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A ROBUST HIERARCHICAL BASIS PRECONDITIONER ON GENERAL MESHES

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Abstract. In this paper, we introduce a multi-level direct sum space decomposition of general, possibly locally refined linear or multi-linear finite element spaces. In contrast to the well-known BPX and hierarchical basis preconditioners, the corresponding additive Schwarz preconditioner will be robust for a class of singularly perturbed elliptic boundary value problems. Important for an efficient implementation is that stable bases of the subspaces defining our decomposition, consisting of functions having small supports can be easily constructed.

1. Background and motivation

This paper deals with additive Schwarz multi-level preconditioners for solving symmetric second order linear elliptic boundary value problems (cf. [Xu92, Yse93, GO95a]). We assume a nested sequence of linear or multi-linear finite element spaces $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_J \subset \ldots$ and discretize the boundary value problem on $\mathcal{M}_J$ using Galerkin's method. Additive Schwarz preconditioners are based on a subspace decomposition $\mathcal{M}_J = \sum_k \mathcal{H}_k$. The corresponding preconditioner can be described by the following steps: For all k, project the right-hand side (or current defect) onto $\mathcal{H}_k$; solve the reduced boundary value problem on $\mathcal{H}_k$ approximately, that is, e.g. apply a point solver as one step of Richardson or Jacobi iteration; finally, sum the obtained approximate solutions from all spaces $\mathcal{H}_k$.

Well-known examples of additive Schwarz preconditioners of multi-level type are the BPX-preconditioner ([BPX90]) and the hierarchical basis preconditioner ([Yse86]). The BPX-preconditioner is based on the overlapping decomposition $\mathcal{M}_J = \sum_{k=0}^{J} \mathcal{M}_k$. Due to the overlap, the subspace solvers should be chosen carefully. With exact subspace solvers, errors in $\cap_{k=0}^{J} \mathcal{M}_k = \mathcal{M}_0$ would be corrected $(J+1)$ times, whereas errors in $\mathcal{M}_J$ that are orthogonal with respect to the energy scalar product to $\mathcal{M}_{J-1}$ are corrected only once. This would result in a (spectral) condition number κ of the preconditioned system that is at least of order $J+1$. In the actual BPX-preconditioner, Richardson iteration is applied as a subspace solver. For Poisson-like problems, this iteration only corrects the oscillating

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("non-overlapping") errors in each subspace, that is, the errors that are not contained in the subspaces corresponding to coarser levels. It can be proved that as a result κ is bounded uniformly in J.

The reasoning that Richardson iteration only corrects oscillating errors is however not applicable for zeroth order problems. This explains why the BPX-preconditioner is not robust for singularly perturbed operators like $-\triangle + \frac{1}{\epsilon}I$ ($\epsilon > 0$). Yet, it can be shown that $\kappa = \mathcal{O}(J+1)$ uniformly in ϵ. Adaptations of the BPX-preconditioner aiming at full robustness for this type of singular perturbation problems are discussed in [Bor92, GO95b].

As we have seen, redundancy in the subspace decomposition can have certain drawbacks. The hierarchical basis preconditioner is based on the direct sum decomposition $\mathcal{M}_J = \bigoplus_{k=0}^J (I_k - I_{k-1})\mathcal{M}_J$, where $I_k : \mathcal{M}_J \to \mathcal{M}_k$ is the nodal value interpolant ($I_{-1} := 0$). A direct sum decomposition yields an optimum preconditioner only if the decomposition is (equivalent to) a decomposition that is orthogonal with respect to the energy scalar product. For Poisson-like problems in more than one dimension, the hierarchical decomposition does not have this property. Yet, assuming exact subspace solvers, it can be proved that $\kappa \sim (J+1)^2$ in two dimensions, so in that case the preconditioner is close to optimal, but in more dimensions κ grows exponentially as a function of J. The reduced boundary value problems on the subspaces are well-conditioned, and so results for exact subspace solvers carry over to the practical preconditioner based on simple point solvers. For zeroth order problems, the hierarchical basis preconditioner gives exponentially growing condition numbers and so clearly this preconditioner is not robust for $-\triangle + \frac{1}{\epsilon}I$.

A promising direct sum decomposition is given by $\mathcal{M}_J = \bigoplus_{k=0}^J \mathcal{W}_k$, where $\mathcal{W}_k = \mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$ ($\mathcal{W}_0 := \mathcal{M}_0$). This decomposition is not only L_2-orthogonal but it is also equivalent to an H^1-orthogonal decomposition (cf. [Osw92, Zha92, DK92, BY93, Xu92, Osw94, Wan94]). Furthermore, since the reduced boundary value problems on the subspaces $\mathcal{H}_k$ appear to be well-conditioned, application of point solvers with respect to some bases of these subspaces would result into an optimum preconditioner for problems like $-\triangle + \frac{1}{\epsilon}I$ even uniformly in ϵ (robustness), in case these bases are uniformly stable.

For one-dimensional linear finite element spaces on a regular grids, such L_2-stable bases are given by a set of prewavelets. Tensor products of these bases give L_2-stable bases of multi-linear finite element spaces on regular grids. However, on general finite element spaces suitable bases are not easily find. For linear finite element spaces in two dimensions on regular grids L_2-stable prewavelet bases are constructed in [KO95] (cf. [Jun94] for the bilinear case). However, each basis function of $\mathcal{W}_k$ is a linear combination of not less than ten nodal basis functions of $\mathcal{M}_k$, which makes the preconditioner expensive to implement.

Subject of this paper is an additive Schwarz multi-level preconditioner based on the direct sum decomposition $\mathcal{M}_J = \bigoplus_{k=0}^J \mathcal{V}_k$, where $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{M}_{k-1}$ ($\mathcal{V}_0 := \mathcal{M}_0$) and $(\cdot,\cdot)_{\mathcal{M}_k}$ is a discrete but L_2-equivalent scalar product on $\mathcal{M}_k$. This decomposition is similar to the L_2-orthogonal decomposition, but in contrast to that decomposition, on general, possibly locally refined finite element spaces, L_2-stable bases of the subspaces $\mathcal{V}_k$ are easily constructed consisting of basis functions that are linear combinations of, in case of linear finite elements only three nodal basis functions on that level. We do not expect that there exists an L_2-stable hierarchical basis of $\mathcal{M}_J$ consisting of basis functions that are linear combinations of a smaller number of nodal basis functions. Note that the spaces $\mathcal{V}_k$ are not mutually orthogonal with respect to one scalar product, which complicates the development of theory.

We will prove that if our decomposition is equivalent to an L_2-orthogonal decomposition then it is also equivalent to an H^1-orthogonal decomposition, which means that then the corresponding preconditioner based on point solvers is optimal for second and zeroth order elliptic problems and thus also that it is robust for problems like $-\triangle+\frac{1}{\epsilon}I$. For model cases of finite element spaces on quasi-uniform two-dimensional meshes, we are able to prove this L_2-stability. Our numerical experiments with locally refined finite element spaces show L_2-, and thus H^1-stability also in that case. We obtain similar results for an example with a discontinuous coefficient, where we apply discrete scalar products that are equivalent to the resulting weighted L_2-scalar product.

Although not subject of this paper, we mention here that our space decomposition also will be useful for solving general H^s-elliptic problems for $s \in [0,1]$ as integral or pseudo-differential equations. We refer to [BCR91, DKPS94, DKPS].

Another direct sum multi-level decomposition close to the L_2-orthogonal one is discussed in [VW95]. In that paper, from each standard hierarchical basis function from $(I_k - I_{k-1})\mathcal{M}_J$ an approximate L_2-orthogonal projection onto $\mathcal{M}_{k-1}$ is subtracted. There is a tradeoff between the accuracy of this approximation and the size of the supports of the resulting basis functions. At least theoretically, to get a uniformly bounded condition number for Poisson-like problems, the approximate L_2-projection has to converge to the exact one if the level tends to infinity.

Anisotropic equations. The idea of a space decomposition based on orthogonality with respect to discrete scalar products was introduced by Hackbusch in the development of the Frequency Decomposition Multi-Level Method ([Hac89, Hac92]) aiming at robustness for anisotropic operators $-\sum_{i=1}^{d} a_i\partial_i^2$, where the a_i are non-negative but differ possibly largely in size. Both the BPX- as the hierarchical basis preconditioner show exponentially growing condition numbers applied to these operators in case of strong anisotropy. A robust BPX-preconditioner can be obtained when the coarser finite element spaces are constructed by means of semi-coarsening in directions guided by the relative sizes of the a_i ([TCJK92, GO95b]). This approach is necessarily restricted to a_i that are essentially constants.

For *multi-linear* finite element tensor product spaces on regular grids, in [Ste94, Ste95] it was proved that our decomposition $\mathcal{M}_J = \bigoplus_{k=0}^{J} \mathcal{V}_k$ is equivalent, uniformly in the size of the anisotropy, to a decomposition that is orthogonal with respect to the energy scalar product related to the anisotropic boundary value problem. However, the reduced boundary value problems on the subspaces $\mathcal{V}_k$ are not uniformly well-conditioned with respect to the anisotropy. So simple point solvers cannot be applied. Therefore, following ideas from [Hac89, Hac92], in [Ste94, Ste95] we made a further direct sum decomposition of each $\mathcal{V}_k$ into $(k+1)^d - k^d$ subspaces, using tensor products and again based on orthogonality with respect to discrete scalar products. Now the reduced boundary value problems on these smaller subspaces can be solved using point solvers yielding a robust preconditioner. A similar method and results were obtained by Griebel and Oswald in [GO95b]. Their preconditioner is based on the L_2-orthogonal decomposition which results into a somewhat less efficient algorithm since the required prewavelet bases have larger masks.

In this paper we consider the question of robustness for anisotropic equations on *linear* finite element spaces in two dimensions. In this case, a decomposition of $\mathcal{V}_k$ as discussed above cannot be made since now the finite element space is not a tensor product space. Instead, we make a direct sum decomposition of $\mathcal{V}_k$ into three subspaces again orthogonal with respect to the discrete scalar product. Although the reduced boundary value problems on these spaces are still not uniformly well-conditioned and thus point solvers cannot be applied, it will turn out that it is sufficient to solve these problems with a line solver. At least for uniform triangulations, the resulting preconditioner will be robust with respect to anisotropy in the three directions of the grid lines.

The remainder of this paper is organized as follows. In Section 2, we discuss general additive Schwarz preconditioners. The direct sum decomposition central in this paper together with various theoretical and practical aspects of the corresponding hierarchical basis preconditioner are treated in Section 3. In Section 4, further direct sum decompositions of the subspaces generating our decomposition are introduced yielding robust preconditioners for anisotropic equations. Numerical examples are given at the end of Sections 3 and 4.

Following [Xu92], we shall use the notations $\lesssim$, $\gtrsim$ and $\eqsim$. When we write

$$x_1 \lesssim y_1, x_2 \gtrsim y_2 \text{ and } x_3 \eqsim y_3,$$

then there exist constants C_1, c_2, c_3 and C_3, that are *independent* of relevant parameters as the level or the value of a singular perturbation parameter, such that

$$x_1 \leq C_1 y_1, x_2 \geq c_2 y_2 \text{ and } c_3 x_3 \leq y_3 \leq C_3 x_3.$$

2. Preliminaries

2.1. Finite element spaces. Throughout this paper, we consider a sequence of nested linear or multi-linear finite dimensional finite element spaces

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_J \subset H_0^1(\Omega),$$

where Ω is a domain in $\mathbf{R}^d$. For use in §3 we impose the following conditions

(T1) The diameter of a underlying finite element of $\mathcal{M}_k$ (a d-simplex or d-parallelogram) that was generated by refinement in the transition from $\mathcal{M}_{k-1}$ to $\mathcal{M}_k$, divided by the diameter of the enclosing finite element of $\mathcal{M}_0$ is equivalent to 2^{-k}.

(T2) Elements that were not refined in the transition from $\mathcal{M}_{k-1}$ to $\mathcal{M}_k$ are never refined further.

(T3) The diameter of an element τ divided by the radius of the largest ball contained in τ is bounded, uniformly in all elements on all levels.

Note that we do not make an assumption whether the subdivision of Ω into finite elements is conforming or non-conforming.

2.2. Additive Schwarz preconditioners. Following presentations from [Xu92, Yse93, GO95a], in this subsection we discuss general additive Schwarz preconditioners. Suppose we have to solve the variational problem find $u \in \mathcal{M}_J$ such that

$$a(u, v) = f(v) \qquad (v \in \mathcal{M}_J), \tag{2.1}$$

where a is a scalar product on $\mathcal{M}_J$ and $\overline{f}$ is a linear functional on $\mathcal{M}_J$.

Let $\mathcal{M}_J = \sum_{k=0}^{N(J)} \mathcal{H}_k$ some space decomposition and, for $0 \leq k \leq N(J) < \infty$, let b_k be an auxiliary scalar product on $\mathcal{H}_k$. These ingredients define the preconditioner which we will introduce in connection to a defect computation, that is, as a defect correction step. In doing so, we avoid introducing an additional scalar product for transferring the variational problem (2.1) into an operator equation, whereas at the same time the discussion of the implementation becomes easier. Given some approximation u^{old} of the solution u, this defect correction step is defined by

$$u^{\text{new}} = u^{\text{old}} + \sum_k u_k, \tag{2.2}$$

where for $0 \leq k \leq N(J)$, u_k is the solution of

$$b_k(u_k, w) = f(w) - a(u^{\text{old}}, w) \qquad (w \in \mathcal{H}_k). \tag{2.3}$$

The use of the scalar products b_k replacing a models the application of inexact solvers on the subspaces $\mathcal{H}_k$.

Defining operators $T_k : \mathcal{M}_J \to \mathcal{H}_k$ by $b_k(T_k u, w) = a(u, w)$ $(w \in \mathcal{H}_k)$, we have $u_k = T_k(u^* - u^{\text{old}})$ where u^* is the solution of (2.1), and thus

$$u^* - u^{\text{new}} = (I - \sum_k T_k)(u^* - u^{\text{old}}).$$

One easily verifies that the preconditioned operator $\sum_k T_k$ is symmetric positive definite (SPD) with respect to $a(\cdot, \cdot)$. We conclude that the convergence speed of the Conjugate Gradient method applied to the preconditioned system is determined in a well-known manner by the spectral condition number $\kappa(\sum_k T_k)$.

Following [GO95a], we define a norm $|\!|\!| \cdot |\!|\!|$ on $\mathcal{M}_J$ by

$$|\!|\!| u |\!|\!|^2 = \inf_{u_k \in \mathcal{H}_k, u = \sum_k u_k} \sum_k b_k(u_k, u_k)$$

and introduce the values

$$\lambda_{\min} = \inf_{0 \neq u \in \mathcal{M}_J} \frac{a(u,u)}{|\!|\!| u |\!|\!|^2}, \quad \lambda_{\max} = \sup_{0 \neq u \in \mathcal{M}_J} \frac{a(u,u)}{|\!|\!| u |\!|\!|^2} \quad \text{and} \quad \kappa(\{\mathcal{M}_J; a\}; \sum_k \{\mathcal{H}_k; b_k\}) = \frac{\lambda_{\max}}{\lambda_{\min}}.$$

It can be proved that

$$\kappa(\sum_k T_k) = \kappa(\{\mathcal{M}_J; a\}; \sum_k \{\mathcal{H}_k; b_k\}) \tag{2.4}$$

(cf. [GO95a] and the references cited there), which reveals the direct relation between the effectiveness of the preconditioner and the stability of the space decomposition.

In case of a *direct sum* space decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{H}_k$ as we will consider in this paper, there holds

$$|\!|\!| \sum_k T_k u |\!|\!|^2 = \sum_k b_k(T_k u, T_k u) = \sum_k a(u, T_k u) = a(\sum_k T_k u, u)$$

and so

$$\lambda_{\min} \leq \frac{a(\sum_k T_k u, \sum_k T_k u)}{a(\sum_k T_k u, u)} \leq \lambda_{\max},$$

where the bounds are attained. By substituting $u = (\sum_k T_k)^{-\frac{1}{2}} w$, we conclude that indeed $\kappa(\sum_k T_k) = \frac{\lambda_{\max}}{\lambda_{\min}}$, which gives an easy proof of (2.4) in this case.

A direct consequence of (2.4) is that for obtaining a qualitative estimate of $\kappa(\sum_k T_k)$ we may replace the scalar products a and b_k by equivalent scalar products, since by doing that, the right-hand side of (2.4) only changes by a constant factor.

Particularly interesting for singularly perturbed boundary value problems is the following: Suppose we have a *direct sum* space decomposition, two scalar products $a^{(1)}$ and $a^{(2)}$ on $\mathcal{M}_J$, and on each space $\mathcal{V}_k$ two auxiliary scalar products $b_k^{(1)}$ and $b_k^{(2)}$. Let us denote the characteristic values of the decomposition using the scalar products $a^{(i)}$ and $b_k^{(i)}$ on $\mathcal{M}_J$ and $\mathcal{V}_k$ respectively by $\lambda_{\max}^{(i)}$ and $\lambda_{\min}^{(i)}$. Then for $\nu, \mu \geq 0$ with $\nu + \mu > 0$, there holds

$$\kappa(\{\mathcal{M}_J; \nu a^{(1)} + \mu a^{(2)}\}; \sum_k \{\mathcal{H}_k; \nu b_k^{(1)} + \mu b_k^{(2)}\}) \leq \frac{\max\{\lambda_{\max}^{(1)}, \lambda_{\max}^{(2)}\}}{\min\{\lambda_{\min}^{(1)}, \lambda_{\min}^{(2)}\}}. \tag{2.5}$$

2.3. Implementation. Let $\{\phi_i\}$ be a basis of $\mathcal{M}_J$. We denote by U the vector representing $u \in \mathcal{M}_J$ with respect to $\{\phi_i\}$. Analogously, by U_k we denote the representation of $u_k \in \mathcal{H}_k$ with respect to some basis $\{\psi_i\}$ of $\mathcal{H}_k$ $(0 \leq k \leq N(J))$. Let p_k be the representation of the trivial embedding $\mathcal{H}_k \to \mathcal{M}_J$ with respect to above bases, that is, $p_k U_k$ is the representation of $u_k \in \mathcal{H}_k \subset \mathcal{M}_J$ with respect to $\{\phi_i\}$. The stiffness matrices $A = A_J$ and $\tilde{A}_k$ are defined by

$$<AU, V>= a(u, v) \qquad (u, v \in \mathcal{M}_J)$$

and

$$<\tilde{A}_k U_k, V_k>= a(u_k, v_k) \qquad (u_k, v_k \in \mathcal{H}_k)$$

respectively, where $<,>$ denotes a Euclidean scalar product. Note that $\tilde{A}_k = r_k A p_k$, where r_k is the matrix adjoint of p_k. With F being the vector such that $<F, V>= f(v)$ $(v \in \mathcal{M}_J)$, we conclude that the representation of the variational problem (2.1) with respect to $\{\phi_i\}$ is given by

$$AU = F. \tag{2.6}$$

Let B_k be the matrix satisfying

$$<B_k U_k, V_k>= b_k(u_k, v_k) \qquad (u_k, v_k \in \mathcal{H}_k).$$

This relation is usually used to *define* b_k by taking B_k some easily invertible approximation of $\tilde{A}_k$. Analogously to (2.6), we find that the representation of (2.3) with respect to $\{\psi_i\}$

is given by

$$B_k U_k = G_k,$$

where G_k is defined by

$$\begin{aligned} <G_k, V_k> &= f(v_k) - a(u^{\text{old}}, v_k) \\ &= <F, p_k V_k> - <AU^{\text{old}}, p_k V_k> \\ &= <r_k(F - AU^{\text{old}}), V_k> \end{aligned}$$

($v_k \in \mathcal{H}_k$), or $G_k = r_k(F - AU^{\text{old}})$. So we have $U_k = B_k^{-1} r_k(F - AU^{\text{old}})$. In order to correct U^{old}, also the function represented by U_k should be represented with respect to $\{\phi_i\}$. We conclude that the implementation of (2.2) is given by

$$U^{\text{new}} = U^{\text{old}} + \sum_k p_k B_k^{-1} r_k (F - AU^{\text{old}}).$$

In our applications, $\mathcal{H}_k$ will be a subspace of $\mathcal{M}_k$. Because of the nesting $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_J$, the embeddings $\mathcal{M}_k \to \mathcal{M}_J$ can be implemented efficiently in a recursive way using the nodal bases on all finite element spaces. To obtain an efficient implementation of the embedding $\mathcal{H}_k \to \mathcal{M}_k$ and thus of the embedding $\mathcal{H}_k \to \mathcal{M}_J$, it is important that the basis functions of $\mathcal{H}_k$ can be expressed as linear combinations of a small number of nodal basis functions from $\mathcal{M}_k$.

3. A new hierarchical basis preconditioner

3.1. The space decomposition. In this section, we replace the general scalar product a by the scalar product

$$D(u, v) := \int_\Omega \varphi \sum_{i=1}^{d} \partial_i u \overline{\partial_i v} dx,$$

where φ is a positive function that is constant on the interiors of the finite elements corresponding to $\mathcal{M}_0$. This function φ has the task to cover jumps of the coefficients of the differential operator across boundaries of the elements on the coarsest level. Let us denote the collection of these elements by Ξ_0. To make our estimates independent of the jumps and the sizes of the $\tau_0 \in \Xi_0$, we consider the weighted L_2-scalar product

$$(u, v)_{L_2} = \sum_{\tau_0 \in \Xi_0} \tfrac{1}{\text{diam}(\tau_0)^2} \int_{\tau_0} \varphi u \overline{v} dx,$$

and define norms $\|\cdot\|_{L_2} = (\cdot,\cdot)_{L_2}^{\frac{1}{2}}$ and $|\cdot|_{H^1} = D(\cdot,\cdot)^{\frac{1}{2}}$. [1]

We start with collecting some well-known tools for analyzing multi-level methods (cf. e.g. [BY93]). On $\mathcal{M}_k$, we have

$$|u|_{H^1} \lesssim 2^k \|u\|_{L_2} \qquad (\textit{inverse inequality}). \tag{3.1}$$

Further, there exists a $\gamma < 1$ such that for $k \geq l$

$$D(u, v) \lesssim \gamma^{k-l} |u|_{H^1} 2^k \|v\|_{L_2} \qquad (u \in \mathcal{M}_l,\ v \in \mathcal{M}_k) \tag{3.2}$$

[1] A large φ or a small $\tau_0 \in \Xi_0$ are different faces of the same problem. If Ω is (linearly) enlarged with a factor $\sqrt{\lambda}$, then a differential operator $-\partial_i^2$ with respect to new coordinates is given by $-\lambda\partial_i^2$.

(*strengthened Cauchy-Schwarz inequality*).

The L_2-orthogonal decomposition $\mathcal{M}_J = \bigoplus_{k=0}^J \mathcal{W}_k$ is defined by $\mathcal{W}_k = \mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$ ($k \geq 1$) and $\mathcal{W}_0 = \mathcal{M}_0$. Let $Q_k : H_0^1(\Omega) \to \mathcal{M}_k$ be the L_2-orthogonal projection onto $\mathcal{M}_k$ ($k \geq 0$) and $Q_{-1} := 0$. Then $Q_k - Q_{k-1} = (I - Q_{k-1})Q_k$ maps onto $\mathcal{W}_k$ and we conclude that $u = \sum_{k=0}^J (Q_k - Q_{k-1})u$ is the unique splitting of $u \in \mathcal{M}_J$ in terms of functions from the spaces $\mathcal{W}_k$. Several authors have proved that

$$|u|_{H^1}^2 \eqsim |Q_0 u|_{H^1}^2 + \sum_{k=1}^J 4^k \|(Q_k - Q_{k-1})u\|_{L_2}^2 \qquad (u \in \mathcal{M}_J), \tag{3.3}$$

that is, $\kappa(\{\mathcal{M}_J; D(,)\}; \{\mathcal{W}_0; D(,)\} + \sum_{k=1}^J \{\mathcal{W}_k; 4^k(,)_{L_2}\}) \lesssim 1$ (cf. [Osw92, Zha92, Xu92, Osw94] for uniform refinements, [Wan94, BY93] for non-constant φ, and [DK92, BY93] for non-uniform refinements assuming additionally to (T1)-(T3) some standard conditions on the refinement procedure that prevent neighboring elements to differ more than a constant factor in size). Note that (3.1) and (3.3) imply that for $k \geq 1$ and $u \in \mathcal{W}_k$

$$2^k \|u\|_{L_2} \eqsim |u|_{H^1}. \tag{3.4}$$

Our hierarchical basis preconditioner is based on a space decomposition very similar to the L_2-orthogonal decomposition stated above. Let Ω_k be the set of (free-) nodal points belonging to $\mathcal{M}_k$. We equip $\mathcal{M}_k$ with a discrete scalar product

$$(u, v)_{\mathcal{M}_k} = \sum_{x \in \Omega_k} w_{k,x} u(x) \overline{v(x)} \tag{3.5}$$

and norm $\|u\|_{\mathcal{M}_k} = (\cdot, \cdot)_{\mathcal{M}_k}^{\frac{1}{2}}$, with weights $w_{k,x}$ such that

$$\| \cdot \|_{\mathcal{M}_k} \eqsim \| \cdot \|_{L_2} \tag{3.6}$$

(uniformly in k).

Using that $\frac{\varphi}{\operatorname{diam}(\tau_0)^2}$ is constant on each finite element τ underlying $\mathcal{M}_k$, an obvious approach for constructing $(\cdot, \cdot)_{\mathcal{M}_k}$ is to apply the quadrature formula exact on $\mathcal{M}_k|_\tau$

$$\int_\tau \frac{\varphi}{\operatorname{diam}(\tau_0)^2} u dx = \frac{\varphi \cdot \operatorname{vol}(\tau)}{\operatorname{diam}(\tau_0)^2} \sum_{i=1}^n u(x_i),$$

where $x_1, \ldots, x_n$ are the vertices of τ. By summing over the elements τ, we get a quadrature formula exact on $\mathcal{M}_k$. Possible resulting weights belonging to slave nodes can then be distributed over the free nodes on that side, since by definition of a slave node the value of $u \in \mathcal{M}_k$ in that node is some (convex) linear combination of the values in the free nodes. The resulting $(\cdot, \cdot)_{\mathcal{M}_k}$ satisfies (3.6) (even without (T3)).

Now we define $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{M}_{k-1}$ ($k \geq 1$) and $\mathcal{V}_0 = \mathcal{M}_0$. Then the space decomposition that underlies our hierarchical basis preconditioner is given by

$$\mathcal{M}_J = \bigoplus_{k=0}^J \mathcal{V}_k.$$

In contrast to the L_2-orthogonal decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{W}_k$, in the next subsection we will see that for general, possibly locally refined meshes, stable and local bases of the spaces $\mathcal{V}_k$ can easily be constructed. Moreover, also in model cases for which such bases of the spaces $\mathcal{W}_k$ are known, the basis functions of $\mathcal{V}_k$ can be expressed as a linear combination of a considerably smaller number of nodal basis functions than the basis functions of $\mathcal{W}_k$. As pointed out in §2.3, this will result in a more efficient implementation.

Let $Y_k : \mathcal{M}_{k+1} \to \mathcal{M}_k$ be the $(\cdot,\cdot)_{\mathcal{M}_{k+1}}$-orthogonal decomposition onto $\mathcal{M}_k$ ($k \geq 0$) and $Y_{-1} := 0$. For $0 \leq k \leq J$, on $\mathcal{M}_J$ we define projections

$$Z_k \ (= Z_k^{(J)}) \ = (I - Y_{k-1})Y_k \cdots Y_{J-1}. \tag{3.7}$$

It is easily verified that Z_k maps onto $\mathcal{V}_k$, $\sum_{k=0}^J Z_k = I$ and so $u = \sum_{k=0}^J Z_k u$ is the unique splitting of $u \in \mathcal{M}_J$ in terms of functions from the spaces $\mathcal{V}_k$. We note here that the mappings Z_k are important for theory, but that they do not enter into the implementation of the preconditioner that is discussed in §2.3 and also at the end of §3.2.

We impose the condition

$$\|Z_k\|_{L_2 \leftarrow L_2} \lesssim 1 \tag{3.8}$$

(uniformly in $k \leq J$). One might think that the property (3.6) already implies (3.8), but unfortunately this is generally not true. Until now, we were able to verify (3.8) in model cases discussed in the next subsection concerning quasi-uniform meshes and $\varphi \equiv 1$. In fact, for these cases we prove the stronger result

$$\|u\|_{L_2}^2 \eqsim \sum_{k=0}^J \|Z_k u\|_{L_2}^2 \qquad (u \in \mathcal{M}_J). \tag{3.9}$$

Yet, our numerical experiments discussed in §3.5 indicate that (3.9) is also valid in local refinement cases and in cases where φ has large jumps.

Assuming (3.3) and (3.8), we can prove the following analogue of (3.3) for our space decomposition.

Theorem 3.1. *There holds*

$$|u|_{H^1}^2 \eqsim |Z_0 u|_{H^1}^2 + \sum_{k=1}^J 4^k \|Z_k u\|_{L_2}^2 \qquad (u \in \mathcal{M}_J).$$

Proof. (Based on [Xu92, Appendix].) To avoid treating the term $|Z_0 u|_{H^1}^2$ every time separately, we start with showing that for $k \geq 1$ and $u \in \mathcal{V}_k$,

$$2^k \|u\|_{L_2} \eqsim |u|_{H^1}, \tag{3.10}$$

so that $|Z_0 u|_{H^1}^2 + \sum_{k=1}^J 4^k \|Z_k u\|_{L_2}^2 \eqsim \sum_{k=0}^J |Z_k u|_{H^1}^2$. Let $k \geq 1$. By (3.4), we have that on $\mathcal{V}_k$,

$$\|u\|_{L_2} \eqsim \|u\|_{\mathcal{M}_k} = \|(I - Y_{k-1})u\|_{\mathcal{M}_k} \leq \|(I - Q_{k-1})u\|_{\mathcal{M}_k} \eqsim \|(I - Q_{k-1})u\|_{L_2} \lesssim 2^{-k}|u|_{H^1}.$$

This result together with (3.1) shows (3.10).

Let $u \in \mathcal{M}_J$ and put $u_l = (Q_l - Q_{l-1})u$. Then $u_l \in \mathcal{M}_l$ and so for $l < k \leq J$, we have $Z_k u_l = 0$. For $k \leq l \leq J$, it follows from (3.1), (3.8) and (3.4) that

$$|Z_k u_l|_{H^1} \lesssim 2^k \|Z_k u_l\|_{L_2} \lesssim 2^k \|u_l\|_{L_2} \lesssim 2^{k-l} |u_l|_{H^1}.$$

Writing $u = \sum_{k=0}^{J} u_l$, we now get that

$$\begin{aligned}
\sum_{k=0}^{J} |Z_k u|_{H^1}^2 &= \sum_{k=0}^{J}\sum_{l,m=k}^{J} D(Z_k u_l, Z_k u_m) = \sum_{l,m=0}^{J}\sum_{k=0}^{\min\{l,m\}} D(Z_k u_l, Z_k u_m) \\
&\lesssim \sum_{l,m=0}^{J}\sum_{k=0}^{\min\{l,m\}} 2^{2k-l-m}|u_l|_{H^1}|u_m|_{H^1} \\
&\lesssim \sum_{l,m=0}^{J} 2^{2\min\{l,m\}-l-m}|u_l|_{H^1}|u_m|_{H^1} \\
&\lesssim \sum_{l=0}^{J} |u_l|_{H^1}^2 \lesssim |u|_{H^1}^2
\end{aligned}$$

by (3.3).

On the other hand, from (3.2) and (3.10), we obtain

$$|u|_{H^1}^2 = \sum_{k,l=0}^{J} D(Z_k u, Z_l u) \lesssim \sum_{k,l=0}^{J} \gamma^{|k-l|}|Z_k u|_{H^1}^2 |Z_l u|_{H^1}^2 \lesssim \sum_{k=0}^{J} |Z_k u|_{H^1}^2$$

which completes the proof. □

3.2. Bases. In this subsection, we select bases of the spaces $\mathcal{M}_J$ and $\mathcal{V}_k$. The union over k of the bases of the spaces $\mathcal{V}_k$ will also be a basis of $\mathcal{M}_J$, which is in particular a *hierarchical basis.*

For $0 \leq k \leq J$, we equip $\mathcal{M}_k$ with nodal basis $\{\phi_{k,x} : x \in \Omega_k\}$, scaled in such a way that it is *orthonormal* with respect to $(\cdot,\cdot)_{\mathcal{M}_k}$. The space $\mathcal{V}_0$ equals $\mathcal{M}_0$, and we equip both spaces with the same basis. For $1 \leq k \leq J$, we equip $\mathcal{V}_k$ with basis

$$\{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\}, \tag{3.11}$$

where

$$\psi_{k,x} = \phi_{k,x} - \sum_{y \in \Omega_{k-1}} \frac{(\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k}}{(\phi_{k,y}, \phi_{k-1,y})_{\mathcal{M}_k}} \phi_{k,y}. \tag{3.12}$$

Indeed, from $\psi_{k,x}(z) = \begin{cases} \phi_{k,x}(x) & \text{if } z = x \\ 0 & \text{if } x \neq z \in \Omega_k \backslash \Omega_{k-1} \end{cases}$, we see that (3.11) is an independent set, whereas using $(\phi_{k,y}, \phi_{k-1,z})_{\mathcal{M}_k} = 0$ for $y, z \in \Omega_{k-1}$ and $y \neq z$, one may check that $\psi_{k,x} \in \mathcal{V}_k$. Note that e.g. for linear finite elements the sum over $y \in \Omega_{k-1}$ in (3.12) has only (at most) two non-zero terms, that is, in that case $\psi_{k,x}$ can be expressed as a linear combination of (at most) three nodal basis functions of $\mathcal{M}_k$. See Figure 1 for a two-dimensional example. Representations of elements of $\mathcal{M}_k$ or $\mathcal{V}_k$ ($k > 0$) with respect to above bases will be most naturally identified with grid functions on Ω_k or $\Omega_k \backslash \Omega_{k-1}$ respectively.

Proposition 3.2. *The basis of $\mathcal{V}_k$ defined in (3.11,3.12) ($k > 0$) is uniformly L_2-stable (Riesz-basis).*

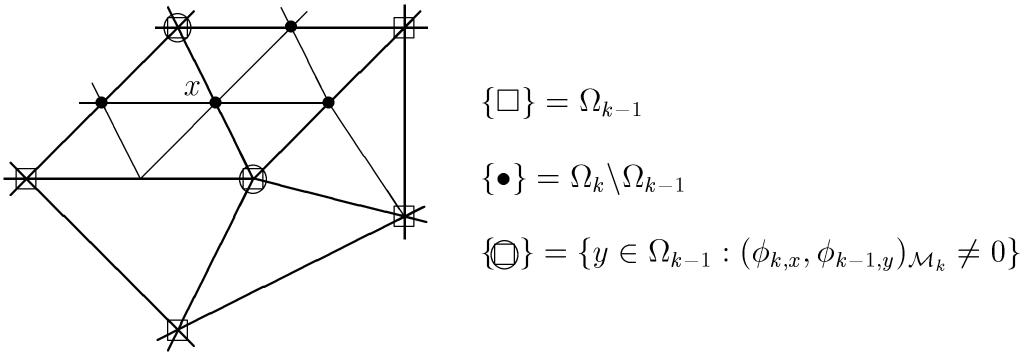


FIGURE 1. $y \in \Omega_{k-1}$ for which $(\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} \neq 0$

Proof. The stability of the basis of $\mathcal{V}_0$ follows from (3.6).

Now let $k \geq 1$. For an underlying finite element τ_k of $\mathcal{M}_k$, let $\tau_{k-1}[\tau_k]$ be the enclosing element of $\mathcal{M}_{k-1}$. The conditions (T1)-(T3) imply that

$$\frac{\mathrm{vol}(\tau_k)}{\mathrm{vol}(\tau_{k-1}[\tau_k])} \gtrsim 1. \tag{3.13}$$

For $y \in \Omega_k$, define $\delta_{k,y} \in \mathcal{M}_k$ by $\delta_{k,y}(x) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \in \Omega_k \end{cases}$. From (3.13), we have $\frac{\|\delta_{k,y}\|_{L^2}}{\|\delta_{k-1,y}\|_{L^2}} \gtrsim 1$ $(y \in \Omega_{k-1})$. Since $\phi_{k,y} = \frac{\delta_{k,y}}{\|\delta_{k,y}\|_{\mathcal{M}_k}}$ and $\|\delta_{k,y}\|_{\mathcal{M}_k} = \sqrt{w_{k,y}}$ (cf. (3.5)), we conclude using (3.6) that for $y \in \Omega_{k-1}$,

(3.14)
$$(\phi_{k,y}, \phi_{k-1,y})_{\mathcal{M}_k} = \frac{w_{k,y}}{\|\delta_{k,y}\|_{\mathcal{M}_k} \|\delta_{k-1,y}\|_{\mathcal{M}_{k-1}}} = \frac{\|\delta_{k,y}\|_{\mathcal{M}_k}}{\|\delta_{k-1,y}\|_{\mathcal{M}_{k-1}}} \eqsim \frac{\|\delta_{k,y}\|_{L_2}}{\|\delta_{k-1,y}\|_{L_2}} \gtrsim 1.$$

Let $u_{k-1} = \sum_{y \in \Omega_{k-1}} c_{k-1,y} \phi_{k-1,y}$ be a function from $\mathcal{M}_{k-1}$. From (3.6) and by writing $\phi_{k-1,y} = \sum_{x \in \Omega_k} (\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} \phi_{k,x}$, we get

$$\begin{aligned} \sum_{y \in \Omega_{k-1}} |c_{k-1,y}|^2 &= (u_{k-1}, u_{k-1})_{\mathcal{M}_{k-1}} \\ &\eqsim (u_{k-1}, u_{k-1})_{\mathcal{M}_k} \\ &= \sum_{x \in \Omega_k} | \sum_{y \in \Omega_{k-1}} (\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} c_{k-1,y}|^2 \\ &\geq \sum_{x \in \Omega_k \backslash \Omega_{k-1}} | \sum_{y \in \Omega_{k-1}} (\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} c_{k-1,y}|^2. \end{aligned} \tag{3.15}$$

Finally, let $v_k = \sum_{x \in \Omega_k \backslash \Omega_{k-1}} d_{k,x} \psi_{k,x}$ be a function from $\mathcal{V}_k$. From (3.6) and the definition of $\psi_{k,x}$ we have

$$\|v_k\|_{L_2}^2 \eqsim \|v_k\|_{\mathcal{M}_k}^2 = \sum_{x \in \Omega_k \backslash \Omega_{k-1}} |d_{k,x}|^2 + \sum_{y \in \Omega_{k-1}} | \sum_{x \in \Omega_k \backslash \Omega_{k-1}} (\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} e_{k,x}|^2,$$

where $e_{k,x} = \frac{d_{k,x}}{(\phi_{k,y},\phi_{k-1,y})_{\mathcal{M}_k}}$. Clearly, there holds $\|v_k\|_{L_2}^2 \gtrsim \sum_{x\in\Omega_k\setminus\Omega_{k-1}} |d_{k,x}|^2$. When we equip coordinate spaces with Euclidean scalar products, the mapping from $(e_{k,x})_{x\in\Omega_k\setminus\Omega_{k-1}}$ to $(\sum_{x\in\Omega_k\setminus\Omega_{k-1}}(\phi_{k,x},\phi_{k-1,y})_{\mathcal{M}_k} e_{k,x})_{y\in\Omega_{k-1}}$ is just the adjoint of the operator that was shown to be bounded in (3.15), and thus it is bounded as well. From this and (3.14) we conclude that

$$\sum_{y\in\Omega_{k-1}} |\sum_{x\in\Omega_k\setminus\Omega_{k-1}} (\phi_{k,x},\phi_{k-1,y})_{\mathcal{M}_k} e_{k,x}|^2 \lesssim \sum_{x\in\Omega_k\setminus\Omega_{k-1}} |e_{k,x}|^2 \lesssim \sum_{x\in\Omega_k\setminus\Omega_{k-1}} |d_{k,x}|^2$$

and thus that $\|v_k\|_{L_2}^2 \lesssim \sum_{x\in\Omega_k\setminus\Omega_{k-1}} |d_{k,x}|^2$. □

3.3. The preconditioner. Suppose the scalar product a is a non-negative linear combination of scalar products $a^{(1)}$ and $a^{(2)}$,

$$a = \nu a^{(1)} + \mu a^{(2)},$$

where $a^{(1)} \eqsim D$ and $a^{(2)} \eqsim (\cdot,\cdot)_{L_2}$. Theorem 3.1 and (3.9) show that

$$\kappa(\{\mathcal{M}_J; a\}; \sum_k \{\mathcal{V}_k; a\}) \lesssim 1$$

uniformly in $\nu+\mu>0$, $\nu,\mu\geq 0$ (and J) (use (2.5)). [Clearly, if instead of (3.9) only (3.8) holds, then this result is still valid for $\mu = 0$.]

Let $\tilde{A}_k$ be the stiffness matrix of a on $\mathcal{V}_k$, that is,

$$<\tilde{A}_k U_k, V_k> = a(u_k, v_k),$$

where U_k (V_k) is the representation of u_k (v_k) with respect to the basis of $\mathcal{V}_k$. From (3.10) and Proposition 3.2, for $k>0$ we have

$$\tilde{A}_k \eqsim \nu 4^k I + \mu I,$$

and thus $\tilde{A}_k \eqsim \rho(\tilde{A}_k)I$. Finally, for $k>0$ defining $b_k(u_k,v_k) =< B_k U_k, V_k >$, with B_k satisfying $B_k \eqsim \rho(\tilde{A}_k)I$, e.g. $B_k = \mathrm{diag}(\tilde{A}_k)$ (Jacobi iteration) or $B_k = \rho(\tilde{A}_k)I$ (Richardson iteration), we obtain $b_k \eqsim a$ on $\mathcal{V}_k$ and so

$$\kappa(\{\mathcal{M}_J; a\}; \{\mathcal{V}_0; a\} + \sum_{k=1}^J \{\mathcal{V}_k; b_k\}) \lesssim 1.$$

We denote the representations of the trivial embeddings $\mathcal{M}_{k-1} \to \mathcal{M}_k$ and $\mathcal{V}_k \to \mathcal{M}_k$ with respect to nodal bases on $\mathcal{M}_{k-1}$ and $\mathcal{M}_k$ and the basis (3.11) on $\mathcal{V}_k$ by p^0 and p^1 respectively. The operator p^0 maps grid functions on Ω_{k-1} to grid functions on Ω_k whereas p^1 maps grid functions on $\Omega_k\setminus\Omega_{k-1}$ to grid functions on Ω_k. Apart from possible different scalings, the operator p^0 is the familiar multi-grid prolongation, whereas e.g. for linear finite elements p^1 is given by a three-point stencil. We note that since the nodal basis on $\mathcal{M}_k$ is orthonormal with respect to $(\cdot,\cdot)_{\mathcal{M}_k}$, the relation $\mathcal{M}_{k-1} \oplus^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{V}_k = \mathcal{M}_k$ is equivalent to

$\operatorname{Im} p^0 \perp \operatorname{Im} p^1$ with respect to the Euclidean scalar product. For the representation p_k of the trivial embedding $\mathcal{V}_k \to \mathcal{M}_J$, we have

$$p_k = \begin{cases} \underbrace{p^0 \cdots p^0}_{(J-k)\times} p^1 & \text{if } 0 < k \leq J \\ \underbrace{p^0 \cdots p^0}_{J\times} & \text{if } k = 0 \end{cases}.$$

Denoting by A_k the stiffness matrix of a on $\mathcal{M}_k$, for $k > 0$ there holds

$$A_{k-1} = r^0 A_k p^0, \quad \tilde{A}_k (= r_k A_J p_k) = r^1 A_k p^1$$

($\tilde{A}_0 = A_0$), where r^0 and r^1 are matrix adjoints of p^0 and p^1 respectively. Using these formulas, necessary stiffness matrices can be computed efficiently and we obtain the following implementation of our preconditioner:

Algorithm 3.3.

function HB(J,F)
  begin
    if $J = 0$ **then** $U = A_0^{-1} F$;
    else
      $U = p^1 B_J^{-1} r^1 F + p^0 \mathrm{HB}(J-1, r^0 F)$;
    return U;
  end

The only difference with the classical hierarchical basis preconditioner is the choice of p^1 (and as a consequence that of r^1, A_0 and the B_k's). For the classical hierarchical preconditioner, p^1 is just the trivial injection from the grid $\Omega_k \backslash \Omega_{k-1}$ to Ω_k.

3.4. Examples. In this subsection, we give examples of finite element spaces with discrete scalar products for which we were able to prove $\|u\|_{L_2}^2 \eqsim \sum_{k=0}^J \|Z_k u\|_{L_2}^2$ $(u \in \mathcal{M}_J)$ ((3.9)) implying $\|Z_k\|_{L_2 \leftarrow L_2} \lesssim 1$ ((3.8)), which latter result was used as a condition for Theorem 3.1.

Example 3.4. In this example, $\mathcal{M}_k \subset H_0^1((0,1)^d)$ is the multi-linear finite element space based on the subdivision of $[0,1]^d$ into $\{(\alpha + [0,1]^d)h_k : 0 \leq \alpha_i < h_k^{-1}, \alpha \in \mathbf{N}^d\}$, where $h_k = 2^{-(k+1)}$. Taking $\varphi \equiv 1$ and following the general construction of $(\cdot,\cdot)_{\mathcal{M}_k}$ explained below formula (3.6), we define $(u,v)_{\mathcal{M}_k} = h_k^d \sum_{x \in \Omega_k} u(x)\overline{v(x)}$, where $\Omega_k = h_k \mathbf{N}^d \cap (0,1)^d$ is the set of nodal points. The choice of $(\cdot,\cdot)_{\mathcal{M}_k}$ determines the decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{V}_k$ as well as the mappings p^0 and p^1 introduced at the end of the previous subsection. In Figure 2, p^0 and p^1 are represented in stencil notation for $d = 2$. These stencils show the representations of basis functions of $\mathcal{M}_{k-1}$ or $\mathcal{V}_k$ in terms of basis functions of $\mathcal{M}_k$.

For $d = 1$, let us denote $\mathcal{M}_k$, $\mathcal{V}_k$ and $(\cdot,\cdot)_{\mathcal{M}_k}$ by $\mathcal{M}_k^{(1)}$, $\mathcal{V}_k^{(1)}$ and $(\cdot,\cdot)_{\mathcal{M}_k^{(1)}}$ respectively. Then for the general case, we have $\mathcal{M}_k = \bigotimes_{j=1}^d \mathcal{M}_k^{(1)}$, $(\bigotimes_{j=1}^d u_j, \bigotimes_{j=1}^d v_j)_{\mathcal{M}_k} =$

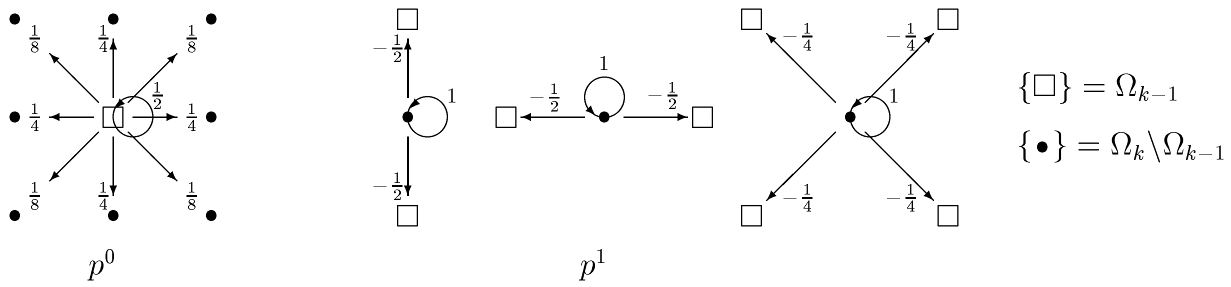


FIGURE 2. p^0 and p^1 in case of *bilinear* finite elements on a regular grid.

$\prod_{j=1}^{d}(u_j, v_j)_{\mathcal{M}_k^{(1)}}$, and one may check that as a consequence

$$\mathcal{V}_k = \bigoplus_{\alpha \in \{0,\dots,k\}^d \backslash \{0,\dots,k-1\}^d} \mathcal{V}_\alpha,$$

where $\mathcal{V}_\alpha = \bigotimes_{j=1}^{d} \mathcal{V}_{\alpha_j}^{(1)}$ (cf. also Figure 6 for the case $d = 2$). In [Ste94, §3.2] and [Ste95, §4] it was proved that the decomposition $\mathcal{M}_J = \bigoplus_{\alpha \in \{0,\dots,J\}^d} \mathcal{V}_\alpha$ of $\mathcal{M}_J$ into $(J+1)^d$ subspaces is L_2-stable, that is, $\kappa(\{\mathcal{M}_J; (\cdot,\cdot)_{L_2}\}; \sum_{\alpha \in \{0,\dots,J\}^d} \{\mathcal{V}_\alpha; (\cdot,\cdot)_{L_2}\}) \lesssim 1$. The clustering lemma stated below says that stability of a direct sum decomposition cannot be lost by clustering several subspaces together. We conclude that (3.9) is valid.

Lemma 3.5 is a special case of a result mentioned in [GO95a, §5].

Lemma 3.5. *Let $\mathcal{M}_J = \bigoplus_{k=0}^{N(J)} \sum_{l=0}^{M(k)} \mathcal{H}_{k,l}$ be a decomposition of $\mathcal{M}_J$ into $\sum_{k=0}^{N(J)} (M(k)+1)$ subspaces and let a be a scalar product on $\mathcal{M}_J$. Then for the space decomposition obtained by for each k clustering the spaces $\mathcal{H}_{k,0}, \dots, \mathcal{H}_{k,M(k)}$, we have*

$$\kappa(\{\mathcal{M}_J; a\}; \sum_{k=0}^{N(J)} \{ \sum_{l=0}^{M(k)} \mathcal{H}_{k,l}; a\}) \leq \kappa(\{\mathcal{M}_J; a\}; \sum_{k=0}^{N(J)} \sum_{l=0}^{M(k)} \{\mathcal{H}_{k,l}; a\})^2.$$

Proof. Put $\mathcal{H}_k = \sum_{l=0}^{M(k)} \mathcal{H}_{k,l}$. From the fact that $\mathcal{M}_J$ is a *direct* sum of the spaces $\mathcal{H}_0, \dots, \mathcal{H}_{N(J)}$, it follows that for each k the characteristic values $\lambda_{\min}^{(k)}$, $\lambda_{\max}^{(k)}$ of the decomposition $\mathcal{H}_k = \sum_l \mathcal{H}_{k,l}$ are in the interval $[\lambda_{\min}, \lambda_{\max}]$, where $\lambda_{\min}$, $\lambda_{\max}$ are the characteristic values of the decomposition $\mathcal{M}_J = \bigoplus_k \sum_l \mathcal{H}_{k,l}$. The remainder of the proof follows easily from the definitions. ∎

Example 3.6. Let Ξ_0 be a finite collection of triangles of type $\alpha + \{(x,y) \in [0, \frac{1}{2}]^2 : y \leq x\}$ or $\alpha + \{(x,y) \in [0, \frac{1}{2}]^2 : y \geq x\}$ where $\alpha \in \frac{1}{2}\mathbf{Z}^2$. Let $\Omega \subset \mathbf{R}^2$ be the interior of $\cup_{\tau \in \Xi_0} \tau$, where we assume that Ω is connected and contains at least one vertex of some $\tau \in \Xi_0$. For $k > 0$, we define Ξ_k to be the collection of triangles that arises by breaking each $\tau \in \Xi_{k-1}$ into four congruent triangles. For $k \geq 0$, $\mathcal{M}_k \subset H_0^1(\Omega)$ is the linear finite element space corresponding to Ξ_k.

Again taking $\varphi \equiv 1$ and following the general construction of $(\cdot,\cdot)_{\mathcal{M}_k}$ proposed below formula (3.6), we define $(u,v)_{\mathcal{M}_k} = h_k^2 \sum_{x \in \Omega_k} u(x)\overline{v(x)}$, where $\Omega_k = h_k \mathbf{Z}^2 \cap \Omega$ with $h_k = 2^{-(k+1)}$ is the set of nodal points. The resulting p^0 and p^1 are represented in Figure 3.

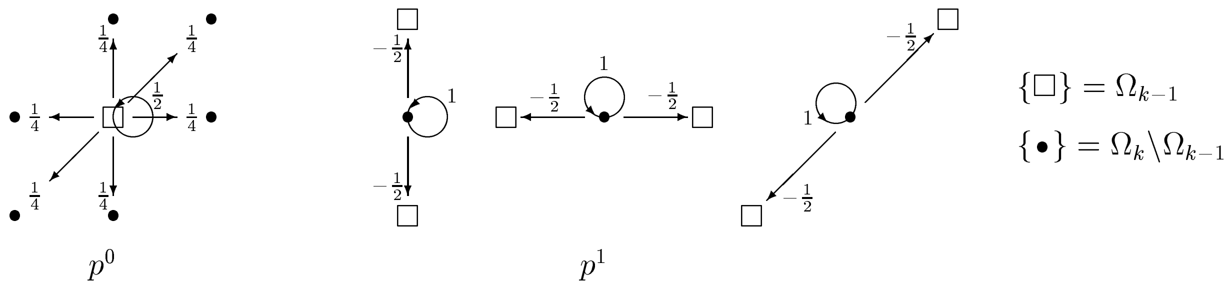


FIGURE 3. p^0 and p^1 in case of *linear* finite elements on a regular grid.

For comparison, we note that in [KO95] for this example stable bases of the subspaces of the L_2-orthogonal decomposition have been constructed consisting of basis functions that are linear combinations of ten nodal basis functions. Moreover, there it was argued that a number smaller than ten cannot be expected.

For a proof of (3.9) in this case we refer to Appendix A.

The same decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{V}_k$ and bases on the spaces $\mathcal{V}_k$ as in this example were used by Dahmen, Kleemann, Prössdorf and Schneider in [DKPS]. Their starting point was not the orthogonality of spaces with respect to discrete scalar products, but the so-called moment conditions; for "interior" $x \in \Omega_k \backslash \Omega_{k-1}$, the basis functions $\psi_{k,x}$ (cf. (3.12)) appear to be L_2-orthogonal to polynomials of degree less or equal to one. As a consequence, with respect to the hierarchical basis of $\mathcal{M}_J$ defined as the union of the bases of the $\mathcal{V}_k$, the entries of the stiffness matrices of certain integral and pseudo-differential operators show a fast decay away from the diagonal (see [DKPS, DKPS94]). Therefore, these (dense) matrices can be well approximated by sparse matrices (matrix compression). Our stability result (3.9) seems to be new.

Both Examples 3.4 and 3.6 deal with regular elements (hypercubes or equilateral, right-angled triangles). In both cases the employed discrete scalar products are equivalent to $(\cdot,\cdot)_{L_2}$ ((3.6)), and it appeared that the corresponding decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{V}_k$ is L_2-stable ((3.9)). Finally in this section, we show that both properties are maintained under two types of transformations of the elements.

Let $\Phi : \mathbf{R}^d \to \mathbf{R}^d$ be an invertible mapping. Define $\tilde{\Omega} = \Phi^{-1}\Omega$ and $\Phi^{\times}$, mapping functions on Ω onto functions on $\tilde{\Omega}$, by $\Phi^{\times} u = u \circ \Phi$. Put $\tilde{\mathcal{M}}_k = \Phi^{\times}\mathcal{M}_k$ and $\tilde{\Omega}_k = \Phi^{-1}\Omega_k$ which is the set of (free) nodal points belonging to $\tilde{\mathcal{M}}_k$. For some discrete scalar product $(\cdot,\cdot)_{\tilde{\mathcal{M}}_k}$ as in (3.5), let $\tilde{\mathcal{V}}_k = \tilde{\mathcal{M}}_k \ominus^{\perp_{(,)_{\tilde{\mathcal{M}}_k}}} \tilde{\mathcal{M}}_{k-1}$ $(k \geq 1)$ and $\tilde{\mathcal{V}}_0 = \tilde{\mathcal{M}}_0$.

First, we consider the case that Φ is some linear (or affine) transformation. We take

$$(\tilde{u}, \tilde{v})_{\tilde{\mathcal{M}}_k} = |D\Phi^{-1}| \sum_{\tilde{x} \in \tilde{\Omega}_k} w_{k,\Phi(\tilde{x})} \tilde{u}(\tilde{x}) \overline{\tilde{v}(\tilde{x})} \qquad (\tilde{u}, \tilde{v} \in \tilde{\mathcal{M}}_k).$$

Then from $(u \circ \Phi, v \circ \Phi)_{\tilde{\mathcal{M}}_k} = |D\Phi^{-1}|\,(u,v)_{\mathcal{M}_k}$ and $(u \circ \Phi, v \circ \Phi)_{L_2(\tilde{\Omega})} = |D\Phi^{-1}|\,(u,v)_{L_2(\Omega)}$, we have $\|\cdot\|_{\tilde{\mathcal{M}}_k} \eqsim \|\cdot\|_{L_2(\tilde{\Omega})}$ uniformly in Φ and k, $\tilde{\mathcal{V}}_k = \Phi^\times \mathcal{V}_k$ and[2]

$$\kappa(\{\tilde{\mathcal{M}}_J; (\cdot,\cdot)_{L_2(\tilde{\Omega})}\}; \textstyle\sum_k\{\tilde{\mathcal{V}}_k; (\cdot,\cdot)_{L_2(\tilde{\Omega})}\}) = \kappa(\{\mathcal{M}_J; (\cdot,\cdot)_{L_2(\Omega)}\}; \textstyle\sum_k\{\mathcal{V}_k; (\cdot,\cdot)_{L_2(\Omega)}\}).$$

As a second case, we consider continuous Φ that are C^1-diffeomorphism on the interiors of $\Phi^{-1}(\tau)$ for each underlying element τ of $\mathcal{M}_0$. We take

$$(\tilde{u},\tilde{v})_{\tilde{\mathcal{M}}_k} = \sum_{\tilde{x}\in\tilde{\Omega}_k} w_{k,\Phi(\tilde{x})}\tilde{u}(\tilde{x})\overline{\tilde{v}(\tilde{x})} \qquad (\tilde{u},\tilde{v}\in\tilde{\mathcal{M}}_k).$$

Note that this scalar product does not correspond to the application of quadrature formulas on $\Phi^{-1}(\tau)$ that are exact on $\tilde{\mathcal{M}}_k|_{\Phi^{-1}(\tau)}$. Clearly, we have $(u\circ\Phi, v\circ\Phi)_{\tilde{\mathcal{M}}_k} = (u,v)_{\mathcal{M}_k}$ and thus $\tilde{\mathcal{V}}_k = \Phi^\times\mathcal{V}_k$. Furthermore, there holds

$$\frac{1}{|||D\Phi|||_\infty} \le \frac{\|u\circ\Phi\|^2_{L_2(\tilde{\Omega})}}{\|u\|^2_{L_2(\Omega)}} \le |||D\Phi^{-1}|||_\infty,$$

and so

$$\begin{aligned}&\kappa(\{\tilde{\mathcal{M}}_J; (\cdot,\cdot)_{L_2(\tilde{\Omega})}\}; \textstyle\sum_k\{\tilde{\mathcal{V}}_k; (\cdot,\cdot)_{L_2(\tilde{\Omega})}\})\\ &\quad\le (|||D\Phi|||_\infty |||D\Phi^{-1}|||_\infty)^2\,\kappa(\{\mathcal{M}_J; (\cdot,\cdot)_{L_2(\Omega)}\}; \textstyle\sum_k\{\mathcal{V}_k; (\cdot,\cdot)_{L_2(\Omega)}\}).\end{aligned}$$

We conclude that the results (3.6) and (3.9) for Examples 3.4 and 3.6 carry over to quasi-uniform meshes.

3.5. Numerical results. We present numerical computed condition numbers of our additive Schwarz preconditioner for a local refinement example and an example with a discontinuous coefficient. Both examples concern linear finite elements on a two-dimensional domain.

Example 3.7. Let Ξ_{-1} be the collection of two triangles $\{(x,y)\in[0,1]^2 : y\le x\}$ and $\{(x,y)\in[0,1]^2 : y\ge x\}$. For $k\ge 0$, we define Ξ_k as the collection of triangles that arises by for $k\le\ell$ breaking each triangle from Ξ_{k-1} into four congruent triangles and for $k>\ell$ by breaking only triangles from Ξ_{k-1} contained in $[0,(\frac{1}{2})^{k-\ell}]^2$ into four congruent ones. For $k\ge 0$, let $\mathcal{M}_k\subset H^1_0((0,1)^2)$ be the linear finite element space corresponding to Ξ_k. We take $\phi\equiv 1$ and define the scalar products $(\cdot,\cdot)_{\mathcal{M}_k}$ according to the construction exposed below formula (3.6) (see Figure 4). The choice of $(\cdot,\cdot)_{\mathcal{M}_k}$ determines the decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{V}_k$ as well as the mappings p^0 and p^1.

We numerically computed the characteristic values $\lambda_{\max}$, $\lambda_{\min}$ and the condition number κ of the decomposition for the bilinear forms $a(u,v)=\int u\overline{v}$ and $a(u,v)=\int\sum_{i=1}^2 \partial_i u\overline{\partial_i v}$ using auxiliary scalar products b_k on $\mathcal{V}_k$ defined by one step of Jacobi iteration. The results are given in Tables 1 and 2. The first column ($J-\ell=0$) of both tables concerns the

[2]Although (3.6) and (3.9) apparently do hold uniformly in $|D\Phi^{-1}|$, this is not true for (3.1), (3.3) and (3.4) since these results rely on (T3). As a consequence, Theorem 3.1 cannot be applied. Remarkably, the results of the forthcoming section 4 can coarsely speaking be summarized by saying that nevertheless in case of Examples 3.4 and 3.6, the decomposition $\tilde{\mathcal{M}}_J = \bigoplus_k\tilde{\mathcal{V}}_k$ is $|\cdot|_{H^1}$-stable uniformly for linear transformations Φ that do not corrupt a “maximum angle condition” (which is weaker than (T3)).

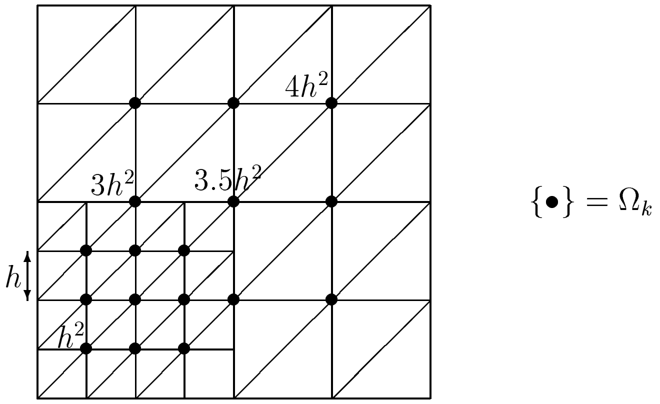


FIGURE 4. Weights $w_{x,k}$ in $(u,v)_{\mathcal{M}_k} = \sum_{x\in\Omega_k} w_{x,k} u(x)\overline{v(x)}$ in case of Example 3.7 with $k=2$ and $\ell = 1$.

ℓ \ $J-\ell$		0			1			2			3		
	"h_J"	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
3	$\frac{1}{16}$	.48	2.2	4.7	.46	2.3	4.9	.46	2.3	5.0	.45	2.3	5.0
4	$\frac{1}{32}$	.45	2.3	5.1	.44	2.3	5.3	.43	2.3	5.3	.43	2.3	5.3
5	$\frac{1}{64}$	.43	2.3	5.4	.42	2.3	5.5	.42	2.3	5.5	.42	2.3	5.5
6	$\frac{1}{128}$	.42	2.3	5.6	.41	2.4	5.7	.41	2.4	5.7	.41	2.4	5.7
7	$\frac{1}{256}$	.41	2.4	5.7	.40	2.4	5.9	.40	2.4	5.9	.40	2.4	5.9

TABLE 1. Condition numbers for Example 3.7 for $a(u,v) = \int u\overline{v}$ with one Jacobi iteration as a subspace solver.

ℓ \ $J-\ell$		0			1			2			3		
	"h_J"	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
3	$\frac{1}{16}$	.25	2.8	12	.23	2.8	12	.23	2.9	13	.23	2.9	13
4	$\frac{1}{32}$	.22	2.9	14	.21	3.0	14	.21	3.0	14	.21	3.0	14
5	$\frac{1}{64}$	.21	3.1	15	.20	3.1	16	.20	3.2	16	.20	3.2	16
6	$\frac{1}{128}$	.20	3.2	16	.20	3.3	17	.20	3.3	17	.20	3.3	17
7	$\frac{1}{256}$	.20	3.3	17	.20	3.4	17	.20	3.4	17	.20	3.4	17

TABLE 2. As Table 1, but now for $a(u,v) = \int \sum_{i=1}^{2} \partial_i u \overline{\partial_i v}$.

uniform refinement case from Example 3.6 and the bounded numbers for $a(u,v) = \int u\overline{v}$ from Table 1 illustrate the theoretical result proved in Appendix A. The other columns of this table show that in case of additional local refinement levels the condition numbers are hardly larger. Table 2 shows that the condition numbers for $a(u,v) = \int \sum_{i=1}^{2} \partial_i u \overline{\partial_i v}$ are bounded as well illustrating Theorem 3.1. From (2.5) we conclude robustness of the additive

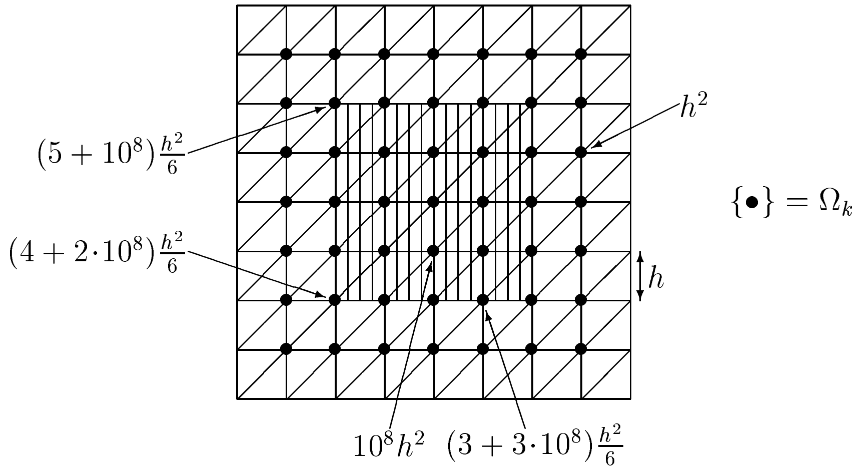


FIGURE 5. Weights $w_{x,k}$ in $(u,v)_{\mathcal{M}_k}=\sum_{x\in\Omega_k} w_{x,k}u(x)\overline{v(x)}$ in case of Example 3.8 with $k=1$.

Schwarz preconditioner for $a(u,v)=\nu\int\sum_{i=1}^2 \partial_i u\overline{\partial_i v}+\mu\int u\overline{v}$ with $\nu,\mu\geq 0$, $\nu+\mu>0$.

Example 3.8. In this example we take $\phi\equiv 10^8$ on $[\frac{1}{4},\frac{3}{4}]^2$ and $\phi\equiv 1$ on $[0,1]\backslash[\frac{1}{4},\frac{3}{4}]^2$. To satisfy the condition that ϕ is constant on the interiors of the finite elements corresponding to $\mathcal{M}_0$ we need a finer initial triangulation than in the previous example. We define $\mathcal{M}_k$ as the finite element space $\mathcal{M}_{k+1}$ from Example 3.7 assuming uniform refinements. We define $(\cdot,\cdot)_{\mathcal{M}_k}$ again following the construction below (3.6) (see Figure 5). Numerical computed condition numbers for $a(u,v)=\int\phi u\overline{v}$ and $a(u,v)=\int\phi\sum_{i=1}^2\partial_i u\overline{\partial_i v}$ are given in Table 3. A comparison between Table 3 and the first column of Tables 1 and 2 shows that the

J	2	3	4	5	6	2	3	4	5	6
"h_J"	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
$\lambda_{\min}$	.44	.41	.39	.38	.37	.23	.20	.19	.18	.19
$\lambda_{\max}$	2.2	2.3	2.3	2.3	2.3	2.7	2.8	2.9	3.1	3.2
κ	5.0	5.5	5.8	6.1	6.4	11	14	16	17	17
	$a(u,v)=\int\phi u\overline{v}$					$a(u,v)=\int\phi\sum_{i=1}^2\partial_i u\overline{\partial_i v}$				

TABLE 3. Condition numbers for Example 3.8 with one Jacobi iteration as a subspace solver.

condition numbers for this discontinuous ϕ are hardly larger than those for $\phi\equiv 1$.

We also computed condition numbers for the function ϕ defined by $\phi\equiv 10^8$ on $[\frac{1}{4},\frac{1}{2}]^2\cup[\frac{1}{2},\frac{3}{4}]^2$ and $\phi\equiv 1$ on $[0,1]^2\backslash([\frac{1}{4},\frac{1}{2}]^2\cup[\frac{1}{2},\frac{3}{4}]^2)$. In this case the jump interface has a so-called cross-point. For $a(u,v)=\int\phi u\overline{v}$ we found condition numbers that are almost equal to those in the left-hand part of Table 3, but for $a(u,v)=\int\phi\sum_{i=1}^2\partial_i u\overline{\partial_i v}$ the condition numbers appeared to be growing as function of J. Yet, this result is not in conflict with

Theorem 3.1, since for a jump interface with a cross-point estimate (3.3), which was used as a condition for Theorem 3.1, is not necessarily valid (cf. [Wan94]). Tests with the BPX-preconditioner showed condition numbers that grew in a similar way.

4. Robustness for anisotropic problems

In this section we study the question of robustness for anisotropic equations of our space decomposition $\mathcal{M}_J = \bigoplus_k \mathcal{V}_k$, in particular for the case of two-dimensional linear finite element spaces. The reduced boundary value problems on the subspaces appear to be not uniformly well-conditioned as function of the anisotropy and so anyhow simple point solvers cannot be applied. Therefore, we will apply the idea of constructing an approximate solver based on a space decomposition orthogonal with respect to discrete scalar products again by decomposing each $\mathcal{V}_k$ into a number of subspaces that are orthogonal with respect to $(\cdot,\cdot)_{\mathcal{M}_k}$.

4.1. Multi-linear finite element spaces. Mainly as an introduction for the case of linear finite element spaces treated in the next subsection, in this subsection we consider the multi-linear finite element spaces $\mathcal{M}_k \subset H_0^1((0,1)^d)$ and scalar products $(\cdot,\cdot)_{\mathcal{M}_k}$ from Example 3.4 and review results obtained in [Ste94, Ste95] for a further decomposition of the spaces $\mathcal{V}_k$. As in Example 3.4, we use notations $\mathcal{M}_k^{(1)}$, $\mathcal{V}_k^{(1)}$, $\Omega_k^{(1)}$, $\phi_{k,x}^{(1)}$ and $\psi_{k,x}^{(1)}$ to denote concepts corresponding to the one-dimensional case. To avoid making exceptions for the case $k=0$ every time, we use additionally the notations $\psi_{0,x}^{(1)} = \phi_{0,x}^{(1)}$ and $\Omega_{-1}^{(1)} = \emptyset$. In [Ste94, Ste95] it was proved that the decomposition

$$\mathcal{M}_J = \bigoplus_{\alpha\in\{0,\dots,J\}^d} \mathcal{V}_\alpha, \text{ where } \mathcal{V}_\alpha := \bigotimes_{j=1}^{d} \mathcal{V}_{\alpha_j}^{(1)}, \tag{4.1}$$

is robust for anisotropic forms

$$a(u,v) = c_0 \int_{(0,1)^d} u\overline{v} + \sum_{j=1}^{d} c_j \int_{(0,1)^d} \partial_j u \overline{\partial_j v}, \text{ with } \sum_j c_j > 0,\ c_j \geq 0, \tag{4.2}$$

that is,

$$\kappa(\{\mathcal{M}_J; a\}; \sum_\alpha \{\mathcal{V}_\alpha; a\}) \lesssim 1. \tag{4.3}$$

Moreover, it was shown that if $\mathcal{V}_\alpha$ is equipped with tensor product basis

$$\{\bigotimes_j \psi_{\alpha_j,x_j}^{(1)} : x \in \bigtimes_{j=1}^{d} \Omega_{\alpha_j}^{(1)} \backslash \Omega_{\alpha_j-1}^{(1)}\},$$

then the corresponding stiffness matrix is uniformly well-conditioned, which means that point solvers can be applied, or equivalently, that

$$\kappa(\{\mathcal{M}_J; a\}; \sum_\alpha \sum_x \{\operatorname{span} \bigotimes_j \psi_{\alpha_j,x_j}^{(1)}; a\}) \lesssim 1. \tag{4.4}$$

The resulting preconditioner is a modification of the Frequency Decomposition Multi-Level Method introduced by Hackbusch ([Hac89, Hac92]). For a similar preconditioner based on

L_2-orthogonal decompositions equivalent results were obtained by Griebel and Oswald in [GO95b].

Decompositions like (4.1) and (4.4) are restricted to tensor product spaces and can therefore not be applied to *linear* finite element spaces. Still for the *multi-linear* case from Example 3.4 but now with $d = 2$, we consider therefore the decomposition

$$\mathcal{M}_J = \mathcal{V}_0 \bigoplus \bigoplus_{k=1}^{J} \bigoplus_{\beta \in \{(1,0),(0,1),(1,1)\}} \mathcal{U}_{k,\beta} \tag{4.5}$$

of $\mathcal{M}_J$ into $(3J + 1)$ subspaces, where

$$\mathcal{U}_{k,(1,0)} = \mathcal{V}_k^{(1)} \bigotimes \mathcal{M}_{k-1}^{(1)}, \ \mathcal{U}_{k,(0,1)} = \mathcal{M}_{k-1}^{(1)} \bigotimes \mathcal{V}_k^{(1)} \ \text{ and } \ \mathcal{U}_{k,(1,1)} = \mathcal{V}_k^{(1)} \bigotimes \mathcal{V}_k^{(1)}.$$

The spaces $\mathcal{U}_{k,(1,0)}$, $\mathcal{U}_{k,(0,1)}$ and $\mathcal{U}_{k,(1,1)}$ are mutually orthogonal with respect to $(\cdot,\cdot)_{\mathcal{M}_k}$ and in particular there holds $\mathcal{V}_k = \bigoplus_{\beta}^{\perp_{(,)\mathcal{M}_k}} \mathcal{U}_{k,\beta}$. The decomposition (4.5) arises from (4.1) by for each $k > 0$ clustering the spaces $\mathcal{V}_{k,0}, \ldots, \mathcal{V}_{k,k-1}$ yielding $\mathcal{U}_{k,(1,0)}$ and $\mathcal{V}_{0,k}, \ldots, \mathcal{V}_{k-1,k}$ yielding $\mathcal{U}_{k,(0,1)}$ (see Figure 6). From Lemma 3.5 and (4.3) we conclude that the decompo-

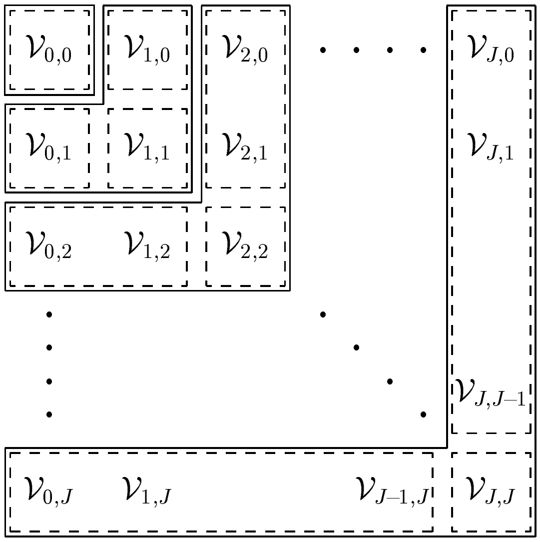
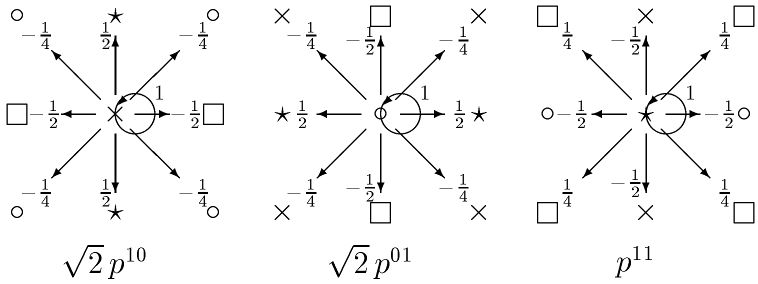


FIGURE 6. Clustering process applied to $\mathcal{M}_J = \bigoplus_{\alpha \in \{0,\ldots,J\}^2} \mathcal{V}_\alpha$ yielding the decompositions $\mathcal{M}_J = \mathcal{V}_0 \oplus \bigoplus_{k=1}^{J} \bigoplus_{\beta \in \{(1,0),(0,1),(1,1)\}} \mathcal{U}_{k,\beta}$ (, ,) and $\mathcal{M}_J = \bigoplus_{k=0}^{J} \mathcal{V}_k$ ( )

sition (4.5) is robust assuming exact subspace solvers.

We equip the spaces $\mathcal{U}_{k,\beta}$ with tensor product bases using the bases on $\mathcal{M}_{k-1}^{(1)}$ and $\mathcal{V}_k^{(1)}$. The representation of an element of $\mathcal{U}_{k,(1,0)}$, $\mathcal{U}_{k,(0,1)}$ or $\mathcal{U}_{k,(1,1)}$ with respect to these bases is most naturally identified with a grid function on $(\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)}) \times \Omega_{k-1}^{(1)}$, $\Omega_{k-1}^{(1)} \times (\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)})$ or $(\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)})^2$ respectively. The representations p^β of the trivial embeddings $\mathcal{U}_{k,\beta} \to \mathcal{M}_k$ are depicted in Figure 7.



$$\{\square\} = \Omega_{k-1}$$
$$\{\times\} = (\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)}) \times \Omega_{k-1}^{(1)}$$
$$\{\circ\} = \Omega_{k-1}^{(1)} \times (\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)})$$
$$\{\star\} = (\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)})^2$$

FIGURE 7. Representations of the basis functions of $\mathcal{U}_{k,\beta}$ in terms of the nodal basis functions of $\mathcal{M}_k$ in case of *bilinear* elements.

The stiffness matrices corresponding to $\mathcal{U}_{k,(1,0)}$ and $\mathcal{U}_{k,(0,1)}$ appear not to be uniformly well-conditioned. Yet, we do obtain a robust preconditioner for problem (4.2), that is, possible anisotropy in one of the two directions of the grid lines, when the reduced boundary value problems on $\mathcal{U}_{k,(1,0)}$ and $\mathcal{U}_{k,(0,1)}$ are solved with *y-line* and *x-line* Jacobi respectively, whereas on $\mathcal{U}_{k,(1,1)}$ point Jacobi can be applied. Indeed, note that these Jacobi solvers correspond to the application of additive Schwarz preconditioners with respect to the decompositions $\mathcal{U}_{k,(1,0)} = \bigoplus_{x_1 \in \Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)}} \psi_{k,x_1}^{(1)} \otimes \mathcal{M}_{k-1}^{(1)}$, $\mathcal{U}_{k,(0,1)} = \bigoplus_{x_2 \in \Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)}} \mathcal{M}_{k-1}^{(1)} \otimes \psi_{k,x_2}^{(1)}$ and $\mathcal{U}_{k,(1,1)} = \bigoplus_{x \in (\Omega_k^{(1)} \backslash \Omega_{k-1}^{(1)})^2} \psi_{k,x_1}^{(1)} \otimes \psi_{k,x_2}^{(1)}$. The resulting space decomposition of $\mathcal{M}_J$ can be obtained by applying clustering to the decomposition from (4.4) and so robustness appears from Lemma 3.5.

4.2. Linear finite element spaces. In this subsection we consider the linear finite element spaces $\mathcal{M}_k$ based on regular triangulations and scalar products $(\cdot,\cdot)_{\mathcal{M}_k}$ from Example 3.6. Following ideas from §4.1, for $k > 0$ we make a direct sum decomposition of $\mathcal{V}_k$ into three spaces denoted by $\mathcal{U}_{k,(1,0)}$, $\mathcal{U}_{k,(0,1)}$ and $\mathcal{U}_{k,(1,1)}$. In Figure 8 we define these spaces $\mathcal{U}_{k,\beta} \subset \mathcal{V}_k$ together with bases on these spaces by giving representations of the basis functions in terms of the basis functions of $\mathcal{V}_k$. Note that the stencils defining $\mathcal{U}_{k,(1,0)}$ and $\mathcal{U}_{k,(0,1)}$

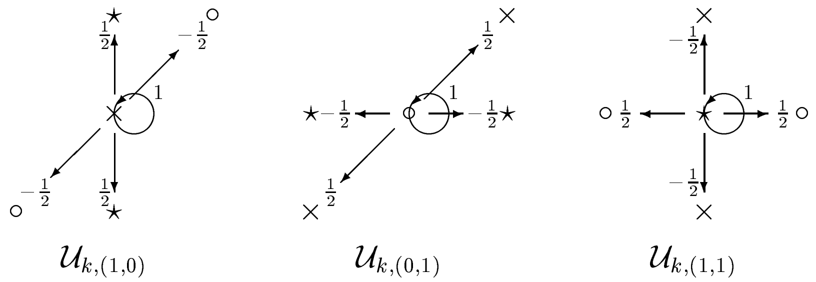


FIGURE 8. Representations of the basis functions of $\mathcal{U}_{k,\beta}$ in terms of the basis functions of $\mathcal{V}_k$ in case of *linear* finite elements (symbols analogue to Figure 7). (As usual, possible contributions from grid points outside Ω should be left out.)

cannot be obtained from each other by exchanging x- and y-coordinates. Yet, replacing $\frac{1}{2}$

by $-\frac{1}{2}$ and vice versa in all three stencils would give equivalent results. By expressing the basis functions of $\mathcal{V}_k$ in terms of the basis functions of $\mathcal{M}_k$ (use Figure 3), we obtain the representations of the basis functions of $\mathcal{U}_{k,\beta}$ in terms of the basis functions of $\mathcal{M}_k$ depicted in Figure 9. The same stencils were also applied in [Kat94].

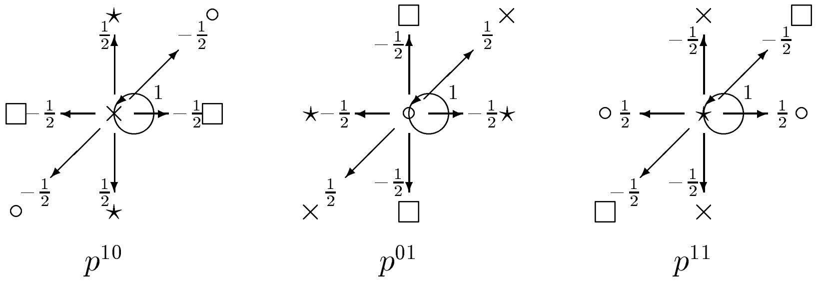


FIGURE 9. Representations of the basis functions of $\mathcal{U}_{k,\beta}$ in terms of the nodal basis functions of $\mathcal{M}_k$ (p^β) in case of *linear* finite elements.

The remainder of this subsection is mainly devoted to a proof of the following theorem.

Theorem 4.1. *Consider the linear finite element spaces $\mathcal{M}_k$ with scalar products $(\cdot,\cdot)_{\mathcal{M}_k}$ from Example 3.6 but now assume that $\Omega = \mathbf{R}^2$. Let*

$$a(u,v) = c_0 \int_{\mathbf{R}^2} u\overline{v} + \int_{\mathbf{R}^2} < \left(c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} \partial_1 u \\ \partial_2 u \end{bmatrix}, \begin{bmatrix} \partial_1 v \\ \partial_2 v \end{bmatrix} >,$$

where $\sum_k c_k > 0$, $c_k \geq 0$, that is, possible anisotropy in one of the three *directions of the grid lines, and let $b_{k,\beta}$ be the auxiliary scalar product on $\mathcal{U}_{k,\beta}$ derived from $a(,)$ corresponding to one y-line ($\updownarrow$), $(x+y)$-line ($\nearrow$) or x-line ($\leftrightarrow$) Jacobi iteration for $\beta = (1,0)$, $(0,1)$ or $(1,1)$ respectively. Then*

$$\kappa(\{\mathcal{M}_J; a\}; \{\mathcal{V}_0; a\} \;+\; \sum_{k=1}^{J} \sum_{\beta\in\{(1,0),(0,1),(1,1)\}} \{\mathcal{U}_{k,\beta}; b_{k,\beta}\}) \lesssim (J+1).$$

A practical implementation of the preconditioner described in above theorem is given by the following algorithm, where r^β denotes the matrix adjoint of p^β.

Algorithm 4.2.

function HB1(J,F)
  begin
    if $J = 0$ **then** $U = A_0^{-1}F$;
    else
      $U = p^{10}(y\text{-line blockdiag}(r^{10}A_J^{-1}p^{10}))^{-1}r^{10}F +$
        $p^{01}((x+y)\text{-line blockdiag}(r^{01}A_J^{-1}p^{01}))^{-1}r^{01}F +$
        $p^{11}(x\text{-line blockdiag}(r^{11}A_J^{-1}p^{11}))^{-1}r^{11}F +$
        $p^0\mathrm{HB1}(J-1, r^0F)$;
    return U;
  end

Remarks 4.3. • The proof of Theorem 4.1 involves estimates of the characteristic values $\lambda_{\max}$ and $\lambda_{\min}$ of the decomposition (cf. §2.2). Here we note that the estimate of $\lambda_{\max}$ that we are going to derive can be generalized to bounded domains by expanding "boundary perturbations" around a zero boundary value using a discrete Taylor formula. However, until now we were not able to give a suitable estimate for $\lambda_{\min}$ in case of a bounded domain.

• Although Theorem 4.1 promises only sub-optimality of the preconditioner, our numerical experiments on a bounded domain reported in the next subsection show bounded condition numbers uniformly in J and the anisotropy.

• The rotated anisotropic form

$$a(u,v) = \int < \begin{bmatrix} \epsilon \mathrm{c}^2 + \mathrm{s}^2 & (\epsilon - 1)\mathrm{sc} \\ (\epsilon - 1)\mathrm{sc} & \epsilon \mathrm{s}^2 + \mathrm{c}^2 \end{bmatrix} \begin{bmatrix} \partial_1 u \\ \partial_2 u \end{bmatrix}, \begin{bmatrix} \partial_1 v \\ \partial_2 v \end{bmatrix} >,$$

where $\mathrm{c} = \cos(\phi)$, $\mathrm{s} = \sin(\phi)$, $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2})$ and $\epsilon \in [0,1]$, satisfies the conditions of Theorem 4.1 when $\phi \in [-\frac{\pi}{2}, 0]$ and $\epsilon \geq 1 - \frac{2}{1+\sqrt{2}\sin(2|\phi - \frac{\pi}{4}| + \frac{\pi}{4})}$, or equivalently, when the resulting stiffness matrix on $\mathcal{M}_k$ is an L-matrix. On $[-\frac{\pi}{2}, 0]$, the range of $\phi \mapsto 1 - \frac{2}{1+\sqrt{2}\sin(2|\phi - \frac{\pi}{4}| + \frac{\pi}{4})}$ is equal to $[0, 3 - 2\sqrt{2}]$ with minima in 0, $-\frac{\pi}{4}$ and $-\frac{\pi}{2}$.

A simple transformation of coordinates shows that the rotated anisotropic equation discretized on $\mathcal{M}_k$ leads to the same system of equations as Laplace' equation discretized on the linear finite element space corresponding to the grid that is rotated over an angle $-\phi$ and afterwards is horizontally enlarged with a factor $\frac{1}{\sqrt{\epsilon}}$ (cf. [Ste93]). Clearly this procedure results in a grid that does not satisfy the minimum angle condition (T3) uniformly in ϵ. The special angles $\phi \in \{-\frac{\pi}{2}, -\frac{\pi}{4}, 0\}$ correspond to those transformed grids that nevertheless do have maximum angles that are bounded below π uniformly in ϵ (cf. [2]).

• A decomposition $\mathcal{V}_k = \bigoplus_{\beta \in \{(1,0),(0,1),(1,1)\}} \mathcal{U}_{k,\beta}$ according to Figure 8 can also be made in case of non-uniform or locally refined triangulations. In view of the preceding considerations, assuming approximate solvers on the spaces $\mathcal{U}_{k,\beta}$ that solve for the "strong couplings", we may hope for robustness for the described anisotropic equations, or equivalently, for isotropic equations discretized on grids that possibly have small angles but that do satisfy an maximum angle condition.

To prove Theorem 4.1, it is sufficient to show that for $\vec{c} := (c_0, c_1, c_2, c_3) = (1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ the condition number κ of the decomposition is $\mathcal{O}(J + 1)$. The stiffness matrices on the spaces $\mathcal{M}_k$, $\mathcal{U}_{k,(1,0)}$, $\mathcal{U}_{k,(0,1)}$ and $\mathcal{U}_{k,(1,1)}$ are given in stencil notation in Table 4, the stencils referring to underlying grids $h_k\mathbf{Z}^2$, $h_k((2\mathbf{Z}+1) \times 2\mathbf{Z})$, $h_k(2\mathbf{Z} \times (2\mathbf{Z}+1))$ and $h_k(2\mathbf{Z}+1)^2$ respectively. From this table we conclude that for all four choices of $\vec{c}$, the auxiliary scalar products $b_{k,\beta}$ on $\mathcal{U}_{k,\beta}$ defined in Theorem 4.1 are equivalent, or for $\vec{c} = (1,0,0,0)$ even equal to a, which means that for our analysis we may replace them by a.

The proof of the next lemma follows directly from the form of the p^β (Figure 9) and that of p^0 (Figure 3).

$\vec{c}$	$\mathcal{M}_k$	$\mathcal{U}_{k,(1,0)}$	$\mathcal{U}_{k,(0,1)}$	$\mathcal{U}_{k,(1,1)}$
$(1,0,0,0)$ (mass-matrix)	$\frac{1}{12}\begin{bmatrix} & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} & & 1 \\ & 6 & \\ 1 & & \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} 1 & 6 & 1 \end{bmatrix}$
$(0,1,0,0)$	$h_k^{-2}\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & 1 & 1 \\ 2 & 8 & 2 \\ 1 & 1 & \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & 1 & 1 \\ 2 & 8 & 2 \\ 1 & 1 & \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}$
$(0,0,1,0)$	$h_k^{-2}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & 2 & 1 \\ 1 & 8 & 1 \\ 1 & 2 & \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & 2 & 1 \\ 1 & 8 & 1 \\ 1 & 2 & \end{bmatrix}$
$(0,0,0,1)$	$h_k^{-2}\begin{bmatrix} & & -1 \\ & 2 & \\ -1 & & \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & 1 & 2 \\ 1 & 8 & 1 \\ 2 & 1 & \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & & -1 \\ & 2 & \\ -1 & & \end{bmatrix}$	$h_k^{-2}\begin{bmatrix} & 1 & 2 \\ 1 & 8 & 1 \\ 2 & 1 & \end{bmatrix}$

TABLE 4. Stiffness matrices for the bilinear form from Theorem 4.1

Lemma 4.4. *There holds*[3] $\mathcal{V}_k = \bigoplus_{\beta\in\{(1,0),(0,1),(1,1)\}}^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{U}_{k,\beta}$. *Further, for* $T_k^\gamma : \mathcal{M}_k \to \mathcal{M}_k$ *defined by* $(T_k^\gamma u)(x) = u(x+\gamma h_k)$ $(\gamma \in \mathbf{Z}^2)$, *we have*

$$T_k^{(0,\gamma_2)}\mathcal{U}_{k,(1,0)},\ T_k^{(\gamma_1,\gamma_1)}\mathcal{U}_{k,(0,1)},\ T_k^{(\gamma_1,0)}\mathcal{U}_{k,(1,1)} \perp_{(\cdot,\cdot)_{\mathcal{M}_k}} \mathcal{M}_{k-1}.$$

For the case $\vec{c} = (1,0,0,0)$, from $\kappa(\{\mathcal{M}_J;(,)_{L_2}\};\sum_k\{\mathcal{V}_k;(,)_{L_2}\}) \lesssim 1$ (Appendix A), $\mathcal{V}_k = \bigoplus_\beta^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{U}_{k,\beta}$ (Lemma 4.4) and $\|\cdot\|_{\mathcal{M}_k} \eqsim \|\cdot\|_{L_2}$ ((3.6)) we immediately obtain that $\kappa(\{\mathcal{M}_J;a\};\{\mathcal{V}_0;a\}+\sum_{k=1}^J\sum_\beta\{\mathcal{U}_{k,\beta};a\}) \lesssim 1$.

In the sequel, we take $\vec{c} = (0,1,0,0)$, that is,

$$a(u,v) = \int \partial_2 u \overline{\partial_2 v}.$$

The remaining two cases can be treated analogously.

From Lemma 4.4 and the form of the stiffness matrix on $\mathcal{M}_k$, we find that

$$\mathcal{U}_{k,(1,0)} \perp_{a(\cdot,\cdot)} \mathcal{M}_{k-1}. \tag{4.6}$$

Analogously to (3.2), one may prove that that there exists a $\gamma < 1$ such that for $k \geq l$

$$a(u,v) \lesssim \gamma^{k-l} a(u,u)^{\frac{1}{2}}\, 2^k \|v\|_{L_2} \qquad (u \in \mathcal{M}_l,\ v \in \mathcal{M}_k). \tag{4.7}$$

On both $\mathcal{U}_{k,(0,1)}$ and $\mathcal{U}_{k,(1,1)}$ the stiffness- and mass-matrices are well-conditioned and so on these spaces we have

$$a(v,v) \eqsim 2^k \|v\|_{L_2}. \tag{4.8}$$

[3] In contrast to the bilinear case, in case of a bounded domain the spaces $\mathcal{U}_{k,\beta}$ are generally not mutually $(,)_{\mathcal{M}_k}$-orthogonal. Yet, by construction, also on bounded domains we have $\mathcal{U}_{k,\beta} \subset \mathcal{V}_k$, and so $\mathcal{U}_{k,\beta} \perp_{(,)_{\mathcal{M}_k}} \mathcal{M}_{k-1}$.

From (4.6), (4.7) and (4.8) we conclude that for $k \geq l$ and $\beta \in \{(1,0),(0,1),(1,1)\}$

$$a(u,v) \lesssim \gamma^{k-l} a(u,u)^{\frac{1}{2}} a(v,v)^{\frac{1}{2}} \qquad (u \in \mathcal{M}_l,\ v \in \mathcal{U}_{k,\beta}).$$

From this result, we directly obtain that for $\mathcal{M}_J \ni u = v_0 + \sum_k \sum_\beta v_{k,\beta}$, with $v_0 \in \mathcal{V}_0$ and $v_{k,\beta} \in \mathcal{U}_{k,\beta}$, there holds

$$a(u,u) \lesssim a(v_0,v_0) + \sum_{k=1}^{J} \sum_{\beta \in \{(1,0),(0,1),(1,1)\}} a(v_{k,\beta}, v_{k,\beta})$$

or $\lambda_{\max} \lesssim 1$.

Conversely, for $1 \leq k \leq J$ and $\beta \in \{(1,0),(0,1),(1,1)\}$, let $Z_{k,\beta} : \mathcal{M}_J \to \mathcal{U}_{k,\beta}$ the projection mapping u to $v_{k,\beta}$. We will prove that $a(Z_{k,\beta}u, Z_{k,\beta}u) \lesssim a(u,u)$ and $a(Z_0 u, Z_0 u) \lesssim a(u,u)$ and so $\frac{1}{\lambda_{\min}} \lesssim (3J+1)$ which completes the proof of Theorem 4.1.

For $k \geq 1$, there holds $Z_{k,\beta} = Y_{k,\beta} Z_k$ with Z_k defined in (3.7) and where $Y_{k,\beta}$ is the projection from $\mathcal{V}_k$ onto $\mathcal{U}_{k,\beta}$ orthogonal with respect to $(\cdot,\cdot)_{\mathcal{M}_k}$. Clearly, from (3.6) we have $\|Y_{k,\beta}\|_{L_2 \leftarrow L_2} \eqsim 1$. Now from the inverse inequality (3.1) and (4.8) it follows that for $\beta \in \{(0,1),(1,1)\}$

$$a(Y_{k,\beta} v_k, Y_{k,\beta} v_k) \lesssim a(v_k, v_k) \qquad (v_k \in \mathcal{V}_k). \tag{4.9}$$

Since on $\mathcal{V}_k$ there holds $Y_{k,(1,0)} = I - Y_{k,(0,1)} - Y_{k,(1,1)}$, we conclude that (4.9) is also valid for $\beta = (1,0)$. The proof of the estimate $\frac{1}{\lambda_{\min}} \lesssim (3J+1)$ is completed by the following theorem.

Theorem 4.5. *In the situation of Theorem 4.1 there holds*

$$a(Z_k u, Z_k u) \lesssim a(u,u) \qquad (u \in \mathcal{M}_J).$$

Proof. Recall that $Z_k = (I - Y_{k-1}) Y_k \cdots Y_{J-1}$ where $Y_k : \mathcal{M}_{k+1} \to \mathcal{M}_k$ is the $(\cdot,\cdot)_{\mathcal{M}_{k+1}}$-orthogonal projection onto $\mathcal{M}_k$ (and $Y_{-1} = 0$). So it suffices to prove that

$$a(Y_k \cdots Y_{J-1} u, Y_k \cdots Y_{J-1} u) \lesssim a(u,u) \qquad (u \in \mathcal{M}_J).$$

Again it is sufficient to consider $a(u,v) = \int \partial_2 u \overline{\partial_2 v}$.

Let us denote by U_l the representation of $u_l \in \mathcal{M}_l$ with respect to the nodal basis on $\mathcal{M}_l$ that is orthonormal with respect $(\cdot,\cdot)_{\mathcal{M}_l}$. From

$$(Y_k u_{k+1}, u_k)_{\mathcal{M}_{k+1}} = (u_{k+1}, u_k)_{\mathcal{M}_{k+1}} = < U_{k+1}, p^0 U_k > = < p^0 (r^0 p^0)^{-1} r^0 U_{k+1}, p^0 U_k >$$

we see that $(r^0 p^0)^{-1} r^0$ is the representation of $Y_k : \mathcal{M}_{k+1} \to \mathcal{M}_k$ with respect to nodal bases on $\mathcal{M}_{k+1}$ and $\mathcal{M}_k$. Using the stiffness matrix on $\mathcal{M}_l$ corresponding to a given in Table 4, we conclude that we have to prove that

$$\tfrac{1}{2}^{J-k} \| A_k^{\frac{1}{2}} \overbrace{(r^0p^0)^{-1} \underbrace{r^0 (r^0p^0)^{-1}} \underbrace{r^0 \cdots} \cdots \underbrace{\cdots (r^0p^0)^{-1}} \underbrace{r^0 (r^0p^0)^{-1}} r^0}^{(J-k)\times} A_J^{-\frac{1}{2}} \| \lesssim 1$$

where $\|\cdot\| =<\cdot>^{\frac{1}{2}}$ is the Euclidean norm, $A_l = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$ and where we have used that $\frac{h_J}{h_k} = \frac{1}{2}^{J-k}$.

By application of Fourier analysis we will show that

$$\|A_l^{\frac{1}{2}} r^0 (r^0 p^0)^{-1} A_{l+1}^{-\frac{1}{2}}\| \leq 2.$$

Since proving $\|A_k^{\frac{1}{2}}(r^0p^0)^{-1}A_{k+1}^{-\frac{1}{2}}\| \lesssim 1$ and $\|A_{J-1}^{\frac{1}{2}} r^0 A_J^{-\frac{1}{2}}\| \lesssim 1$ gives no additional troubles, we may then conclude that the theorem is valid.

Using notations as in Appendix A, straightforward computations show that the symbol of $A_l^{\frac{1}{2}} r^0 (r^0p^0)^{-1} A_{l+1}^{-\frac{1}{2}}$ is given by

$$\xi \mapsto a(\xi)^{\frac{1}{2}} \begin{bmatrix} \frac{2n(\xi^{(11)})-1}{n(\xi^{(11)})a(\xi^{(11)})^{\frac{1}{2}}} & \frac{2n(\xi^{(01)})-1}{n(\xi^{(01)})a(\xi^{(01)})^{\frac{1}{2}}} & \frac{2n(\xi^{(10)})-1}{n(\xi^{(10)})a(\xi^{(10)})^{\frac{1}{2}}} & \frac{2n(\xi^{(00)})-1}{n(\xi^{(00)})a(\xi^{(00)})^{\frac{1}{2}}} \end{bmatrix},$$

where

$$a(\xi) = 2 - 2\cos(\xi_2) = 4(1-\cos^2\tfrac{\xi_2}{2}) \text{ and } n(\xi) = \tfrac{1}{8}(5 + \cos\xi_1 + \cos\xi_2 + \cos(\xi_1+\xi_2)).$$

So, we have

$$\|A_l^{\frac{1}{2}} r^0(r^0p^0)^{-1}A_{l+1}^{-\frac{1}{2}}\|^2 = \sup_\xi a(\xi) \sum_{\alpha\in\{0,1\}^2} \frac{(2-\frac{1}{n(\xi^{(\alpha)})})^2}{a(\xi^{(\alpha)})} =$$

$$\sup_\xi 4(1-\cos^2\tfrac{\xi_2}{2})\ \Big\{\frac{1}{2-2\cos\frac{\xi_2}{2}} \sum_{\substack{\alpha_1\in\{0,1\}\\ \alpha_2=0}} (2-\tfrac{1}{n(\xi^{(\alpha)})})^2 + \frac{1}{2+2\cos\frac{\xi_2}{2}} \sum_{\substack{\alpha_1\in\{0,1\}\\ \alpha_2=1}} (2-\tfrac{1}{n(\xi^{(\alpha)})})^2\Big\}.$$

The function $x \mapsto (2-\frac{1}{x})^2$ is increasing on $[\frac{1}{2},\infty)$, $n(\xi)\geq\frac{1}{2}$ and $n(\xi) \leq \frac{3}{4} + \frac{1}{8}(\cos\xi_1 + \cos(\xi_1+\xi_2))$. Furthermore, since $-\frac{1}{4} \leq \frac{1}{8}(\cos\xi_1+\cos(\xi_1+\xi_2)) \leq \frac{1}{4}$, we have for $\alpha_2\in\{0,1\}$ that

$$\sum_{\alpha_1\in\{0,1\}} (2-\tfrac{1}{n(\xi^{(\alpha)})})^2 \leq \sup_{x\in[-\frac{1}{4},\frac{1}{4}]} \{(2-\tfrac{1}{\frac{3}{4}+x})^2 + (2-\tfrac{1}{\frac{3}{4}-x})^2\} = 1.$$

The proof is completed by $4(1-y^2)\{\frac{1}{2-2y}+\frac{1}{2+2y}\} = 4$. □

4.3. Numerical results. For the linear finite element spaces $\mathcal{M}_k$ based on regular triangulations from Example 3.6 with $\Omega = [0,1]^2$, scalar product a defined by

$$a(u,v) = c_0\int_\Omega u\overline{v} + \int_\Omega <\left(c_1\begin{bmatrix}1&0\\0&0\end{bmatrix} + c_2\begin{bmatrix}0&0\\0&1\end{bmatrix} + c_3\begin{bmatrix}1&1\\1&1\end{bmatrix}\right)\begin{bmatrix}\partial_1 u\\ \partial_2 u\end{bmatrix}, \begin{bmatrix}\partial_1 v\\ \partial_2 v\end{bmatrix}>$$

and decomposition $\mathcal{M}_J = \mathcal{V}_0 \oplus \bigoplus_{k=1}^J \bigoplus_{\beta\in\{(1,0),(0,1),(1,1)\}} \mathcal{U}_{k,\beta}$ and auxiliary scalar products $b_{k,\beta}$ on $\mathcal{V}_{k,\beta}$ introduced in the previous subsection, we give numerical computed condition numbers for $\vec{c} = (1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$. For all these four choices of $\vec{c}$ the condition numbers given in Table 5 appear to be bounded uniformly in J, which is an even better result than was proved in Theorem 4.1 for the case $\Omega = \mathbf{R}^2$. From (2.5) we conclude robustness for $\sum_k c_k > 0$, $c_k \geq 0$. The pure anisotropic equations

J("h_J")	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ
$3(\frac{1}{16})$	.48	1.4	3.0	.15	2.2	15	.15	2.2	15	.23	2.1	8.9
$4(\frac{1}{32})$	.44	1.5	3.3	.13	2.3	18	.13	2.3	17	.19	2.2	11
$5(\frac{1}{64})$	.41	1.5	3.6	.11	2.4	22	.11	2.4	22	.17	2.4	14
$6(\frac{1}{128})$	.39	1.5	3.7	.10	2.5	25	.10	2.5	24	.16	2.5	15
$7(\frac{1}{256})$	.38	1.5	3.9	.098	2.6	26	.099	2.6	26	.15	2.6	17
$8(\frac{1}{512})$	.37	1.5	4.0	.095	2.6	28	.096	2.6	28	.15	2.6	18
$a(u,v) =$	$\int u\overline{v}$			$\int \partial_1 u \overline{\partial_1 v}$			$\int (\partial_1 u + \partial_2 u)\cdot(\overline{\partial_1 v} + \overline{\partial_2 v})$			$\int \partial_2 u \overline{\partial_2 v}$		

TABLE 5. Condition numbers using the preconditioner given by Algorithm 4.2.

from Table 5 appear to be worst cases. For example for the Poisson equation the condition numbers were always less than 11.

APPENDIX A. L_2-STABILITY OF THE DECOMPOSITION $\mathcal{M}_J = \bigoplus_{k=0}^J \mathcal{V}_k$ IN CASE OF LINEAR FINITE ELEMENTS

In this appendix, for the space decomposition $\mathcal{M}_J = \bigoplus_{k=0}^J \mathcal{V}_k$ from Example 3.6, we will prove that

$$\kappa(\{\mathcal{M}_J; (\,,\,)_{L_2}\}; \textstyle\sum_{k=0}^J \{\mathcal{V}_k; (\,,\,)_{L_2}\}) \lesssim 1, \tag{A.1}$$

that is, estimate (3.9).

Let $\mathcal{E} = (\mathcal{E}_{kl})_{0\leq k,l\leq J}$ be the Gram matrix defined by

$$\mathcal{E}_{kl} = \cos\phi(\mathcal{V}_k, \mathcal{V}_l) := \sup_{0\neq v_k\in\mathcal{V}_k, 0\neq v_l\in\mathcal{V}_l} \frac{|(v_k, v_l)_{L_2}|}{\|v_k\|_{L_2}\|v_l\|_{L_2}}.$$

Let $v_k \in \mathcal{V}_k$ and $u = \sum_{k=0}^J v_k$. From

$$\|u\|_{L_2}^2 = (\sum_{k=0}^J v_k, \sum_{l=0}^J v_l)_{L_2} = \sum_{k=0}^J \|v_k\|_{L_2}^2 + \sum_{0=k\neq l}^J (v_k, v_l)_{L_2}$$

and

$$|\sum_{0=k\neq l}^J (v_k, v_l)_{L_2}| \leq \sum_{0=k\neq l}^J \cos\phi(\mathcal{V}_k, \mathcal{V}_l)\|v_k\|_{L_2}\|v_l\|_{L_2} \leq \rho(\mathcal{E} - I)\sum_{k=0}^J \|v_k\|_{L_2}^2,$$

we conclude that for proving (A.1) it is sufficient to show that $\rho(\mathcal{E} - I) < 1$ uniformly in J. Below, for $k > l$ we will estimate $\cos\phi(\mathcal{V}_k, \mathcal{M}_l)$ which is clearly greater of equal to $\cos\phi(\mathcal{V}_k, \mathcal{V}_l)$.

At this place, we note that the technique of proving stability of a decomposition by estimating the entries of the Gram matrix is restricted to some model cases. More sophisticated general criteria for stability of a direct sum space decomposition are given in [Dah94]. Yet, an assumption made in [Dah94] reformulated for our problem is that $\|Z_k\|_{L_2\leftarrow L_2} \lesssim 1$

((3.8)). Again for our problem, we did not succeed to check this assumption except than by proving (A.1).

For $u_m \in \mathcal{M}_m$, let U_m be its representation with respect to the $(\cdot,\cdot)_{\mathcal{M}_m}$-orthonormal nodal basis $\{\phi_{x,x} : x \in \Omega_m\}$ of $\mathcal{M}_m$, where $\Omega_m = h_m\mathbf{Z}^2 \cap \Omega$ and $h_m = 2^{-(m+1)}$. The representation of $v_m \in \mathcal{V}_m$ with respect to the basis $\{\psi_{m,x} : x \in \Omega_m \backslash \Omega_{m-1}\}$ of $\mathcal{V}_m$ (cf. (3.11)) is denoted by V_m. We will identify the coordinate vectors U_m and V_m with grid functions on Ω_m and $\Omega_m \backslash \Omega_{m-1}$ respectively, that is, with elements of the spaces $\ell_2(\Omega_m)$ and $\ell_2(\Omega_m \backslash \Omega_{m-1})$ equipped with Euclidean scalar products. The trivial embeddings $\mathcal{M}_{m-1} \to \mathcal{M}_m$ and $\mathcal{V}_m \to \mathcal{M}_m$ are denoted by p^0 and p^1 respectively. From subsection 3.2 recall that $\operatorname{Im} p^1 \perp \operatorname{Im} p^0$.

We define the mass matrix M_m by $<M_m U_m^{(1)}, U_m^{(2)}> = (u_m^{(1)}, u_m^{(2)})_{L_2}$ ($u_m^{(1)}$, $u_m^{(2)} \in \mathcal{M}_m$). This matrix M_m can be represented by the stencil $\frac{1}{12}\begin{bmatrix} & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & \end{bmatrix}$. For $k > l$ we obtain

$$\begin{aligned}
\cos\phi(\mathcal{V}_k, \mathcal{M}_l) &= \sup_{0 \neq v_k \in \mathcal{V}_k, 0 \neq u_l \in \mathcal{M}_l} \frac{|(v_k, u_l)_{L_2}|}{\|v_k\|_{L_2}\|u_l\|_{L_2}} \\
&= \sup_{0 \neq v_k \in \mathcal{V}_k, 0 \neq u_l \in \mathcal{M}_l} \frac{|<M_k p^1 V_k, \overbrace{p^0 \cdots p^0}^{(k-l)\times} U_l>|}{<M_k p^1 V_k, p^1 V_k>^{\frac{1}{2}} <M_l U_l, U_l>^{\frac{1}{2}}} \\
&= \sup_{0 \neq v_k \in \mathcal{V}_k, 0 \neq u_l \in \mathcal{M}_l} \frac{|<(M_k - \lambda I) p^1 V_k, \overbrace{p^0 \cdots p^0}^{(k-l)\times} U_l>|}{<M_k p^1 V_k, p^1 V_k>^{\frac{1}{2}} <M_l U_l, U_l>^{\frac{1}{2}}} \qquad (\lambda \in \mathbf{R}) \\
&\leq \sup_{0 \neq u_k \in \mathcal{M}_k, 0 \neq u_l \in \mathcal{M}_l} \frac{|<(M_k - \lambda I) U_k, \overbrace{p^0 \cdots p^0}^{(k-l)\times} U_l>|}{<M_k U_k, U_k>^{\frac{1}{2}} <M_l U_l, U_l>^{\frac{1}{2}}}.
\end{aligned}$$

Since M_k is given by a compact 7 point stencil, the last expression is less or equal to the expression we would have found if M_k, p^0 and M_l were defined on underlying infinite grids. [Without the trick of removing p^1 this would not be true since $M_k p^1$ has "boundary perturbations". Note that by $\mathcal{M}_l \subset \mathcal{M}_k$ the last expression without the term λI would be equal to one. So also the preceding inclusion of λI is essential.] By replacing U_k by $M_k^{-\frac{1}{2}} U_k$ and U_l by $M_l^{-\frac{1}{2}} U_l$, we obtain that

$$\text{(A.2)} \qquad \cos\phi(\mathcal{V}_k, \mathcal{M}_l) \leq c_{k-l} := \inf_\lambda \|M_k^{-\frac{1}{2}} (M_k - \lambda I) \overbrace{p^0 \cdots p^0}^{(k-l)\times} M_l^{-\frac{1}{2}}\|,$$

where we now use the symbols M_k, p^0 and M_l to denote operators on underlying infinite grids. We conclude that for obtaining (A.1), it is sufficient to prove that $2\sum_{n>0} c_n < 1$.

The right hand side of (A.2) can now be analyzed by applying Fourier transformations. In the following we recall some well-known results. Let d be the space dimension and $q = \frac{h_{m-1}}{h_m}$. In our applications both d and q will be equal to two. We equip the spaces

$\ell_2(h_m\mathbf{Z}^d)$ and $L_2(\mathbf{T}^d)$ ($\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$) with scalar products

$$<U_m, W_m> = \sum_{x\in h_m\mathbf{Z}^d} U_m(x)\overline{W_m(x)}$$

and

$$(u,w)_{L_2} = \int_{(-\pi,\pi)^d} u(\xi)\overline{w(\xi)}d\xi$$

respectively. Then the Fourier transform $\mathcal{F}_m : \ell_2(h_m\mathbf{Z}^d) \to L_2(\mathbf{T}^d)$ defined by

$$(\mathcal{F}_m U_m)(\xi) = (2\pi)^{-\frac{d}{2}} \sum_{j\in\mathbf{Z}^d} U_m(jh_m)e^{-i<j,\xi>}$$

is an isomorphism and

$$(\mathcal{F}_m^* u)(jh_m)(= (\mathcal{F}_m^{-1}u)(jh_m)) = (2\pi)^{-\frac{d}{2}} \int_{(-\pi,\pi]^d} u(\xi)e^{i<j,\xi>}d\xi.$$

For the *shift* operator $T_\gamma : \ell_2(h_m\mathbf{Z}^d) \to \ell_2(h_m\mathbf{Z}^d)$ defined by $T_\gamma U_m(x) = U_m(x + h_m\gamma)$ ($\gamma \in \mathbf{Z}^d$), it is easily verified that

$$(\mathcal{F}_m T_\gamma \mathcal{F}_m^* u)(\xi) = e^{i<\gamma,\xi>}u(\xi).$$

For the *flat prolongation* operator $\overline{p} : \ell_2(h_{m-1}\mathbf{Z}^d) \to \ell_2(h_m\mathbf{Z}^d)$ defined by

$$(\overline{p}U_{m-1})(jh_m) = \begin{cases} U_{m-1}(jh_m) & j \in q\mathbf{Z}^d \\ 0 & \text{elsewhere} \end{cases}$$

we have

$$(\mathcal{F}_m\overline{p}\mathcal{F}_{m-1}^* u)(\xi^{(\alpha)}) = u(\xi) \qquad (\alpha \in \{0,\dots,q-1\}^d)$$

where $\xi^{(\alpha)} := \frac{\xi+2\pi\alpha}{q}$. Finally, the *trivial restriction* operator $\overline{r} : \ell_2(h_m\mathbf{Z}^d) \to \ell_2(h_{m-1}\mathbf{Z}^d)$ defined by $(\overline{r}U_m)(jh_{m-1}) = U_m(jqh_m)$ is the adjoint of $\overline{p}$ and it satisfies

$$(\mathcal{F}_{m-1}\overline{r}\mathcal{F}_m^* u)(\xi) = q^{-d} \sum_{\alpha\in\{0,\dots,q-1\}^d} u(\xi^{(\alpha)}).$$

It will appear that any of the operators that we will encounter mapping functions on an infinite grid to functions on an infinite grid possibly with different mesh widths can be written as a linear combination or a composition of shifts, flat prolongations and trivial restrictions. As a consequence the norm of such an operator can be expressed as the supremum over the "frequencies" ξ of a, generally matrix valued "symbol" that represents the operator in the frequency domain. For example, from $\|U_{m-1}\|^2 = \int_{(-\pi,\pi]^d} |(\mathcal{F}_{m-1}U_{m-1})(\xi)|^2 d\xi$

and

$$\begin{aligned}
\|\overline{p}U_{m-1}\|^2 &= \int_{(-\pi,\pi]^d} |(\mathcal{F}_m\overline{p}U_{m-1})(\eta)|^2 d\eta \\
&= \sum_{\alpha\in\{0,\dots,q-1\}^d} \int_{(-\pi,\pi]^d} q^{-d}|(\mathcal{F}_m\overline{p}U_{m-1})(\xi^{(\alpha)})|^2 d\xi \\
&= \sum_{\alpha\in\{0,\dots,q-1\}^d} \int_{(-\pi,\pi]^d} q^{-d}|(\mathcal{F}_{m-1}U_{m-1})(\xi)|^2 d\xi \\
&= \int_{(-\pi,\pi]^d} \|q^{-\frac{d}{2}}[1\cdots 1]^T(\mathcal{F}_{m-1}U_{m-1})(\xi)\|^2 d\xi
\end{aligned}$$

we see that $\|\overline{p}\| = \sup_\xi \|q^{-\frac{d}{2}}[1\cdots1]^T \in \mathbf{C}^{1\times q^d}\|$ $(=1)$ where we call $q^{-\frac{d}{2}}[1\cdots1]^T$ the symbol of $\overline{p}$. In a similar way, we find that $q^{-\frac{d}{2}}[1\cdots1] \in \mathbf{C}^{q^d\times 1}$ is the symbol of $\overline{r}$ and that $\xi \mapsto e^{i<\gamma,\xi>}$ is the symbol of T^γ.

To show the structure of symbols of more general operators we consider an example. Our operator $p^0 = \frac{1}{4}\left]\begin{matrix} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \end{matrix}\right[: \ell_2(h_{m-1}\mathbf{Z}^2) \to \ell_2(h_m\mathbf{Z}^2)$ $(q = d = 2)$ can be written as $S\circ\overline{p}$, where $S = \frac{1}{4}\left[\begin{matrix} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \end{matrix}\right]$ represents a linear combination of shifts $\ell_2(h_m\mathbf{Z}^2) \to \ell_2(h_m\mathbf{Z}^2)$ with symbol $s(\xi) = \frac{1}{2}(1+\cos(\xi_1)+\cos(\xi_2)+\cos(\xi_1+\xi_2))$. Now the symbol of p^0 can be found to be $\xi \mapsto \begin{bmatrix} s(\xi^{(11)}) & & & \\ & s(\xi^{(01)}) & & \\ & & s(\xi^{(10)}) & \\ & & & s(\xi^{(00)}) \end{bmatrix} \circ \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, which implies that $\|p^0\| = \sup_\xi \sqrt{\sum_{\alpha\in\{0,1\}^2} p(\xi^{(\alpha)})^2}$ where $p(\xi) = \frac{1}{4}(1+\cos(\xi_1)+\cos(\xi_2)+\cos(\xi_1+\xi_2))$.

Now we return to our problem of estimating c_n (cf. (A.2)). The mass-matrix M_m has symbol $m(\xi) = \frac{1}{2} + \frac{1}{6}(\cos(\xi_1)+\cos(\xi_2)+\cos(\xi_1+\xi_2))$. Using the preceding considerations one verifies that

$$c_0 = \inf_\lambda \sup_\xi \sqrt{\frac{(m(\xi)-\lambda)^2}{m(\xi)^2}}, \qquad c_1 = \inf_\lambda \sup_\xi \sqrt{\sum_{\alpha\in\{0,1\}^2} \frac{(m(\xi^{(\alpha)})-\lambda)^2 p(\xi^{(\alpha)})^2}{m(\xi^{(\alpha)})m(\xi)}}$$

and generally, that

$$c_n = \inf_\lambda \sup_\xi \sqrt{\frac{q(\xi,\lambda,n)}{m(\xi)}},$$

where

$$q(\xi,\lambda,n) = \begin{cases} \dfrac{(m(\xi)-\lambda)^2}{m(\xi)} & n = 0 \\ \displaystyle\sum_{\alpha\in\{0,1\}^2} q(\xi^{(\alpha)},\lambda,n-1)p(\xi^{(\alpha)})^2 & n > 0 \end{cases}.$$

Now our approach will be to find an upper bound for c_n analytically, which is possibly crude but useful for large n, and to compute c_n numerically for small n. We start with the first task. For

$$\tilde{q}(\xi, n) := \frac{q(\xi, n, 1)}{(m(\xi)-1)^2},$$

we get the recursion

$$\tilde{q}(\xi, n) = \sum_{\alpha \in \{0,1\}^2} \tilde{q}(\xi^{(\alpha)}, n-1) p(\xi^{(\alpha)})^2 \frac{(m(\xi^{(\alpha)})-1)^2}{(m(\xi)-1)^2} \qquad (n > 0).$$

Some calculations show that $\inf_\xi |m(\xi)| = \frac{1}{4}$ and

$$\sup_\xi \left| \sum_{\alpha \in \{0,1\}^2} p(\xi^{(\alpha)})^2 \frac{(m(\xi^{(\alpha)})-1)^2}{(m(\xi)-1)^2} \right| = \tfrac{7}{48}.$$

Taking $\lambda = 1$ and using $\tilde{q}(\xi, 0) = \frac{1}{m(\xi)}$, we get

(A.3)
$$c_n \leq \sup_\xi \sqrt{\frac{(m(\xi)-1)^2 \tilde{q}(\xi, n)}{m(\xi)}} \leq \sup_\xi \sqrt{\frac{(m(\xi)-1)^2}{m(\xi)}} \left(\sqrt{\tfrac{7}{48}}\right)^n \sup_\xi \sqrt{\frac{1}{m(\xi)}} = 3 \left(\sqrt{\tfrac{7}{48}}\right)^n.$$

Our numerical computations yielded the sharper bounds

$$c_1 \leq .3200 \;\; (\lambda = .68), \qquad c_2 \leq .1821 \;\; (\lambda = .821).$$

Unfortunately, recalling that we had to prove that $2\sum_{n>0} c_n < 1$, already $2(.3200 + .1821)$ is larger than 1.

To find a way out, we consider the cosine of the angle between $\mathcal{V}_{2k+1} + \mathcal{V}_{2k}$ and $\mathcal{V}_{2l+1} + \mathcal{V}_{2l}$ $(k > l)$. Clearly, L_2-stability of the decomposition $\mathcal{M}_J = \sum_{k=0}^{(J-1)/2} (\mathcal{V}_{2k+1} + \mathcal{V}_{2k})$ implies L_2-stability of $\mathcal{M}_J = \sum_{k=0}^{J} \mathcal{V}_k$ (the case of even J causes no additional troubles). Let $v_{2k+1} \in \mathcal{V}_{2k+1}$, $v_{2k} \in \mathcal{V}_{2k}$ and $u_{2l+1} \in \mathcal{M}_{2l+1}$. Put $x = \begin{bmatrix} \|v_{2k+1}\| & \|v_{2k}\| \end{bmatrix}^{\mathrm{T}}$ and $A = \begin{bmatrix} 1 & -c_1 \\ -c_1 & 1 \end{bmatrix}$. Then

$$\|v_{2k+1} + v_{2k}\|_{L_2}^2 \geq \|v_{2k+1}\|_{L_2}^2 + \|v_{2k}\|_{L_2}^2 - 2c_1 \|v_{2k+1}\|_{L_2} \|v_{2k}\|_{L_2} = <Ax, x>$$

and so with $c = \begin{bmatrix} c_{2k-2l} & c_{2k-2l-1} \end{bmatrix}^{\mathrm{T}}$,

$$\begin{aligned} \frac{(v_{2k+1} + v_{2k}, u_{2l+1})_{L_2}}{\|v_{2k+1} + v_{2k}\|_{L_2} \|u_{2l+1}\|_{L_2}} &= \frac{\|v_{2k+1}\|_{L_2} \frac{(v_{2k+1}, u_{2l+1})_{L_2}}{\|v_{2k+1}\|_{L_2} \|u_{2l+1}\|_{L_2}} + \|v_{2k}\|_{L_2} \frac{(v_{2k}, u_{2l+1})_{L_2}}{\|v_{2k}\|_{L_2} \|u_{2l+1}\|_{L_2}}}{\|v_{2k+1} + v_{2k}\|_{L_2}} \\ &\leq \frac{<x, c>}{<Ax, x>^{\frac{1}{2}}} \\ &\leq <A^{-1}c, c>^{\frac{1}{2}}. \end{aligned}$$

Using $c_n \leq 3\left(\sqrt{\frac{7}{48}}\right)^n$ and $c_1 \leq .3200$, we obtain

$$\cos\phi(\mathcal{V}_{2k+1} + \mathcal{V}_{2k}, \mathcal{V}_{2l+1} + \mathcal{V}_{2l}) \leq \cos\phi(\mathcal{V}_{2k+1} + \mathcal{V}_{2k}, \mathcal{M}_{2l+1}) < d\left(\sqrt{\tfrac{7}{48}}\right)^{2k-2l-1},$$

where $d = 3.734$. [Note that by $\frac{d(\sqrt{\frac{7}{48}})^{2n+1}}{3(\sqrt{\frac{7}{48}})^{2n+1}+3(\sqrt{\frac{7}{48}})^{2n+2}} \approx .9$, we already gained something compared to the estimate (A.3).]

Our next step is to estimate $\cos\phi(\mathcal{V}_{2k+1}+\mathcal{V}_{2k}, \mathcal{M}_{2l+1})$ for $k \in \{l+1, l+2\}$ directly, that is, not by using estimates of $\cos\phi(\mathcal{V}_{k'}, \mathcal{M}_{l'})$. We have

$$\cos\phi(\mathcal{V}_{2k+1}+\mathcal{V}_{2k}, \mathcal{M}_{2l+1}) = \sup_{\substack{0 \neq v_{2k} \in \mathcal{V}_{2k} \\ 0 \neq v_{2k+1} \in \mathcal{V}_{2k+1} \\ 0 \neq u_{2l+1} \in \mathcal{M}_{2l+1}}} \frac{(v_{2k+1}+v_{2k}, u_{2l+1})_{L_2}}{\|v_{2k+1}+v_{2k}\|_{L_2}\|u_{2l+1}\|_{L_2}}$$

$$= \sup_{\substack{0 \neq v_{2k} \in \mathcal{V}_{2k} \\ 0 \neq v_{2k+1} \in \mathcal{V}_{2k+1} \\ 0 \neq u_{2l+1} \in \mathcal{M}_{2l+1}}} \frac{< \begin{bmatrix} M_{2k+1} & 0 \\ 0 & M_{2k} \end{bmatrix} \begin{bmatrix} p^1 V_{2k+1} \\ p^1 V_{2k} \end{bmatrix}, \begin{bmatrix} p^0 \\ I \end{bmatrix} \overbrace{p^0 \cdots p^0}^{(2k-2l-1)\times} U_{2l+1} >}{< \begin{bmatrix} M_{2k+1} & M_{2k+1}p^0 \\ r^0 M_{2k+1} & M_{2k} \end{bmatrix} \begin{bmatrix} p^1 V_{2k+1} \\ p^1 V_{2k} \end{bmatrix}, \begin{bmatrix} p^1 V_{2k+1} \\ p^1 V_{2k} \end{bmatrix} >^{\frac{1}{2}} < M_{2l+1} U_{2l+1}, U_{2l+1} >^{\frac{1}{2}}}$$

$$\leq \sup_{\substack{0 \neq u_{2k} \in \mathcal{M}_{2k} \\ 0 \neq u_{2k+1} \in \mathcal{M}_{2k+1} \\ 0 \neq u_{2l+1} \in \mathcal{M}_{2l+1}}} \frac{< \begin{bmatrix} U_{2k+1} \\ U_{2k} \end{bmatrix}, \begin{bmatrix} (M_{2k+1}-\lambda_2 I)p^0 \\ M_{2k}-\lambda_3 I \end{bmatrix} \overbrace{p^0 \cdots p^0}^{(2k-2l-1)\times} U_{2l+1} >}{< \begin{bmatrix} M_{2k+1} & (M_{2k+1}-\lambda_1 I)p^0 \\ r^0(M_{2k+1}-\lambda_1 I) & M_{2k} \end{bmatrix} \begin{bmatrix} U_{2k+1} \\ U_{2k} \end{bmatrix}, \begin{bmatrix} U_{2k+1} \\ U_{2k} \end{bmatrix} >^{\frac{1}{2}} < M_{2l+1} U_{2l+1}, U_{2l+1} >^{\frac{1}{2}}}$$

by $\operatorname{Im} p^1 \perp \operatorname{Im} p^0$ ($\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}$). Again, the last expression is less or equal to the expression we would have found if all operators were defined on underlying infinite grids. We conclude that

$$\cos\phi(\mathcal{V}_{2k+1}+\mathcal{V}_{2k}, \mathcal{M}_{2l+1}) \leq$$

$$\tilde{c}_{k-l} := \inf_{\lambda_1,\lambda_2,\lambda_3} \| \begin{bmatrix} M_{2k+1} & (M_{2k+1}-\lambda_1 I)p^0 \\ r^0(M_{2k+1}-\lambda_1 I) & M_{2k} \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} (M_{2k+1}-\lambda_2 I)p^0 \\ M_{2k}-\lambda_3 I \end{bmatrix} \overbrace{p^0 \cdots p^0}^{(2k-2l-1)\times} M_{2l+1}^{-\frac{1}{2}} \|,$$

where now M_{2k+1}, M_{2k}, M_{2l+1}, p^0 and r^0 denote operators on underlying infinite grids.

We estimate $\tilde{c}_{k-l}$ by applying Fourier transformations. Let $t(\xi, 0, \lambda_1, \lambda_2, \lambda_3)$ be the sum of the squares of the elements of the 5×1 matrix

$$\begin{bmatrix} \begin{bmatrix} m(\xi^{(11)}) & & \\ & \ddots & \\ & & m(\xi^{(00)}) \end{bmatrix} & \begin{bmatrix} (m(\xi^{(11)})-\lambda_1)p(\xi^{(11)}) \\ \vdots \\ (m(\xi^{(00)})-\lambda_1)p(\xi^{(00)}) \end{bmatrix} \\ \begin{bmatrix} \quad \end{bmatrix}^* & m(\xi) \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} (m(\xi^{(11)})-\lambda_2)p(\xi^{(11)}) \\ \vdots \\ (m(\xi^{(00)})-\lambda_2)p(\xi^{(00)}) \\ m(\xi)-\lambda_3 \end{bmatrix}$$

and define $t(\xi, n, \lambda_1, \lambda_2, \lambda_3) = \sum_{\alpha \in \{0,1\}^2} t(\xi^{(\alpha)}, n-1, \lambda_1, \lambda_2, \lambda_3) p(\xi^{(\alpha)})^2$ ($n > 0$), then

$$\tilde{c}_n = \inf_{\lambda_1,\lambda_2,\lambda_3} \sup_{\xi} \sqrt{\frac{t(\xi, n, \lambda_1, \lambda_2, \lambda_3)}{m(\xi)}}.$$

Numerically, we found

$$\tilde{c}_1 \leq .3373 \quad (\lambda_1 = .425,\ \lambda_2 = .885,\ \lambda_3 = .675)$$

(which is only a little larger than .3200) and

$$\tilde{c}_2 \leq .0925 \quad (\lambda_1 = .535,\ \lambda_2 = .99,\ \lambda_3 = .96).$$

The L_2-stability of $\mathcal{M}_J = \sum_{k=0}^{(J-1)/2}(\mathcal{V}_{2k+1} + \mathcal{V}_{2k})$ and thus of $\mathcal{M}_J = \sum_{k=0}^{J} \mathcal{V}_k$ now follows from

$$2(.3373 + .0925 + d\sum_{n=2}^{\infty}\left(\sqrt{\tfrac{7}{48}}\right)^{2n+1}) < .9304 < 1.$$

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