T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 03 Tuesday, 24 March 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

2 Tools from the Theory of Sobolev Spaces

12 Let us consider the function

$$u(x) = \begin{cases} 1, & -1 \leq x \leq 0 \\ -1, & 0 \leq x \leq -1 \end{cases},$$

Obviously, $u \in L_p(\Omega) \subset L_{loc}(\Omega) \subset D'(\Omega)$, but $u \notin C(\overline{\Omega})$!
Compute

1. $u' = \partial^1 u \in$?
2. $u'' = \partial^2 u \in$?
3. $u''' = \partial^3 u \in$?

in the distributive sense !

13 Show that

$$|g|_{H^{1/2}(\Gamma)} = \inf_{u \in H^1(\Omega):\, \gamma_0 u = g} |u|_{H^1(\Omega)} \tag{11}$$

defines a semi-norm in $H^{1/2}(\Gamma) := \gamma_0 H^1(\Omega)$ (check the norm semi-axioms), where $|u|_{H^1(\Omega)} := \|\nabla u\|_{L_2(\Omega)}$ denotes the standard semi-norm in $H^1(\Omega)$! The infimum in (11) is realized. Characterize the minimizer $u^* \in H^1(\Omega)$ as a unique solution of a variational problem !

14 Show that

$$\|u\|^*_{W_2^2(\Omega)} = \left(\int_{\Gamma_D} |u|^2 ds + \int_{\Gamma_D} |\partial_n u|^2 ds + |u|^2_{W_2^2(\Omega)} \right)^{1/2}$$

defines a new norm in $W_2^2(\Omega)$ that is equivalent to the standard norm

$$\|u\|_{W_2^2(\Omega)} = \left(\sum_{|\alpha| \leq 2} \int_\Omega |\partial^\alpha u|^2 dx \right)^{1/2} = \left(\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^2 dx + |u|^2_{W_2^2(\Omega)} \right)^{1/2},$$

where $\Gamma_D \subset \Gamma = \partial\Omega$ with $\text{meas}_{d-1}(\Gamma_D) > 0$, $\partial_n u(x) = \frac{\partial u}{\partial n}(x) = (\nabla u(x), n(x)) = \nabla u(x)^T n(x) = \nabla u(x) \bullet n(x)$, and $|u|_{W_2^2(\Omega)} = (\sum_{|\alpha|=2} \int_\Omega |\partial^\alpha u|^2 dx)^{1/2}$ denotes the standard semi-norm in $W_2^2(\Omega)$.

15 Show that there exists a positive constant $c_0 = \mathit{const} > 0$ and $c_1 = \mathit{const} > 0$ such that

$$\int_\Pi (u(x))^2 dx \le c_0 \left(\int_\Pi u(x)\, dx \right)^2 + c_1 \int_\Pi |\nabla u(x)|^2 dx \quad \forall u \in H^1(\Pi)$$

with $c_0 =?$ and $c_1 =?$, where $\Pi := \{x = (x_1, x_2) \in \mathbf{R}^2 : a_i < x_i < b_i,\ i = 1, 2\}$.

○ <u>Hint:</u> Use the representation

$$u(y_1, y_2) - u(x_1, x_2) = \int_{x_2}^{y_2} \frac{\partial u}{\partial \xi_2}(y_1, \xi_2) d\xi_2 + \int_{x_1}^{y_1} \frac{\partial u}{\partial \xi_1}(\xi_1, x_2) d\xi_1$$

16 Show that the inequalities

$$\inf_{q \in \mathbf{R}} \int_\Omega |u(x) - q|^2 dx \le c^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)$$

and

$$\int_\Omega |u(x)|^2 dx \le \frac{1}{|\Omega|} \left(\int_\Omega u(x)\, dx \right)^2 + c^2 \int_\Omega |\nabla u(x)|^2 dx \quad \forall u \in W_2^1(\Omega) = H^1(\Omega)$$

are equivalent !