T U T O R I A L

“Numerical Methods for the Solution of Elliptic Partial Differential Equations”

to the lecture

“Numerics of Elliptic Problems”

Tutorial 02 Tuesday, 17 March 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

1.2 The linear elasticity problem

07 Show that, for the BVP of the first type ($\Gamma_1 = \Gamma$) and for the mixed BVP ($\text{meas}_2(\Gamma_1) > 0$ and $\text{meas}_2(\Gamma_2) > 0$) of the linear elasticity, the following statements are true:

1. $a(.,.)$ is symmetric, i.e., $a(u,v) = a(v,u) \quad \forall u, v \in V$,
2. $a(.,.)$ is nonnegative, i.e., $a(v,v) \geq 0 \quad \forall v \in V$,
3. $a(.,.)$ is positive on $V_0 := \{v \in V = [H^1(\Omega)]^3 : v = 0 \text{ on } \Gamma_1\}$ provided that $\text{meas}_2(\Gamma_1) > 0$, i.e., $a(v,v) > 0 \quad \forall v \in V_0 : v \not\equiv 0$.

The equivalence of VF $(9)_{VF}$ and MP $(9)_{MP}$ then follows from the statements 1 and 2 above according to Section 1.1 of the lecture.

08 Show that, for the first BVP ($\Gamma_1 = \Gamma$) of 3D linear elasticity in the case of isotrop and homogeneous material, the assumptions of Lax-Milgram's Theorem are fulfiled, i.e.

1) $F \in V_0^*$,

2a) $\exists\, \mu_1 = \text{const} > 0 : \ a(v,v) \geq \mu_1 \parallel v \parallel^2_{H^1(\Omega)} \ \ \forall v \in V_0$,

2b) $\exists\, \mu_2 = \text{const} > 0 : \ |a(u,v)| \leq \mu_2 \parallel u \parallel^2_{H^1(\Omega)} \parallel v \parallel^2_{H^1(\Omega)} \ \ \forall u, v \in V_0$.

Provide the constants μ_1 and μ_2 !

○ Hint: to the proof of the V_0-ellipticity:

- $a(v,v) \geq 2\mu \int_\Omega \sum_{i,j=1}^3 (\varepsilon_{ij}(v))^2 dx$,
- Korn's inequality for $V_0 = [H_0^1(\Omega)]^3$, where $H_0^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ auf } \Gamma\}$,
- Friedrichs' inequality.

09 Formulate the iterative method (3) from Section 1.1 of the lecture for the first BVP of the linear elasticity in case of 3D homogeneous and isotrop material, i.e.,

$$u_{n+1} = u_n - \rho(JAu_n - JF) \text{ in } V_0 = (H_0^1(\Omega))^3, \tag{6}$$

for $n = 0, 1, 2, ...$, and given $u_0 \in V_0$. Derive the weak form, i.e., the variational formulation, for the calculation of $u_{n+1} \in V_0$. Discuss two cases in which the scalar product in V_0 is defined as follows:

$$(u, v)_{V_0}^2 := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V_0, \tag{7}$$

and

$$(u, v)_{V_0}^2 := \int_\Omega (\nabla u \cdot \nabla v + uv) \, dx \quad \forall u, v \in V_0. \tag{8}$$

Derive the corresponding classical formulation of the iteration process (6) !

$\boxed{10^*}$ Let us consider the variational formulation,

$$\text{find } u \in V_g = V_0 \text{ such that } a(u, v) = \langle F, v \rangle \text{ for all } v \in V_0, \tag{9}$$

of a plane linear elasticity problem in $\Omega = (0, 1) \times (0, 1)$, where

$$\begin{aligned}
V_0 = \; & \{ \quad u = (u_1, u_2) \in V = [H^1(\Omega)]^2 : \\
& \qquad u_1 = 0 \text{ on } \Gamma_1 = \{0\} \times [0, 1] \cup \{1\} \times [0, 1], \\
& \qquad u_2 = 0 \text{ on } \Gamma_2 = [0, 1] \times \{0\} \cup [0, 1] \times \{1\}\}, \\
a(u, v) \; &= \; \int_\Omega D_{ijkl} \varepsilon_{ij}(u) \varepsilon_{kl}(v) \, dx = \int_\Omega \sigma_{kl}(u) \varepsilon_{kl}(v) \, dx, \\
\langle F, v \rangle \; &= \; \int_\Omega f_i v_i \, dx + \int_{\Gamma_1} ? \, ds + \int_{\Gamma_2} ? \, ds.
\end{aligned}$$

Impose the right natural boundary conditions ! Give the classical formulation of (9) !

1.3 Scalar elliptic problems of the fourth order

$\boxed{11}$ Show existence and uniqueness of the solution of the first biharmonic BVP

$$u \in V_0 := H_0^2(\Omega) : \int_\Omega \Delta u(x) \Delta v(x) dx = \int_\Omega f(x) v(x) dx \; \forall v \in V_0 \tag{10}$$

by means of the Lax-Milgram-Theorem. Then formulate a minimization problem that is equivalent to the variational formulation (10) above !