T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 05 Tuesday, 21 April 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

3 Galerkin FEM

3.1 Galerkin-Ritz-Method

23 Let us consider the variational problem: Find $u \in V_g = V_0 = L_2(0,1)$:

$$\int_0^1 u(x)v(x)\,dx = \int_0^1 f(x)v(x)dx \quad \forall v \in V_0. \tag{14}$$

Solve this variational problem with the Galerkin-Method using the basis

$$V_{0h} = V_{0n} = \text{span}\{1, x, x^2, \dots, x^{n-1}\},$$

where the right-hand side is given as $f(x) = \cos(k\pi x)$, $k = l + 1$ and l is the last digit from your study code (Matrikelnummer) ! Compute the stiffness matrix K_h analytically and solve the linear system $K_h \underline{u}_h = \underline{f}_h$ numerically using the Gaussian elimination method ! Consider n to be $2, 4, 8, 10, 50, 100$!

3.2 Generation of the System of Finite Element Equations

24 Show that the integration rule

$$\int_\Delta f(\xi,\eta)d\xi d\eta \approx \frac{1}{2}\{\alpha_1 f(\xi_1,\eta_1) + \alpha_2 f(\xi_2,\eta_2) + \alpha_3 f(\xi_3,\eta_3)\} \tag{15}$$

integrates quadratic polynomials exactly, if the the weights α_i and the integration points (ξ_i, η_i) are choosen as follows: $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ und $(\xi_1, \eta_1) = (1/2, 0)$, $(\xi_2, \eta_2) = (1/2, 1/2)$, $(\xi_3, \eta_3) = (0, 1/2)$.

Hint: cf. also Exercise 17 !

25 Let us assume that $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ is a regular triangulation of the polygonally bounded Lipschitz domain $\overline{\Omega} = \cup_{r \in \mathbb{R}_h} \overline{\delta}_r \subset \mathbb{R}^2$ into triangles δ_r, and let $u \in H^2(\Omega)$. Let us now compute the integral

$$I(u) = \int_\Omega u(x)dx$$

by the quadrature rule

$$I_h(u) = \sum_{r \in \mathbb{R}_h} u(x_{\delta_r}(\xi^*))\,|\delta_r|,$$

where $x_{\delta_r}(\cdot)$ maps the unit triangle Δ onto δ_r, and $\xi^* = (1/3, 1/3)$. Show that

$$|I(u) - I_h(u)| \le ch^2 |u|_{H^2(\Omega)},$$

where c is some generic positive constant. Can you weaken the assumption that $u \in H^2(\Omega)$?

Hint: Use the mapping principle and the Bramble-Hilbert Lemma; cf. also Exercise 17 !

$\boxed{26}$ Show the inequality

$$\frac{1}{2}\sin\theta_r\, h_r^2 \le |J_{\delta_r}| \le \frac{\sqrt{3}}{2} h_r^2, \tag{16}$$

where h_r is the largest edge and θ_r the smallest angle of the triangle δ_r.