T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 11 Tuesday, 9 June 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

3.6 Variational Crimes

Consider the one-dimensional BVP to find $u \in V_g = V_0 = H_0^1(0,1)$:

$$\int_0^1 \lambda(x)\, u'(x)\, v'(x) dx \;=\; \int_0^1 f(x)\, v(x)\, dx \qquad \forall v \in V_0\,, \tag{3.36}$$

where $f \in L^2(0,\,1)$ and $\lambda \in L^\infty(0,\,1)$. We assume that there exists positive constants $\underline{\lambda}$ and $\overline{\lambda}$ such that $0 < \underline{\lambda} \le \lambda(x) \le \overline{\lambda} \quad \forall x \in (0,\,1)$ a.e. Let us consider a finite element discretization with continuous linear finite elements ($\mathcal{F}(\Delta) = \mathcal{P}_1$) on an equidistant grid ($x_i = ih,\ i = \overline{0, n+1}, h = 1/(n+1),\ \delta_r = (x_{r-1}, x_r), r = \overline{1, n+1}$). Now we approximate the bilinear form $a(\cdot,\cdot)$ and the linear form $\langle F, \cdot\rangle$ defined in (3.36) using numerical integration, namely the *midpoint rule*:

$$a_h(u_h,\, v_h) \;=\; \sum_{r=1}^{n+1} h\, \lambda(x_r^\star)\, u_h'(x_r^\star)\, v_h'(x_r^\star) \quad \text{and} \quad \langle F_h,\, v_h\rangle_h \;=\; \sum_{r=1}^{n+1} h\, f(x_r^\star)\, v_h(x_r^\star)\,, \tag{3.37}$$

where $x_r^\star = x_{\delta_r}(\frac{1}{2}) = x_{r-1} + \frac{1}{2}h$. To ensure that these expressions are well-defined we assume (for simplicity) that

$$\lambda,\, f \in W_\infty^1(0,\,1)\,.$$

Let $\widetilde{u}_h \in V_{0h}$ be such that $a_h(\widetilde{u}_h,\, v_h) = \langle F_h,\, v_h\rangle_h$ for all $v_h \in V_{0h}$. We are interested whether the error $\|u - \widetilde{u}_h\|_{H^1(0,1)}$ obeys the same asymptotics with respect to h than if we compute $a(\cdot,\cdot)$ and $\langle F,\,\cdot\rangle$ exactly. This investigation will be done using Strang's first lemma. Throughout this tutorial, we choose $\|\cdot\|_{V_0} := |\cdot|_{H^1(0,1)}$ that is a norm in $V_0 = H_0^1(0,1)$.

57 Show that the bilinear and linear forms above fulfill the standard assumptions (33) (the assumptions of Lax-Milgram) and the additional assumption (34) from the lecture notes. The latter is called *uniform ellipticity* of the discrete bilinear form $a_h(\cdot,\cdot)$, i.e. there exists a positive constant $\mu_3 \neq \mu_3(h)$ such that

$$a_h(v_h,\, v_h) \;\ge\; \mu_3\, \|v_h\|_{V_0}^2 \quad \forall v_h \in V_{0h}. \tag{34}$$

Hint: For the uniform ellipticity, use that $\lambda \ge \underline{\lambda} = \text{const} > 0$ and that the midpoint rule is exact for some (which?) polynomials.

We can now apply Strang's first lemma which yields

$$\|u - \widetilde{u}_h\|_{V_0} \;\le\; c\left\{ \inf_{v_h \in V_0} \left[\|u - v_h\|_{V_0} + \sup_{w_h \in V_{0h}} \frac{|a(v_h,\, w_h) - a_h(v_h,\, w_h)|}{\|w_h\|_{V_0}} \right] + \right. \tag{3.38}$$

$$\left. + \sup_{w_h \in V_{0h}} \frac{|\langle F,\, w_h\rangle - \langle F_h,\, w_h\rangle_h|}{\|w_h\|_{V_0}} \right\}$$

$\boxed{58}$ For $\varphi \in H^1(\delta_r)$, prove that

$$\Big| \int_{\delta_r} \varphi(x)\, dx - h\, \varphi(x_r^*) \Big| \;\le\; c\, h^{3/2}\, |\varphi|_{H^1(\delta_r)}\,,$$

similarly to the exercises in Tutorial 07. Then set $\varphi = f\, w_h$ and show that

$$|\langle F,\, w_h\rangle - \langle F_h,\, w_h\rangle_h| \;\le\; c\, h\, \|f\|_{W^{1,\infty}(0,\,1)}\, \|w_h\|_{V_0}\,.$$

$\boxed{59}$ For $v_h,\, w_h \in V_{0h}$, prove that

$$|a(v_h,\, w_h) - a_h(v_h,\, w_h)| \;\le\; c\, h\, |\lambda|_{W^1_\infty(0,\,1)}\, \|v_h\|_{V_0}\, \|w_h\|_{V_0}\,.$$

Hint: Treat each element separately and use that v_h', w_h' are constant on each element, so that we are left with $| \int_{\delta_r} \lambda(x)\, dx - h\, \lambda(x_r^*)|$. To get an error bound for this term, use Bramble-Hilbert on the reference element.

$\boxed{60}$ Show that if $u \in H^2(0,\, 1)$ then

$$\|u - \widetilde{u}_h\|_{V_0} \;\le\; c\, h \left\{ |u|_{H^2(0,\,1)} + |\lambda|_{W^1_\infty(0,\,1)}\, \|u\|_{V_0} + \|f\|_{W^1_\infty(0,\,1)} \right\}.$$

Hint: Choose $v_h = u_h$ in (3.38), where $u_h \in V_{0h}$ is the finite element solution in the exact case, i.e.

$$a(u_h,\, v_h) \;=\; \langle F,\, v_h\rangle \qquad \forall v_h \in V_{0h}\,.$$

Show and use that

$$\|u_h\|_{V_0} \le c(\lambda)\, \|u\|_{V_0}\,.$$