T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 13 Tuesday, 23 June 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

Programming (continued)

L_2-error and H^1-error

62 Write a function

```
double calcElErrorL2 (const Point2D& p0, const Point2D& p1,
                      const Point2D& p2, ScalarField exact,
                      double v0, double v1, double v2);
```

that approximates the element L^2-error $\|v - v_h\|_{L^2(\delta_r)}$, where `exact`$=v$ and $v_h(x_{\delta_r}(\xi)) = \sum_{\alpha\in A} v^{(r,\alpha)}\, p^{(\alpha)}(\xi)$ with `v0`$=v^{(r,1)}$ etc.

Hint: Use the quadrature rule from Exercise 34 to approximate

$$\|v - v_h\|^2_{L^2(\delta_r)} \;=\; \int_{\delta_r} |v(x) - v_h(x)|^2\, dx \;=\; \int_{\Delta} |v(x_{\delta_r}(\xi)) - v_h(x_{\delta_r}(\xi))|^2\, |\det J_{\delta_r}|\, d\xi$$

63 Write a function

```
double calcElErrorH1 (const Point2D& p0, const Point2D& p1,
                      const Point2D& p2,
                      ScalarField Dx1exact, ScalarField Dx2exact,
                      double v0, double v1, double v2);
```

that approximates the element H^1-error $|Dv - \nabla v_h|_{L^2(\delta_r)}$, where $Dv = \nabla v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2})^T$, with `Dx1exact`$=\frac{\partial v}{\partial x_1}$, `Dx2exact`$=\frac{\partial v}{\partial x_2}$ and $v_h(x_{\delta_r}(\xi)) = \sum_{\alpha\in A} v^{(r,\alpha)}\, p^{(\alpha)}(\xi)$ with `v0`$=v^{(r,1)}$ etc.

Hint: Use the quadrature rule from Exercise 34 to approximate

$$|v - v_h|^2_{H^1(\delta_r)} \;=\; \int_{\delta_r} |Dv(x) - \nabla_x v_h(x)|^2\, dx \;=\; \int_{\Delta} |Dv(x_{\delta_r}(\xi)) - J_r^{-T}\nabla_\xi v_h(x_{\delta_r}(\xi))|^2\, |\det J_{\delta_r}|\, d\xi$$

$\boxed{64}$ Write a function

```
double calcErrorL2 (const Mesh& mesh, ScalarField exact,
                    const Vector& solution);
```

that approximates the global L^2-error $\|v - v_h\|_{L^2(\Omega)}$, where `exact`$=v$ and `solution`$=v_h$.

Hint: use `calcElErrorL2` in a loop over all elements.

Show that $u(x_1, x_2) = \frac{1}{4}\cos(2\pi\, x_1)\cos(4\pi\, x_2)$ is the unique solution of (3.22) (see Tutorial 07, Exercise $\boxed{39}$). Compute $\|u - u_h\|_{L^2(\Omega)}$ for each finite element solution u_h from Exercise $\boxed{39}$ for the different meshes.

$\boxed{65}$ Write a function

```
double calcErrorH1 (const Mesh& mesh, ScalarField exact,
                    ScalarField Dx1exact, ScalarField Dx2exact,
                    const Vector& solution);
```

that approximates the global H^1-error $\|v - v_h\|_{H^1(\Omega)}$, where `exact`$=v$, `Dx1exact`$=\frac{\partial v}{\partial x_1}$, `Dx2exact`$=\frac{\partial v}{\partial x_2}$ and `solution`$=v_h$.

Hint: use `calcElErrorL2` and `calcElErrorH1` in a loop over all elements.

Compute $\|u - u_h\|_{H^1(\Omega)}$ for each finite element solution u_h from Exercise $\boxed{39}$ for the different meshes.

The CHIP-Problem

Recall the CHIP-Problem from the lecture (T08a, T08b, T09)!

$\boxed{66}$ Prepare the initial mesh for the CHIP problem as proposed on T09 in your mesh-format, taking care of the appropriate boundary conditions.

Hint: If possible use symmetric reduction.

$\boxed{67}$ Modify your functions from $\boxed{33}$, $\boxed{35}$ and $\boxed{36}$, such that you can assemble the stiffness matrix `K` according to the bilinear form

$$a(u, v) \;=\; \int_\Omega \lambda(x)\nabla u(x)\cdot\nabla v(x) + a(x)u(x)\,v(x)\,dx,$$

where $\lambda(x)$ and $a(x)$ are given coefficient functions.

$\boxed{68}$ Solve the finite element system corresponding to the CHIP problem on T08a with the parameter setting of T08b for the initial mesh of $\boxed{66}$. Solve the same system for uniformly refined meshes with $h/h_0 = 2, 3, 8, 16$ and visualize the solution.

Hint: For incorporating the BC, use the following order: First natural BC, than essential BC.

A posteriori error estimates

69* Implement the residual error estimator for the CHIP-problem as derived in Exercise 61.

70* Compute the residual error for the CHIP-problem for uniformly refined meshes with $h/h_0 = 2, 3, 8, 16$ and visualize the error on each element!