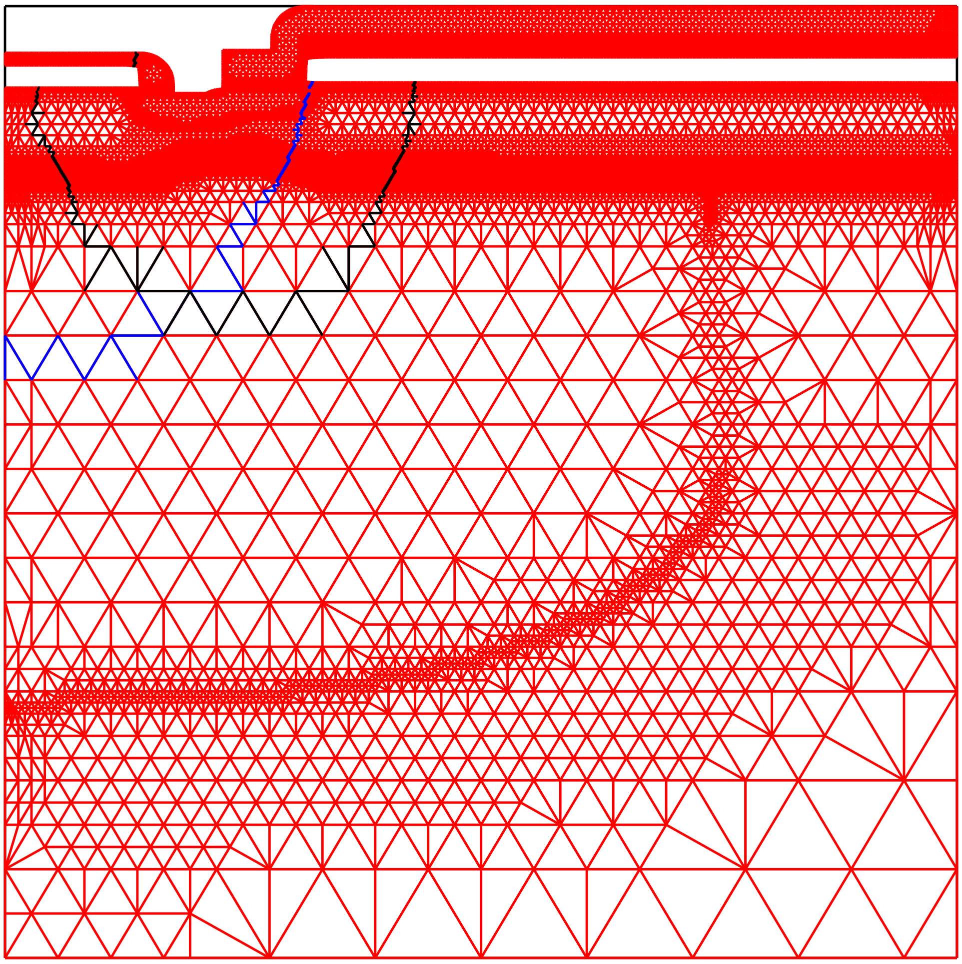
total integer storage: 102960
total real storage : 142837

number	eigenvalue Neumann pr.	# cut edges if '2' parts
1	2.226853e-4	143
2	1.260926e-3	
3	2.423339e-3	177
4	4.462123e-3	
5	6.093297e-3	
6	7.845493e-3	
7	8.923921e-3	

iterations	$\int g_{0.9}(u)dV$	λ_1
1	**108041.6**	**2.13561e-4**
2	**82673.7**	**1.97424e-4**
3	**54928.7**	**1.89490e-4**
4	**37740.6**	**1.85002e-4**
5	**25931.9**	**1.82381e-4**
6	**17703.7**	**1.80917e-4**
7	**12060.2**	**1.79885e-4**
8	**7414.1**	**1.79052e-4**
9	**3580.5**	**1.77466e-4**
10	**171.9**	**1.73065e-4**

Separator related to the Neumann problem and a subset of the first iterations related to a) without any explicit minimization. All separators are larger than that of the Neumann problem - the last one was the starting point for Newtons method.





Separator due to the Neumann eigenvalue problem (black, 143 cut edges) and the best one (blue, 80 cut edges, known until now).

References

[1] B. Hendrickson, R. Leland The Chaco User's Guide SAND94–2692, bah@cs.sandia.gov as one example and the references in.