# STABLE MULTISCALE BASES AND ADAPTIVE TECHNIQUES FOR ELLIPTIC PROBLEMS

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- Some basic concepts
  - Multiscale bases
  - Stability concepts
  - Norm equivalences
  - Compression
- Local a-posteriori error estimators
- a convergent strategy

  (S. Dahlke, R. Hochmuth, R. Schneider)

# SCOPE OF PROBLEMS

$$Au = f$$

$$A : H_1 \to H_2 \quad \|Au\|_{H_2} \sim \|u\|_{H_1}, \quad u \in H_1,$$

**Example 1:** $H_1 = H^t, \quad H_2 = H^{-t} := (H^t)', \quad A$ selfadjoint

$$a(u, v) = (Au, v), \quad u, v \in H^t$$

Ellipticity

$$a(u, u) \sim \|u\|^2_{H^t(\Omega)}, \quad u \in H^t$$

e.g.

$$A = -\Delta, \quad A = -\nabla B \nabla + cI$$

$$H_1 = H_0^1(\Omega), H^1(\Omega), \quad H_2 = H^{-1}(\Omega), \ (H^1(\Omega))'$$

**Example 2:** $\Delta U = 0 \quad \text{auf} \quad I\!R^3 \setminus \Omega, \quad U\,|_{\partial\Omega} = f$

**Indirect Method** $\rightsquigarrow$

$$Au = f \qquad A = I + 2K$$

where

$$Ku(x) := [1/2 - \theta_\Omega(x)]u(x) + \frac{1}{4\pi}\int\limits_{\partial\Omega} \frac{n_y \cdot (x-y)}{|y-x|^3} u(y)dy$$

and

$$U(x) = \frac{1}{4\pi}\int\limits_{\partial\Omega} \frac{n_y \cdot (x-y)}{|x-y|^3} u(y)dy, \quad x \in \Omega$$

- $A$ has order $2t = \rho = 0$, $H_1 = H_2 = L_2(\partial\Omega)$

**Direct Method** $\rightsquigarrow$

$$Au = Vu = (\frac{1}{2} + K)f, \quad Vu(x) = \int\limits_{\partial\Omega} \frac{u(y)}{4\pi|x-y|} dy$$

- integral equation of the first kind
- order $2t = \rho = -1$, $H_1 = H^{-1/2}(\partial\Omega)$, $H_2 = H^{1/2}(\partial\Omega)$

## ASSUMPTIONS

- $$Au(x) = \int_\Omega K_A(x,y)u(y)dy$$

  $$|\partial_x^\alpha \partial_y^\beta K_A(x,y)| \leq c_{\alpha,\beta} \mathrm{dist}\,(x,y)^{-(n+\rho+|\alpha|+|\beta|)}$$

- Galerkin schemes are stable

# DISCRETIZATION

## Multiresolution

$$S_0 \subset S_1 \subset S_2 \subset \ldots H, \qquad \overline{\bigcup_j S_j} = H$$

## Single-scale bases:

$$S_j = \operatorname{span} \Phi_j, \quad \Phi_j = \{\varphi_{j,k} : k \in \Delta_j\}$$

**Stability:** $\mathbf{c}^T \Phi_j := \sum_{k \in \Delta_j} c_k \varphi_{j,k}$

$$\|\mathbf{c}\|_{l_2(\Delta_j)} \sim \|\mathbf{c}^T \Phi_j\|_H$$

## Decomposition:

$$S_{j+1} = S_j \bigoplus W_{j+1}$$

## Complement bases:

$$W_j = \operatorname{span} \Psi_j, \quad \Psi_j = \{\psi_{j,k} : k \in \nabla_j\}$$

**How to choose the $\Psi_j$ ???**

# MULTISCALE-BASES

$$\Psi := \bigcup_{j \in \mathbb{N}_0} \Psi_j \qquad (\Psi_0 := \Phi_0)$$

**Notation:** $\nabla := \{\lambda = (j,k) : k \in \nabla_j,\ j \in \mathbb{N}_0\}, \quad |\lambda| = j$

$$\bigoplus_{l=0}^{j} W_j \quad = \quad S_j$$

$$\sum_{|\lambda| \le j} d_\lambda \psi_\lambda \quad = \quad \sum_{k \in \Delta_j} c_k \varphi_{j,k}$$

$$\mathbf{T}_j \ : \ \mathbf{d} \ \mapsto \quad \mathbf{c}$$

$$\|\mathbf{T}_j\| \ \|\mathbf{T}_j^{-1}\| = \mathcal{O}(1)$$

$$\Longleftrightarrow$$

$$\exists \quad \tilde{\Psi} \quad \text{s.t.} \quad (\psi_\lambda, \tilde{\psi}_{\lambda'}) = \delta_{\lambda,\lambda'}$$

and

$$v = \sum_{\lambda \in \nabla} (v, \tilde{\psi}_\lambda) \psi_\lambda$$

$$\|v\|_H^2 \sim \sum_{\lambda \in \nabla} |(v, \tilde{\psi}_\lambda)|^2 \sim \sum_{\lambda \in \nabla} |(v, \psi_\lambda)|^2$$

## A GENERAL FACT

$\omega(\cdot,t) : H \to I\!R_+$ subadditive, unif. bd., $\omega(f,0^+) = 0$

$$(*) \quad \begin{cases} \inf_{v_j \in S_j} \|v - v_j\|_H & \lesssim \ \omega(v, 2^{-j}), \quad f \in H, \\ \omega(v_j, t) & \lesssim \ (\max\{1, t2^j\})^\gamma \, \|v_j\|_H, \quad v_j \in S_j, \end{cases}$$

**Theorem**

- $H$ Hilbert space
- $\Psi$, $\tilde{\Psi}$ biorthogonal bases
- Let $\mathcal{S} = \{S_j\}_{j=0}^{\infty}$ as above and $\tilde{\mathcal{S}} = \{\tilde{S}_j\}_{j=0}^{\infty}$ where

$$\tilde{S}_j := \mathrm{span}\{\tilde{\psi}_\lambda : |\lambda| \le j\}$$

- $\mathcal{S}, \tilde{\mathcal{S}}$ satisfy $(*)$

$$\Rightarrow$$

$$\|v\|_H \sim \left(\sum_{\lambda\in\nabla} |(v, \tilde{\psi}_\lambda)|^2\right)^{1/2} \sim \left(\sum_{\lambda\in\nabla} |(v, \psi_\lambda)|^2\right)^{1/2}$$

## SOBOLEV-NORMS:

**Direct estimate (J):** $s \leq m$

$$\inf_{v_j \in S_j} \|v - v_j\|_0 \lesssim 2^{-js} \|v\|_s, \quad v \in H^s(\Omega)$$

**Inverse estimate (B):** $0 < \gamma \leq m$ fixed

$$\|v_j\|_s \lesssim 2^{js} \|v_j\|_0, \quad v_j \in S_j$$

$\tilde{S}_j \longleftrightarrow \gamma', m'$

$$Q_j v := \sum_{|\lambda| \leq j} (v, \tilde{\psi}_\lambda) \psi_\lambda$$

$\{Q_j\}$ uniformly bounded

$$\rightsquigarrow$$

$$\begin{aligned}
\|v\|_s &\sim \left( \sum_{\lambda \in \nabla}^{\infty} 2^{2s|\lambda|} |(v, \tilde{\psi}_\lambda)|^2 \right)^{1/2}, \quad s \in (-\gamma', \gamma) \\
&\sim \left( \sum_{j=0}^{\infty} 2^{2sj} \|(Q_j - Q_{j-1})v\|_0^2 \right)^{1/2} \\
&\sim \left( \sum_{\lambda \in \nabla}^{\infty} 2^{2s|\lambda|} |(v, \psi_\lambda)|^2 \right)^{1/2}, \quad s \in (-\gamma, \gamma')
\end{aligned}$$

$$c_1\|Av\|_{-t} \leq \|v\|_t \leq c_2\|Av\|_{-t}$$

$$c_3 \left( \sum_{\lambda \in \nabla} 2^{-t|\lambda|} |(f, \psi_\lambda)|^2 \right)^{1/2} \leq \|f\|_{-t} \leq c_4 \left( \sum_{\lambda \in \nabla} 2^{-t|\lambda|} |(f, \psi_\lambda)|^2 \right)^{1/2}$$

# NON-UNIFORM REFINEMENTS

**Task:** Given $\epsilon > 0$ **find**

$$\Lambda \subset \nabla \quad S_\Lambda := \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda\}$$

s.t. for

$$(Au_\Lambda, v) = (f, v), \qquad v \in S_\Lambda$$

$$\rightsquigarrow$$

$$\|u - u_\Lambda\|_t \leq \epsilon \quad ! \quad \#\Lambda \text{ small}$$

- Local a-posteriori error estimates
- A convergent adaptive strategy
  - **No** explicit a-priori assumptions on type of singularities!
  - Saturation property is **not** assumed.

| **Solve** | **Estimate** | **Refine** |
|---|---|---|
| $u_\Lambda \in S_\Lambda$ | $\|u - u_\Lambda\|_t$ | $\Lambda \to \tilde{\Lambda}$ |

# RESIDUAL ESTIMATES

$$r_\Lambda := Au_\Lambda - f = A(u_\Lambda - u).$$

$$\rightsquigarrow$$

$$c_1 \|r_\Lambda\|_{-t} \le \|u - u_\Lambda\|_t \le c_2 \|r_\Lambda\|_{-t},$$

$$r_\Lambda = \sum_{\lambda \in \nabla} (r_\Lambda, \psi_\lambda) \tilde{\psi}_\lambda = \sum_{\lambda \in \nabla \setminus \Lambda} (r_\Lambda, \psi_\lambda) \tilde{\psi}_\lambda$$

**Norm equivalences** $\rightsquigarrow$

$$c_3 \left( \sum_{\lambda \in \nabla \setminus \Lambda} 2^{-2t|\lambda|} |(r_\Lambda, \psi_\lambda)|^2 \right)^{1/2} \le \|r_\Lambda\|_{-t} \le c_4 \left( \sum_{\lambda \in \nabla \setminus \Lambda} 2^{-2t|\lambda|} |(r_\Lambda, \psi_\lambda)|^2 \right)^{1/2},$$

$$\delta_\lambda := 2^{-t|\lambda|} |(r_\Lambda, \psi_\lambda)| \quad u_\Lambda = \sum_{\lambda' \in \Lambda} u_{\lambda'} \psi_{\lambda'}$$

$$\rightsquigarrow$$

$$\delta_\lambda = 2^{-t|\lambda|} |f_\lambda - \sum_{\lambda' \in \Lambda} (A\psi_{\lambda'}, \psi_\lambda) u_{\lambda'}|, \quad f_\lambda := (f, \psi_\lambda)$$

$$\rightsquigarrow$$

$$c_3 \left( \sum_{\lambda \in \nabla \setminus \Lambda} \delta_\lambda^2 \right)^{1/2} \le \|r_\Lambda\|_{-t} \le c_4 \left( \sum_{\lambda \in \nabla \setminus \Lambda} \delta_\lambda^2 \right)^{1/2}$$

## DECAY ESTIMATES

$$(Av)(x) = \int_\Omega K(x,y)v(y)dy,$$

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \lesssim \mathrm{dist}\,(x,y)^{-(d+2t+|\alpha|+|\beta|)}, \quad d+2t+|\alpha|+|\beta| > 0$$

$$\rightsquigarrow$$

$$2^{-(|\lambda'|+|\lambda|)t}|(A\psi_{\lambda'},\psi_\lambda)| \lesssim \frac{2^{-||\lambda|-|\lambda'||(d/2+\sigma)}}{(1+2^{\min(|\lambda|,|\lambda'|)}\mathrm{dist}(\Omega_\lambda,\Omega_{\lambda'}))^{d+m'+2t}},$$

$\Omega_\lambda := \mathrm{supp}\,\psi_\lambda$

**Moment conditions:** $\tilde{S}_j$ exact of order $m'$ and $W_j \perp \tilde{S}_{j-1}$

$$\rightsquigarrow$$

$$\int_\Omega p(x)\psi_\lambda(x)dx = 0, \quad \deg p < m'$$

## SIGNIFICANT NEIGHBORHOODS

**Lemma 1** *For $\epsilon > 0$, $0 < \delta < \sigma$ choose $\epsilon_1, \epsilon_2 > 0$ s.t.*

$$\epsilon_1^{m'+\rho} + 2^{-\delta/\epsilon_2} \leq \epsilon \ .$$

$$\nabla_{\lambda,\epsilon} := \{\lambda' \in \nabla : ||\lambda| - |\lambda'|| \leq \epsilon_2^{-1} \ \textit{and} \ 2^{\min(|\lambda|,|\lambda'|)} \mathrm{dist}(\Omega_\lambda, \Omega_{\lambda'}) \leq \epsilon_1^{-1}\}.$$

$$e_\lambda := \sum_{\lambda' \in \Lambda \setminus \nabla_{\lambda,\epsilon}} (A\psi_{\lambda'}, \psi_\lambda) u_{\lambda'}, \qquad \lambda \in \nabla \setminus \Lambda,$$

$$\Longrightarrow$$

$$\left( \sum_{\lambda \in \nabla \setminus \Lambda} 2^{-2t|\lambda|} |e_\lambda|^2 \right)^{1/2} \leq c_5 \epsilon \|Q'_\Lambda f\|_{-t} \sim \epsilon \|u_\Lambda\|_t.$$

(S. Bertoluzza)

**Replace** $\delta_\lambda,\ \lambda \in \nabla \setminus \Lambda$ by:

$$d_\lambda(\Lambda,\epsilon) = d_\lambda := 2^{-t|\lambda|}|f_\lambda - \sum_{\lambda' \in \Lambda \cap \nabla_{\lambda,\epsilon}} (A\psi_{\lambda'}, \psi_\lambda) u_{\lambda'}|$$

**Note:**

- $$N_{\Lambda,\epsilon} := \{\lambda \in \nabla \setminus \Lambda : \Lambda \cap \nabla_{\lambda,\epsilon} \neq \emptyset\} \quad \rightsquigarrow \quad \# N_{\Lambda,\epsilon} < \infty$$
- $$d_\lambda(\Lambda,\epsilon) = 2^{-t|\lambda|}|f_\lambda|, \quad \lambda \in \nabla \setminus (\Lambda \cup N_{\Lambda,\epsilon}).$$
- $$\begin{aligned} \sum_{\lambda \in \nabla \setminus (N_{\Lambda,\epsilon} \cup \Lambda)} 2^{-2t|\lambda|}|f_\lambda|^2 &\lesssim \|f - Q'_{\Lambda \cup N_{\Lambda,\epsilon}} f\|_{-t}^2 \sim \inf_{v \in \tilde{S}_{\Lambda \cup N_{\Lambda,\epsilon}}} \|f - v\|_{-t}^2 \\ &\leq \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}^2 \end{aligned}$$

**Computable quantities**

$$a_\lambda(\Lambda,\epsilon) = a_\lambda := 2^{-t|\lambda|}|\sum_{\lambda' \in \Lambda \cap \nabla_{\lambda,\epsilon}} (A\psi_{\lambda'}, \psi_\lambda) u_{\lambda'}|, \quad \lambda \in \nabla \setminus \Lambda$$

**Note:**

$$a_\lambda(\Lambda,\epsilon) \neq 0 \quad \Rightarrow \quad \lambda \in N_{\Lambda,\epsilon}$$

$$\rightsquigarrow$$

# RELIABLE AND EFFICIENT ESTIMATES

**Theorem 1** *Under the assumptions in Lemma 1 one has*

$$\begin{aligned}\|u-u_\Lambda\|_t \;&\le\; c_2c_4\left\{\left(\sum_{\lambda\in\nabla\setminus\Lambda} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} + c_5\epsilon\|Q'_\Lambda f\|_{-t}\right.\\ &+\; \left. c_6\inf_{v\in\tilde S_\Lambda}\|f-v\|_{-t}\right\},\end{aligned}$$

*as well as*

$$\begin{aligned}\left(\sum_{\lambda\in\nabla\setminus\Lambda} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \;&\le\; \frac{1}{c_1c_3}\|u-u_\Lambda\|_t + c_5\epsilon\|Q'_\Lambda f\|_{-t}\\ &+\; c_6\inf_{v\in\tilde S_\Lambda}\|f-v\|_{-t}.\end{aligned}$$

## Assumption:

$$a(\cdot,\cdot) := (A\cdot,\cdot), \quad \|\cdot\| := a(\cdot,\cdot)^{1/2}, \quad \|\cdot\| \sim \|\cdot\|_t$$

# SATURATION PROPERTY

$A$ selfadjoint, $S' \subset S''$, $u', u''$ Galerkin solutions

**Remark:** $\|u'' - u'\| \leq \|u - u'\|$ and for $\beta < 1$ one has

$$\|u - u'\| \leq (1 - \beta^2)^{-\frac{1}{2}} \|u'' - u'\|$$

$$\Longleftrightarrow$$

$$\|u - u''\| \leq \beta \|u - u'\|$$

**Lemma 2** *For* $\Lambda \subset \tilde{\Lambda} \subset \nabla$ *one has*

$$\begin{aligned}\left(\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2\right)^{1/2} &\leq \frac{1}{c_3 c_1} \|u_{\tilde{\Lambda}} - u_\Lambda\|_t \;+\; c_5 \epsilon \|Q'_\Lambda f\|_{-t} \\ &+\; c_6 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t}.\end{aligned}$$

**Strategy:** Find $\tilde{\Lambda} \subset \nabla$ (**small**) such that

$$\|u - u_{\tilde{\Lambda}}\|_t \leq \kappa \|u - u_\Lambda\|_t \quad \kappa < 1$$

**Theorem 2** *Let* eps $> 0$. *Fix any* $\theta^* \in (0,1)$ *and define*

$$C_e := \left(\frac{1}{c_1 c_3} + \frac{1-\theta^*}{2c_2 c_4}\right).$$

*Choose any* $\mu^* > 0$ *such that*

$$\mu^* C_e \leq \frac{1-\theta^*}{2(2-\theta^*)c_2 c_4}$$

*and set*

$$\epsilon := \frac{\mu^* \mathrm{eps}}{2c_5' \|f\|_{-t}}.$$

*Choose* $\Lambda \subset \nabla$ *so that*

$$c_7 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t} < \frac{1}{2}\mu^* \mathrm{eps}$$

*Then for* $\tilde{\Lambda} \subset \nabla$, $\Lambda \subset \tilde{\Lambda}$ *such that*

$$\left(\sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2} \geq (1-\theta^*) \left(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2\right)^{1/2}$$

*there exists* $\kappa = \kappa(\mu^*, \theta^*, c_i) \in (0,1)$ *such that either*

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\|$$

*or* $(\sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2)^{1/2} = (\sum_{\lambda \in \nabla \setminus \Lambda} a_\lambda(\Lambda,\epsilon)^2)^{1/2} <$ eps.

**Initialization:** eps $> 0$, $\theta^* \in (0,1)$, compute $C_e$, $\mu^*$

**Algorithm A:**

**Step 1:** Compute $\epsilon$

**Step 2:** Determine $\Lambda \subset \nabla$ such that

$$c_6 \inf_{v \in \tilde{S}_\Lambda} \|f - v\|_{-t} < \frac{1}{2}\mu^* \text{eps}.$$

**Step 3:** Compute $u_\Lambda$

**Step 4:** Compute

$$\eta_{\Lambda,\epsilon} := \left( \sum_{\lambda \in N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2 \right)^{1/2}.$$

If $\eta_{\Lambda,\epsilon} <$ eps **Stop**, $\rightarrow \quad u_\Lambda$

**Step 5:** Determine $\tilde{\Lambda}$, $\Lambda \subset \tilde{\Lambda} \subset \nabla$ such that

$$\left( \sum_{\lambda \in \tilde{\Lambda} \cap N_{\Lambda,\epsilon}} a_\lambda(\Lambda,\epsilon)^2 \right)^{1/2} \geq (1-\theta^*)\eta_{\Lambda,\epsilon}$$

Set $\tilde{\Lambda} \rightarrow \Lambda$ and go to **Step 3**.

(I) Choose $\text{eps}_0 > 0$ and $\text{eps}_f = 2^{-L}$

$\text{eps}_0 \rightarrow \text{eps}$.

(II) Apply **Algorithm A**

(III) If $\text{eps} \leq \text{eps}_\text{f}$ **STOP**. Otherwise $\text{eps}/2 \rightarrow \text{eps}$, go to (II)

# PRECONDITIONING

**Assume:** $A$ as before and $\quad t < \gamma, \quad -t < \gamma'$

$$c_1 \|Av\|_{-t} \leq \|v\|_t \leq c_2 \|Av\|_{-t}$$

Sufficiently many **low** frequencies in $\Lambda \quad \rightsquigarrow$

$$\|Q'_\Lambda A v_\Lambda\|_{-t} \sim \|v_\Lambda\|_t, \quad v_\Lambda \in S_\Lambda$$

Define

$$\Sigma_s v := \sum_{j=0}^{\infty} 2^{sj}(Q_j - Q_{j-1})v,$$

Note:

$$\|\Sigma_s v\|_\tau \sim \|v\|_{s+\tau}, \quad s + \tau \in (-\gamma', \gamma).$$

$$\rightsquigarrow$$

$$\begin{aligned} w_\Lambda := \Sigma_t v_\Lambda \quad \rightsquigarrow \quad \|w_\Lambda\|_0 \ &\sim \ \|v_\Lambda\|_t \sim \|Q'_\Lambda A v_\Lambda\|_{-t} \\ &\sim \ \|\Sigma'_{-t} Q'_\Lambda A Q_\Lambda \Sigma_{-t} w_\Lambda\|_0, \end{aligned}$$

$$\rightsquigarrow$$

$$\mathrm{cond}_2(\mathbf{D}_\Lambda^{-t} \mathbf{A}_\Lambda \mathbf{D}_\Lambda^{-t}) \sim 1,$$

# SOLUTION STEP

**Note:** $Q'_{\tilde{\Lambda}\setminus\Lambda} A Q_{\tilde{\Lambda}\setminus\Lambda}$ is **well conditioned** on $S_{\tilde{\Lambda}\setminus\Lambda}$

Solve

$$(Au_{\tilde{\Lambda}}, v) = (f, v), \quad v \in S_{\tilde{\Lambda}}$$

with starting value

$$v_{\tilde{\Lambda}} = \sum_{\lambda \in \tilde{\Lambda}} v_\lambda \psi_\lambda$$

$$v_\lambda := \begin{cases} u_\lambda & \text{if } \lambda \in \Lambda \\ w_\lambda & \text{if } \lambda \in \tilde{\Lambda} \setminus \Lambda \end{cases}$$

where $w_{\tilde{\Lambda}\setminus\Lambda}$ solves

$$Q'_{\tilde{\Lambda}\setminus\Lambda} A w_{\tilde{\Lambda}\setminus\Lambda} = Q'_{\tilde{\Lambda}\setminus\Lambda} f$$