Efficient Parallel Solving the Potential Flow Problems on Nonmatching Grids

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Abstract. We consider a parallel application of DD algorithm proposed in [8] to approximate solving the fully potential subsonic flow problem on nonmatching grids. A short description of the method and results of numerical experiments on parallel computers SP2 and Paragon are presented.

1 INTRODUCTION

We consider an application of a domain decomposition (DD) method with Lagrange multipliers [8] to parallel solving the potential and fully potential subsonic flow problems. We use a coarse triangulation of the computational domain to split it into subdomains. The grids on subdomains do not match at the interfaces. For this reason one can generate meshes within a subdomain regardless of meshing in neibouring subdomains, i.e. in parallel. Besides, it brings about more flexible parallel mesh adaptation techniques. By using of Lagrange multipliers we reformulate the differential problem in a saddle-point form and discretize it by the mortar finite elements method [2]. To accelerate the convergence of iterative method which solves arising algebraic systems, we use a block diagonal preconditioner that is in some sence [8] spectrally equivalent to the saddle-point matrix and has the optimal order of arithmetical complexity [8]. The preconditioner consists of the blocks for preconditioning subdomain problems and interface problem. These blocks incorporate the algebraic multigrid preconditioner [7] and a global coarse solver associated with the coarse triangulation. In a parallel application each processor deals with data related to a set of subdomains. The applications of the above approach which exploit up to 56 processors of parallel computers SP2 and Paragon have exhibited computational efficiency of the DD-method as well as the efficiency of its parallel implementation.

2 POTENTIAL FLOW PROBLEMS

We consider the flow around two bodies with boundaries S_1 and S_2 in an unbounded domain at constant speed $\vec{u}_\infty$ at infinity. This flow is approximated numerically by that in a domain Ω with an outer boundary Γ_∞ at a finite distance. Thus, Ω is a two-dimensional bounded domain with a Lipschitz-continuous boundary $\Gamma = \Gamma_\infty \cup S_1 \cup S_2$.

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The region of flow is not simply connected. A global potential on Ω cannot exist unless we introduce smooth slits Σ_1, Σ_2 from the profiles to infinity. For convenience let the slit Σ_i touches S_i at fixed point P_i which is identified with the trailing edge of the wing. Hereafter we consider a simply connected domain Ω with a boundary

$$\partial\Omega = S_1 \cup S_2 \cup \Gamma_\infty \cup \Sigma_1 \cup \Sigma_2.$$

We shall deal with the steady, inviscid, irrotational, adiabatic flows of an ideal gas. Due to irrotationality one can introduce the potential φ of the velocity field $\vec{u}$:

$$\vec{u} = \nabla\varphi. \tag{1}$$

The governing equations for incompressible and compressible flows are

$$\Delta\varphi = 0, \tag{2}$$

$$\nabla(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, \tag{3}$$

respectively. Here ρ is a density function

$$\rho(|\nabla\varphi|^2) = \rho_0 \left(1 - \frac{\gamma - 1}{2a_0^2}|\nabla\varphi|^2\right)^{\frac{1}{\gamma-1}},$$

a_0, ρ_0 stand for speed of sound and density in motionless gas, $\gamma = 1.4$ in a standard atmosphere. From now on we scale ρ_0 to unit.

The above equations are equipped with the boundary conditions of no penetration on the profiles

$$\frac{\partial\varphi(x)}{\partial\vec{n}} = 0, \qquad \forall x \in S_1 \cup S_2, \tag{4}$$

and uniformity of the flow at "infinite" boundary

$$g_{cmp}(x) \equiv \rho(|\vec{u}_\infty|^2)\frac{\partial\varphi(x)}{\partial\vec{n}} = \rho(|\vec{u}_\infty|^2)(\vec{u}_\infty \cdot \vec{n}), \tag{5}$$

$$g_{inc}(x) \equiv \frac{\partial\varphi(x)}{\partial\vec{n}} = (\vec{u}_\infty \cdot \vec{n}), \quad \forall x \in \Gamma_\infty, \tag{6}$$

Here $\vec{n}$ is the outward normal to the boundary $\partial\Omega$.

The artificial slits should not influence the flow, i. e. the flow should be continuous at Σ_1, Σ_2. Therefore, we require that φ has a finite constant jump along Σ_i, $i = 1, 2$, i. e.,

$$\varphi|_{\Sigma_i^+} - \varphi|_{\Sigma_i^-} = \beta_i, \quad \frac{\partial\varphi}{\partial\vec{n}}|_{\Sigma_i^+} - \frac{\partial\varphi}{\partial\vec{n}}|_{\Sigma_i^-} = 0. \tag{7}$$

With $\varphi|_{\Sigma_i^+}, \varphi|_{\Sigma_i^-}$ we denote the respective one-sided limits of φ.

In order to close the arising systems of equations we have to add two more equations which impose some limitations on the unknown values β_i. These are the so-called Kutta-Joukowski conditions which require the flow continuity and provide physically correct solutions. For the case of a wing profile it requires the flow to leave the profile at the trailing edge P_i, which is equivalent to the condition

$$\frac{\partial\varphi}{\partial s^+}|_{P_i^+} = \frac{\partial\varphi}{\partial s^-}|_{P_i^-}, \tag{8}$$

where $\frac{\partial\varphi}{\partial s^+}|_{P_i^+}, \frac{\partial\varphi}{\partial s^-}|_{P_i^-}$ are one-sided limits of the tangential derivative of φ at point P_i, $i = 1, 2$.

The method of tackling system (2),(4),(6)-(8) is as follows: first, one computes function φ_0 which is a solution of (2),(4),(6),(7) with $\beta_1 = \beta_2 = 0$, second, one finds φ_1 and φ_2 satisfying (2), uniform Neumann boundary conditions and possessing the unit jump along $\Sigma_1\,(\beta_1 = 1, \beta_2 = 0)$ and $\Sigma_2\,(\beta_1 = 0, \beta_2 = 1)$, respectively. The desired result is a linear combination of $\varphi_0, \varphi_1, \varphi_2$ which satisfies the Kutta-Joukowski condition (8).

For solving nonlinear system (3)-(5),(7),(8) we apply the following algorithm [9]:

– Given an initial guess φ^0 as a solution of the linear problem

– Find a correction ω^{k+1} for φ^k with zero jumps along Σ_1, Σ_2

$$\nabla\rho(|\nabla\varphi^k|^2)\nabla\omega^{k+1} = -\nabla\rho(|\nabla\varphi^k|^2)\nabla\varphi^k,$$

subject to uniform boundary conditions,

– Modify $\omega^{k+1} + \varphi^k \to \varphi^{k+1}$ in order to satisfy (8) taking the advantages of known functions φ_1, φ_2 and the above trick.

For the case of subsonic flows, the method is robust and fast convergent. The method is used as an outer iteration loop, at each step of which one solves a linear problem by inner iterations with prescribed accuracy.

3 THE MORTAR ELEMENT METHOD

Let τ^h be a triangular covering of Ω, i.e. $\tau^h = \{e_i\}$, where e_i is a triangle, $|e_i \cap e_j| = \emptyset$, $i \neq j$. Let suppose that we are able to gather the triangles into groups which specify the subdomains $\bar{\Omega}_k = \bigcup \bar{e}_i$, $k = 1, \ldots, m$, and let the decomposition of Ω is topologically equivalent to some regular coarse triangulation $\Theta^d = \{\Theta_k\}_{k=1}^m$ of Ω, see Fig.1.

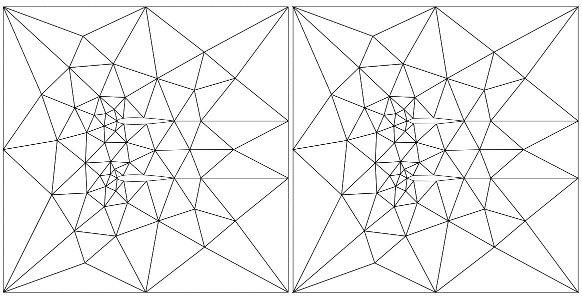

Figure 1. Splitting of computational domain into subdomains $\bar{\Omega} = \bigcup\limits_{k=1}^{m} \bar{\Omega}_k$ and related regular coarse triangulation Θ^d.

Further, we assume that the restriction of τ^h onto Ω_k, $k = 1, \ldots, m$, is a quasi-uniform triangulation with meshstep h_k which is topologically equivalent to a hierarchical grid Θ_k^h on Θ_k. The sets $\Gamma_{kl} = \bar{\Omega}_k \cap \bar{\Omega}_l$, $|\Gamma_{kl}| \neq 0$, are called the interfaces between the subdomains Ω_k and Ω_l. The traces of the grids Ω_{kh} and Ω_{lh} at the interface Γ_{kl} generally do not coincide; such grids are called nonmatching grids.

We denote by V_{kh} the piecewise linear finite element subspace of the space $H^1(\Omega_k)$ defined by the grid Ω_{kh} and by W_{klh} the trace of V_{kh} at each of the interfaces Γ_{kl}. For each of the interfaces $\Gamma_{kl} \equiv \Gamma_{lk}$, $|\Gamma_{kl}| \neq 0$ we define $\Lambda_{klh} \equiv \Lambda_{lkh}$ as a subspace of one of the spaces W_{klh} and W_{lkh} consisting from the mortar elements [2],[8]. These latter distinguish from standard nodal basis functions in W_{klh} by lack of freedom degree at the cross point and the shape of the adjacent to cross point mortar element which is a composition of cross point nodal basis function and its neighbour. We specify the global spaces as a product of local ones:

$$V_h = \prod_{k=1}^{m} V_{kh}, \quad \Lambda_h = \prod_{k=1}^{m} \prod_{l<k, |\Gamma_{kl}| \neq 0} \Lambda_{klh}.$$

Using the above notation and the assumption made, we consider the finite element problems with Lagrange multipliers at the interfaces Γ_{kl}, $l < k$, $k = 1, \ldots, m$, which are counterparts [1],[2],[4] of problems (2),(4),(6)-(8) and (3)-(5),(7),(8), respectively. Find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that u_h satisfies the discrete Kutta-Joukowski conditions [9]

$$\frac{\partial u^h}{\partial s^+}\Big|_{e_i^+} = \frac{\partial u^h}{\partial s^-}\Big|_{e_i^-}, \qquad (9)$$

where the triangles e_i^+, e_i^- are on S_i and have P_i as a vertex $i = 1, 2$, and

$$\begin{aligned} a(u_h, v) + b(\lambda_h, v) &= l(v) \\ b(\mu, u_h) &= l^{\Sigma}(\beta_i, \mu) \end{aligned} \qquad (10)$$

$$\begin{aligned} a^{\rho}(u_h, u_h, v) + b(\lambda_h, v) &= l^{\rho}(v) \\ b(\mu, u_h) &= l^{\Sigma}(\beta_i, \mu) \end{aligned} \qquad (11)$$

for all $(v,\mu)\in V_h\times\Lambda_h$. Here

$$
\begin{aligned}
a(u,v) &= \sum_{k=1}^{m} a_k(u,v),\\
a^\rho(w,u,v) &= \sum_{k=1}^{m} a_k^\rho(w,u,v),\\
b(\lambda,v) &= \sum_{k=1}^{m} \sum_{l=1,|\Gamma_{kl}|\neq 0}^{m} b_{kl}(\lambda,v),\\
l(v) &= \sum_{k=1}^{m} l_k(v),\\
l^\rho(v) &= \sum_{k=1}^{m} l_k^\rho(v),\\
l^\Sigma(\beta_i,\mu) &= \sum_{k=1}^{m} \sum_{l=1,|\Gamma_{kl}|\neq 0}^{m} l_{kl}^\Sigma(\beta_i,\mu),\\
a_k(u,v) &= \int_{\Omega_k} [\nabla u\cdot\nabla v+\varepsilon uv]d\Omega,\\
a_k^\rho(w,u,v) &= \int_{\Omega_k} [\rho(w)\nabla u\cdot\nabla v+\varepsilon uv]d\Omega,\\
l_k(v) &= \int_{\partial\Omega_k\cap\Gamma_\infty} g_{inc} v ds,\\
l_k^\rho(v) &= \int_{\partial\Omega_k\cap\Gamma_\infty} g_{cmp} v ds,\\
b_{kl}(\lambda,v) &= \int_{\Gamma_{kl}} \lambda_{kl} v_k ds,\\
l_{kl}^\Sigma(\beta_i,\mu) &= \int_{\Gamma_{kl}\cap\Sigma_i} \mu_{kl}\beta_i ds,
\end{aligned}
$$

where $\lambda_{kl}=-\lambda_{lk}$ at $l>k$, $\varepsilon>0$ is a small perturbation parameter.

4 DOMAIN DECOMPOSITION METHOD

The above techniques of tackling the finite element problems (10), (11) bring about a problem of solving a system of linear algebraic equations in a saddle-point form:

$$
\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \qquad (12)
$$

or

$$
\begin{bmatrix} A_1 & & & 0 & B_1^T \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ 0 & & & A_m & B_m^T \\ B_1 & \cdot & \cdot & B_m & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_m \\ g \end{bmatrix}
$$

where block representation of matrices A and B associates with the construction of spaces V_h and Λ_h, while matrix A and vectors f, g are specified by bilinear forms $a^\rho(\varphi^k,u,v)$, $a(u,v)$ and functionals $l^\rho(v)$, $l(v)$, $l^\Sigma(\beta_i,\mu)$, respectively.

To solve a linear problem (12) we use the generalized Lanczos method with a preconditioner

$$
\hat{R} = \begin{bmatrix} \hat{R}_A & 0 \\ 0 & \hat{R}_\lambda \end{bmatrix}, \; \hat{R}_A = \begin{bmatrix} \hat{R}_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \hat{R}_m \end{bmatrix}.
$$

Here diagonal blocks $\hat{R}_k$, $k=1,\ldots,m$, incorporate the algebraic multigrid preconditioner $\hat{C}_k$ [7] for the discrete Laplace operator associated with a hierarchical grid Θ_k^h on Θ_k:

$$
\hat{R}_k^{-1} = (I_k-P_k)\hat{C}_k^+(I_k-P_k)+\frac{1}{\varepsilon h_k^2}P_k,
$$

P_k is the orthogonal projector onto a constant vector. Matrix $\hat{C}_k$ is spectrally equivalent to finite element counterpart of the Laplace operator in $\mathrm{Im}(I_k-P_k)$ and multiplication of $\hat{C}_k^+$ by vector is performed as 3-fold W-cycle. This leads to optimal arithmetical complexity for tackling a system with matrix $\hat{R}_A$.

$\hat{R}_\lambda^{-1}$ is an approximate inverse (through inner iterative process with L Chebyshev parameters) of matrix $B\tilde{H}B^T$ which is spectrally equivalent to Schur complement $BA^{-1}B^T$ [8]:

$$\hat{R}_\lambda = B\tilde{H}B^T \left(I_\lambda - \prod_{j=1}^{L} (I_\lambda - \beta_j \tilde{R}_\lambda^{-1} B\tilde{H}B^T) \right)^{-1}$$

$$\begin{aligned} B\tilde{H}B^T &= \sum_{k=1}^{m} B_{\Gamma_k} \Big[\frac{1}{\varepsilon d_k h_k} P_{\Gamma_k} + \\ & (I_{\Gamma_k} - P_{\Gamma_k}) \tilde{T}_k \tilde{C}_k^+ \tilde{T}_k^T (I_{\Gamma_k} - P_{\Gamma_k}) \Big] B_{\Gamma_k}^T \end{aligned}$$

$$\begin{aligned} \tilde{R}_\lambda &= \sum_{k=1}^{m} B_{\Gamma_k} \left((I_{\Gamma_k} - P_{\Gamma_k}) + \frac{1}{\varepsilon d_k h_k} P_{\Gamma_k} \right) B_{\Gamma_k}^T \\ &= \sum_{k=1}^{m} B_{\Gamma_k} B_{\Gamma_k}^T + \sum_{k=1}^{m} \alpha_k B_{\Gamma_k} P_{\Gamma_k} B_{\Gamma_k}^T. \end{aligned}$$

Here $d_k = \operatorname{diam} \Theta_k$, I_{Γ_k} is an identity matrix of order n_{Γ_k}, $B_{\Gamma_k} \in R^{n_{\Gamma_k} \times n_{\Gamma_k}}$ is a nonzero block of B_k associated with nodes on $\Gamma_k = \partial\Theta_k$, n_{Γ_k} is the number of Θ_k^h nodes on Γ_k,

$$P_{\Gamma_k} = \frac{1}{n_{\Gamma_k}} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \in R^{n_{\Gamma_k} \times n_{\Gamma_k}},$$

$\tilde{C}_k \in R^{\tilde{n}_k \times \tilde{n}_k}$ is a preconditioner for the finite element discretization of the Laplace operator $\tilde{A}_k \in R^{\tilde{n}_k \times \tilde{n}_k}$ on a certain sparse mesh $\tilde{\Theta}_k^h$ in a subspace of vectors orthogonal to a constant vector in $R^{\tilde{n}_k}$, $\tilde{T}_k$ is an operator of restriction onto $\partial\Theta_k$

$$\tilde{T}_k = (I_{\Gamma_k} \quad \tilde{O}) \in R^{\tilde{n}_k \times n_{\Gamma_k}}.$$

The traces of Θ_k^h and $\tilde{\Theta}_k^h$ on $\partial\Theta_k$ are coincident, while the number of nodes $\tilde{n}_k$ in $\tilde{\Theta}_k^h$ is proportional to n_{Γ_k}. One can specify $\tilde{\Theta}_k^h$ by the following construction: starting from the coarsest level refine sequentially those triangles which have at least one vertex on $\partial\Theta_k$. Examples of grids $\Theta_k^h, \tilde{\Theta}_k^h$ are shown on Fig.2.

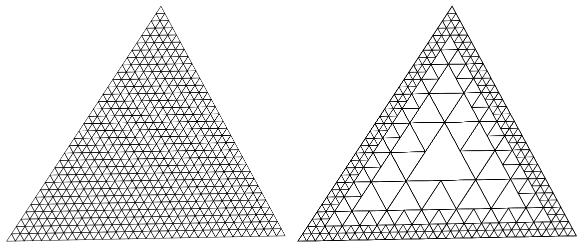

Figure 2. Hierarchical grids used in the preconditioner: uniform Θ_k^h and sparse $\tilde{\Theta}_k^h$.

In our numerical experiments we have chosen $\tilde{C}_k$ to be algebraic multigrid preconditioner for $\tilde{A}_k$ [6, 7] which can be viewed schematically as a 3-fold W-cycle. The arithmetical complexity for $\tilde{C}_k^+$ multiplication by vector is turned out to be equal to $O(n_{\Gamma_k}^{1.58})ops$. Alternative choice is the BPX preconditioner [3, 8] or techniques presented in [5] which lead to arithmetical complexity $O(n_{\Gamma_k})ops$. The usage of operator $\tilde{R}_\lambda$ which is introduced to make condition number of $\tilde{R}_\lambda^{-1} B\tilde{H}B^T$ independent of ε and d_k results in solving a coarse mesh problem associated with an operator $\sum_{k=1}^{m} \alpha_k B_{\Gamma_k} P_{\Gamma_k} B_{\Gamma_k}^T$. However, the condition number of $\tilde{R}_\lambda^{-1} B\tilde{H}B^T$ is still $O\left(\max_k \frac{d_k}{h_k}\right)$, and hence we choose the number of inner iterations L in $\hat{R}_\lambda$ to be proportional to $\left(\max_k \frac{d_k}{h_k}\right)^{\frac{1}{2}}$. It is easy to see that the computational cost of solving a system with $\hat{R}$ is $O\left(\sum_{k=1}^{m} n_k^{1.04}\right) ops$, where n_k be the number of nodes in Ω_k^h. As follows from [8], the convergence rate of Lanczos method is independent of n_k, m, ε.

5 NUMERICAL EXPERIMENTS

In current version of DD algorithm we associate each subdomain Ω_k with triangle Θ_k of certain unstructured mesh. To construct a hierarchical grid in Θ_k we refine uniformly $l-1$ times each "coarse" triangle into 4 equal "finer" ones. The resulting l-levels hierarchical grid is a basis for constructing algebraic multigrid preconditioners $\hat{C}_k$ and $\tilde{C}_k$ for the discrete Laplace operators corresponding to uniform and sparse grids, respectively. To enhance approximation of the curvelinear parts of $\partial\Omega$, we adopt the grid with respect to S_i, moving the middles of triangle sides approximating S_i at each level of refinement onto S_i, $i = 1, 2$. To simulate nonmatching grids, we displace the nodes on interfaces Γ_{kl} randomly shifting them along Γ_{kl} from their natural positions, see Fig.3. The resulting grids Ω_k^h, $k = 1, \ldots, m$, are those which are used for the approximation of the differential problem.

In numerical experiments presented below we use a coarse mesh consisting of m triangles, $m = 56, 111$. On Fig.4 we exhibit the dependency of total execution time per unknown on the number of unknowns $N = \sum_{k=1}^{56} n_k$ for solving potential nonlifting flow problem in the case of a profile NACA0012. One can notice asymptotical optimality of the method. The quality of parallelization is reflected on the second plot of Fig.4 where we show the dependency of the efficiency of parallelization on the number of processors p for the same problem.

Fig.5 depicts isolines of the velocity field and c_p distribution along the profiles for compressible flow around two profiles NACA0012, an angle of attack is chosen to be equal 0, $|\vec{u}_\infty| = 160m/s$, number of levels l in a hierarchical grid equals to 7, $m = 111$.

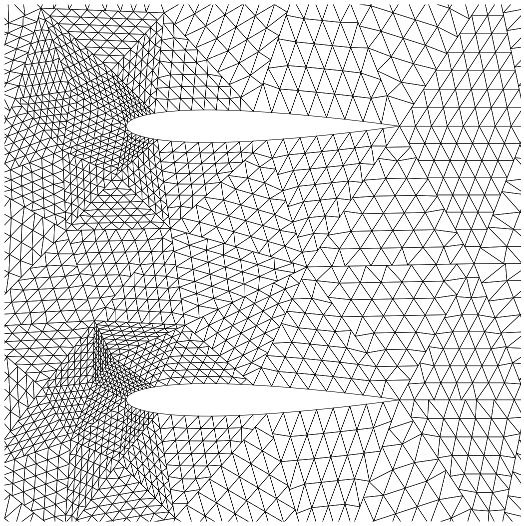

Figure 3. Triangular covering of Ω τ^h.

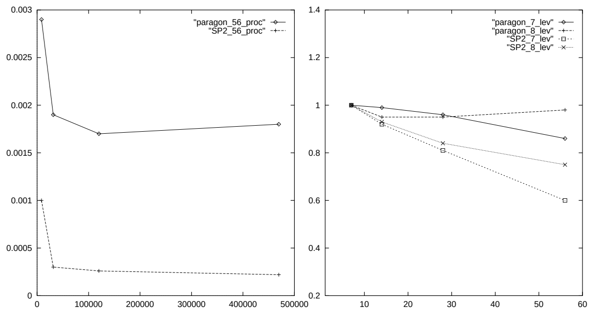

Figure 4. Asymptotical optimality and parallel efficiency.

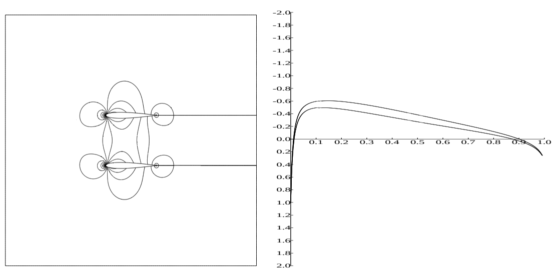

Figure 5. Symmetric compressible flow around two profiles.

In Table 1 we present total execution time for tackling all linear problems to be solved for the cases of incompressible and compressible flows depicted on Fig.5. The stopping criterinon for solving linear problem is the decrease of residual norm in 10^6 whereas the outer iterations terminate when squared residual norm becomes less than 10^{-6}. Besides, we show the weight of time for communications and coarse mesh solver in total execution time. Table 2 summarizes data presented above in terms of efficiency values measured on SP2 and Paragon.

Table 1. Execution time and the weight of communication time in execution one.

	Processors, flow					
	28, incomp		56, incomp		56, comp	
Task	◇	□	◇	□	◇	□
Paragon $\varepsilon = 10^{-3}$	2845	13%	1458	15%	2310	15%
SP2 $\varepsilon = 10^{-3}$	297.6	19%	177.5	34%	356.2	34%

◇ execution time, sec □ communications

Table 2. Efficiency values measured on SP2 and Paragon, $\sum_{k=1}^{111} n_k = 238095$.

		Processors		
Computer	Flow	14	28	56
Paragon	incomp		1	0.98
SP2	incomp	1	0.94	0.79
SP2	comp	1	0.94	0.79

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