9th International GAMM - Workshop on Parallel Multigrid Methods
(St.Wolfgang/Strobl, Austria, May 13-17, 1996)

Sofronov, I.L.

Jump-Keeping and Upwind Transfer in MultiGrid for Upwind Schemes

Transfer operators in multigrid (MG) are generated, as a rule, on the basis of usual central formulas of weighting and linear interpolation to restriction and prolongation, respectively. This is quite naturally to elliptic problems with smooth coefficients, but not so evidently to hyperbolic equations in gasdynamics, where discontinuous solutions with shocks arise often. Nevertheless, beginning from Jameson's [1] and Ni's [2] pioneer papers, the central transfer operators are successively used in numerous Euler solvers. Probably this fact is explained due to including *artificial viscosity* into schemes like [1], [2] to stabilise shocks. The terms of artificial viscosity are represented by elliptic operators and they essentially influence the convergence properties of a relaxation process. However, for *upwind schemes* where there is no artificial viscosity, the MG procedure with standard prolongation and restriction operators can fail. Koren and Hemker [3], and Leclercq and Stoufflet [4] suggest ways for improving transfer operators in MG by taking into account the upwindness of governing equations. Our own experience in generating and using *upwind transfer operators* has given also good results. Nevertheless, in our opinion, the question about what kind of transfer operators is needed for upwind schemes is still open. For instance, the authors of [3] point to upwind *prolongation* operator as the main reason of improving MG, whereas the authors of [4] improve MG only by using upwind *restriction* operator; we obtain a stable MG procedure with *mixed central-upwind* both *restriction* and *prolongation* operators, see Item 2. Note that central transfer operators have smoothing properties. Besides, all modifications are required only for flow cases with shocks. Therefore the main reason of failing standard MG is likely in the discontinuity of solutions. For instance, an appropriate modification of transfer operators is used for elliptic problems with strongly discontinuous coefficients, see [5]. That is why we develop special transfer operators that treat functions with jumps. Such modification turns out enough to obtain stable relaxation process for transonic flow calculations.

1. Jump-keeping transfer. We consider piecewise-continuous functions in d-dimensional unit parametric cube $[0,1]^d$. Let h and $H = 2h$ be spacings of two nested uniform rectangular grids in our cube. We generate multidimensional transfer operators by the product of one-dimensional ones, i.e., at first the given function is transferred for one parametric direction, then the obtained function is transferred for the next direction, and so on. Let index i run from 0 to $I = 1/h$ in somehow parametric direction with the fixed others. We introduce a shock-indicator function q_i; for transonic flows this function can be defined in the following way:

$$\begin{array}{ll} q_{i-1} := q_i := 1 & \text{if} \left[\begin{array}{l} \{M_i > 1 > M_{i-1}\} \& \{P_{i-1} > P_i + \varepsilon\} \ \text{ or} \\ \{M_i < 1 < M_{i-1}\} \& \{P_{i-1} < P_i - \varepsilon\} \end{array} \right. \\ q_i := 0 & \text{otherwise,} \\ q_0 := q_I := 1 & , \end{array}$$

where M is the current Mach number on the h-grid; P is the pressure; $\varepsilon > 0$ is a given threshold magnitude. The one-dimensional transfer operators for function f on the h-grid and function F on the H-grid are given by the formulas:

$$\begin{array}{ll} \text{restriction:} & F_{i/2} := \alpha_i f_{i-1} + (1-2\alpha_i) f_i + \alpha_i f_{i+1}, \quad \alpha_i = (1-q_i)/4; \\ \text{prolongation:} & f_i := F_{i/2}, \\ (\ i \text{ is even}) & f_{i+1} := \beta_i F_{i/2} + (1-\beta_i) F_{i/2+1}, \quad \beta_i = (q_{i+1}(q_{i+2}-q_i)+1)/2; \end{array}$$

It is easy to see that the generated transfer operators have a one-point stencil for grid points near the shock, and usual central stencil for the rest grid points.

2. Upwind transfer. Just as in the previous Section, we give here one-dimensional formulas because the multidimensional upwind transfer is the product of one-dimensional ones.

Let us write 2D Euler equations in the quasilinear form

$$(JU)_t + \mathrm{A} U_\xi + \mathrm{B} U_\eta = 0, \qquad U \equiv (\rho, u, v, P)^T, \qquad J \text{ is Jacobian of s coordinates transform},$$

and factorize the matrix A: $\mathrm{A} = S^{-1}\Lambda S$. Here ξ, η are parametric variables in the cube $[0,1]^2$; Λ is the diagonal matrix of eigenvalues. Denote by S^+ the matrix with the same rows of matrix S if the corresponding diagonal values of Λ are greater than zero, and zero rows otherwise. Define $S^- = S - S^+$. The one-dimensional upwind transfer in ξ-direction is given by formulas:

$$\begin{array}{ll} \text{restriction:} & F_{i/2} := 0.5 f_i + 0.5 S_i^{-1}(S_i^+ f_{i-1} + S_i^- f_{i+1}); \\ \text{prolongation:} & f_i := F_{i/2}, \\ (\ i \text{ is even}) & f_{i+1} := S_{i+1}^{-1}(S_{i+1}^+ F_{i/2} + S_{i+1}^- F_{i/2+1}); \end{array}$$

The stable relaxation process is obtained in practice when we use a mixed central-upwind transfer:

$$\begin{array}{ll} \text{restriction:} & F_{i/2} := 0.5 F_{i/2}|_{upwind} + 0.125(f_{i-1} + 2f_i + f_{i+1}); \\ \text{prolongation:} & f_{i+1} := 0.5 f_{i+1}|_{upwind} + 0.25(F_{i/2} + F_{i/2+1}); \end{array}$$

3. Numerical results. The test was in transonic flow calculation around RAE2822 airfoil with $M_\infty = 0.75$ and angle of attack $\alpha = 3^o$. To better understand the interaction of MG with an upwind scheme, we extract the inviscid subproblem from the problem of viscous flow, i.e., we calculate Euler solution on the Navier-Stokes grid. The code was developed by Radvogin and Zaitsev on the basis of an upwind high-resolution multidimensional half-implicit difference scheme [6]. Figure 1 shows the convergence history for the lift coefficient on a coarse C-type 193*33 grid with one MG level. MG with the central transfer does not give a final steady-state solution in contrast to MG with

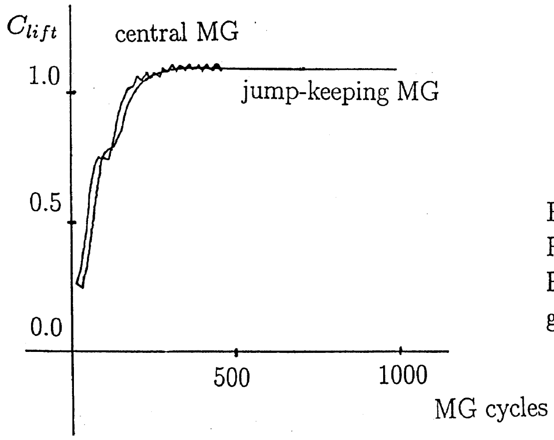

FIGURE 1:
RAE 2822 airfoil, $M_\infty = 0.75, \alpha = 3^o$,
Euler solution on 193*33 Navier–Stokes grid, one MG level

jump-keeping transfer and mixed central-upwind transfer (the curve for the latter is not shown). The calculation on the 385*65 grid with two MG levels gives no solution at all for the central transfer, but gives a stable solution for both modifications of transfer.

4. Parallelization aspects. The proposed operators of jump-keeping and upwind transfer are very suitable for parallelizing because they are defined by the product of correspondent one-dimensional operators.

Acknowledgement: The research was done at Stuttgart University and sponsored by the Alexander von Humboldt Foundation.

References

1 A. Jameson, Solution of the Euler equations for two dimensional transonic flow by a multigrid method, Presented at the International Multigrid Conference, Copper Mountain (1983).
2 R.H. Ni, A Multiple grid scheme for solving the Euler equations, Proc. AIAA 5th Computational Fluid Dynamics Conf., Palo Alto, Paper No. 81-0132 (1981), 257–264.
3 B. Koren, and P.W. Hemker, Damped, direction-dependent multigrid for hypersonic flow computations, Appl. Num. Math., **7** (1991), 309–328.
4 M.P. Leclercq, and B. Stoufflet, Characteristic multigrid method application to solve the Euler equations with unstructured and unnested grids, J. Comput. Phys., **104** (1993), 329–346.
5 R.E. Alcouffe, A. Brandt, J.E. Dendy Jr., and J.W. Painter, The multi-grid method for the diffusion equation with strongly discontinuous coefficients, SIAM J. Sci. Stat. Comput., **2** (1981), 430–454.
6 Yu.B. Radvogin, and N.A. Zaitsev Multidimensional minimal stencil supported second order accurate upwind schemes for solving hyperbolic and Euler systems, Preprint of Keldysh Inst. Appl. Math. RAS. Moscow (1996)

Address: Ivan L. Sofronov, Keldysh Institute of Applied Mathematics RAS, Miusskaya sq. 4, Moscow, 125047, Russia (*permanent*);
Universität Stuttgart, Mathematisches Institut A, LS 6, Postfach 801140, D-70511 Stuttgart (*current*).
E-mail: sofronov@mathematik.uni-stuttgart.de