T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 10 Tuesday, 26 May 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

3.4 Discretization Error Estimates

$\boxed{53}$ Show that for $d = 1$: $\Omega = (0, 1)$, $k = 1$: $\mathcal{S}(\Delta) = \mathcal{P}_1(\Delta)$, and $u(x) = x^2$ there holds

$$\inf_{v_h \in V_h} \int_0^1 |u'(x) - v_h'(x)|^2 dx = \frac{1}{3} h^2 \,, \tag{3.32}$$

where $V_h = \text{span}\{p^{(i)} : i = 0, 1, ..., n\}$ is defined using continuous affine linear finite elements on the mesh $0 = x^{(0)} < ... < x^{(i)} = ih < ... < x^{(n)} = 1$, $h = 1/n$.

$\boxed{54}$ Prove the completeness of the FE-spaces $\{V_h\}_{h \in \Theta}$ in $V = H^1(\Omega)$, i.e.,

$$\lim_{h \to 0} \inf_{v_h \in V_h} ||u - v_h|| = 0 \quad \forall u \in V, \tag{3.33}$$

under the assumptions 1 and 2 of the Approximation Theorem 3.6 , i.e.,

Assumption 1: The bounded Lipschitz domain Ω is provided by a regular triangulation (see Definition 3.3),

Assumption 2: $P_k(\Delta) \subset \mathcal{S}(\Delta) = \text{span}\{p^{(\alpha)} : \alpha \in A\}$.

3.5 Inverse-Inequalities

$\boxed{55}$ Compute the constant $c_A(\Delta)$ in the inequality

$$\max_{\xi \in \overline{\Delta}} \Big| \sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi) \Big| \;\le\; c_A(\Delta) \Big\| \sum_{\alpha \in A} v^{(r,\alpha)} p^{(\alpha)}(\xi) \Big\|_{L_2(\Delta)}, \tag{3.34}$$

used in the proof of Lemma 3.11, for linear triangular elements ($d = 2$, $k = 1$, $\mathcal{S}(\Delta) = \mathcal{P}_1$) !

$\boxed{56}$ Under the assumptions of Lemma 3.11, i.e. assumptions 1 of $\boxed{54}$ and $\dim S(\Delta) = |A_r| < \infty$, prove the inverse inequality

$$||v_h||_{L_\infty(\Omega)} \le c h^{-\frac{d}{p}} ||v_h||_{L_p(\Omega)} \quad \forall v_h \in V_h \tag{3.35}$$

for some given natural number p !