T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 08 Tuesday, 12 May 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

DEFINITION 3.3 *A family* $\{\tau_h\}_{h\in\mathcal{H}}$ *of triangulations* $\tau_h = \{\delta_r : r \in R_h\}$ *is called regular, if there exists positive and h-independent constants* $\underline{c}_1, \overline{c}_1, c_2, c_3 > 0$ *such that*

1. $\underline{c}_1 h^d \le |J_{\delta_r}| \le \overline{c}_1 h^d$
2. $\|J_{\delta_r}\| \le c_2 h$
3. $\|J_{\delta_r}^{-T}\| \le c_3 h^{-1}$

THEOREM 3.4 *Let* $a(\cdot,\cdot) : V \times V \to \mathrm{R}^n$ *be a bilinear form with* $V = H^1(\Omega)$ *and* $\|\cdot\| = \|\cdot\|_1$*, which is symmetric and fulfils the assumptions of Lax Milgram. Moreover, let the triangulation be regular in the sense of Definition 3.3.*
Then the following two statements are valid:

1. *There exists constants* $\underline{c}_E, \overline{c}_E > 0$*, independent of h such that*

$$\underline{c}_E h^d \le \lambda_{min}(K_h) \le \lambda_{max}(K_h) \le \overline{c}_E h^{d-2}$$

2. $\kappa(K_h) = \frac{\lambda_{max}(K_h)}{\lambda_{min}(K_h)} \le \frac{\overline{c}_E}{\underline{c}_E} h^{-2}$

3.3 Properties of the Finite Elements Equations

40 Prove that the inheritance identity

$$(K_h \underline{u}_h, \underline{v}_h) \;=\; a(u_h, v_h) \qquad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h}\ ! \tag{3.23}$$

is valid !

41 Show that the eigenvalue estimates in Theorem 3.4 are sharp with respect to the h-order by proving the following statement. There exist positive constants $\underline{c}'_E$ and $\overline{c}'_E$ independent of h satisfying the estimates

$$\lambda_{\min}(K_h) \;\le\; \underline{c}'_E h^d \qquad \text{and} \qquad \lambda_{\max}(K_h) \;\ge\; \overline{c}'_E h^{d-2}\,. \tag{3.24}$$

For simplicity, consider the 1D case ($d = 1$):

$$\begin{aligned} -u''(x) &= f(x) \qquad \forall x \in (0,1)\,, \\ u(0) = u(1) &= 0\,. \end{aligned}$$

42 Show that, for a regular triangulation according to Definition 3.3, there exist h-independent positive constants $\underline{c}_0$ and $\overline{c}_0$ satisfying the inequalities

$$\underline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \ \leq \ (M_h \underline{v}_h, \underline{v}_h) \ \leq \ \overline{c}_0 h^d (\underline{v}_h, \underline{v}_h) \tag{3.25}$$

for all $\underline{v}_h \in \mathbb{R}^{N_h}$, where M_h denotes the mass-matrix defined by the identity

$$(M_h \underline{u}_h, \underline{v}_h) := \int_\Omega u_h(x) v_h(x) dx \qquad \forall \underline{u}_h, \underline{v}_h \leftrightarrow u_h, v_h \in V_{0h} \tag{3.26}$$

The spectral inequalities (3.26) yield that the mass matrix M_h is well conditioned, i.e the spectral condition number $\mathrm{cond}_2(M_h)$ can be bounded by the h-independent constant $\overline{c}_0 / \underline{c}_0$.

43* Let $\lambda = \lambda_{\max}$ be the maximal eigenvalue of the generalized eigenvalue problem

$$K_h \underline{u}_h = \lambda M_h \underline{u}_h \tag{3.27}$$

and let $\lambda_r = \lambda_{r,\max}$ be the maximal eigenvalues of generalized eigenvalue problems

$$K_h^{(r)} \underline{u}_h^{(r)} = \lambda_r M_h^{(r)} \underline{u}_h^{(r)}, \tag{3.28}$$

where $K_h^{(r)}$ and $M_h^{(r)}$ denote the (local) element stiffness and mass matrices for element number $r = 1, 2, ..., R_h$, i. e., it holds

$$K_h \ = \ \sum_{r=1}^{R_h} C_r K_h^{(r)} C_r^T \qquad \text{and} \qquad M_h \ = \ \sum_{r=1}^{R_h} C_r M_h^{(r)} C_r^T \, .$$

Show the eigenvalue estimate

$$\lambda \ \leq \ \max_{r=1,2,...,R_h} \lambda_r \, . \tag{3.29}$$