T U T O R I A L

"Numerical Methods for the Solution of Elliptic Partial Differential Equations"

to the lecture

"Numerics of Elliptic Problems"

Tutorial 01 Tuesday, 10 March 2020, Time: $10^{\underline{15}}$ – $11^{\underline{45}}$, Room: KEP3.

1 Variational formulation of multi-dimensional elliptic Boundary Value Problems (BVP)

1.1 Scalar Second-order Elliptic BVP

◯ In Section 1.2.1 of our lectures, we considered the BVP in classical formulation

$$
\begin{array}{l}
\text{Find } u \in X := C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1): \\
-\sum\limits_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum\limits_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u(x) = f(x), x \in \Omega \\
+\text{BC:} \quad \bullet \quad u(x) = g_1(x), x \in \Gamma_1, \\
\qquad\quad \bullet \quad \frac{\partial u}{\partial N} := \sum\limits_{i,j=1}^{d} a_{ij}(x)\frac{\partial u(x)}{\partial x_j} n_i(x) = g_2(x), x \in \Gamma_2, \\
\qquad\quad \bullet \quad \frac{\partial u}{\partial N} + \alpha(x)u(x) = \underbrace{g_3(x)}_{\alpha(x)u_A(x)}, x \in \Gamma_3.
\end{array} \tag{1}
$$

and derived the variational formulation

$$
\begin{array}{l}
\text{Find } u \in V_g \text{ such that} \quad a(u,v) = < F, v > \quad \forall v \in V_0, \\
\text{with} \\
a(u,v) \quad := \int\limits_{\Omega} (\sum\limits_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum\limits_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} v + cuv)\, dx + \int\limits_{\Gamma_3} \alpha uv\, ds, \\
< F, v > \quad := \int\limits_{\Omega} fv\, dx + \int\limits_{\Gamma_2} g_2 v\, ds + \int\limits_{\Gamma_3} g_3 v\, ds, \\
V_g := \{v \in V = W_2^1(\Omega) : v = g_1 \text{ on } \Gamma_1\}, \\
V_0 := \{v \in V : v = 0 \text{ on } \Gamma_1\}.
\end{array} \tag{2}
$$

under the assumptions

$$\boxed{\begin{array}{ll} 1) & a_{ij}, b_i, c \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_3), \\ 2) & f \in L_2(\Omega), g_i \in L_2(\Gamma_i), i = 2,3, \\ 3) & g_1 \in H^{\frac{1}{2}}(\Gamma_1), \text{ i.e., } \exists \tilde{g}_1 \in H^1(\Omega) : \tilde{g}_1|_{\Gamma_1} = g_1, \\ 4) & \Omega \subset \mathbf{R}^d (\text{bounded}) : \Gamma = \partial\Omega \in C^{0,1} \text{ (Lip boundary)}, \\ 5) & \text{uniform ellipticity:} \\ & \left.\begin{array}{l} \sum\limits_{i,j=1}^{d} a_{ij}(x)\xi_i;\xi_j \geq \bar{\mu}_1|\xi|^2 \quad \forall \xi \in \mathbf{R}^d \\ a_{ij}(x) = a_{ji}(x) \quad \forall i,j = \overline{1,d} \end{array}\right\} \forall \text{ a.e. } x \in \Omega. \end{array}} \tag{3}$$

01 Formulate the classical assumptions on $\{ a_{ij}, b_i, c, \alpha, f, g_i, \Omega \text{ resp. } \partial\Omega \}$ for (1) !

02 Show that, for sufficiently smooth data, a the generalized solution $u \in V_g \cap X \cap H^2(\Omega)$ of the Boundary Value Problem (2) is also a classical solution, i.e. a solution of (1) !

$$(1) \quad \begin{cases} \text{Find } u \in X = C^2(\Omega) \cap C^1(\Omega \cup \Gamma_2 \cup \Gamma_3) \cap C(\Omega \cup \Gamma_1) : \\ -\Delta u(x) + c(x)u(x) = f(x), x \in \Omega \subset \mathbf{R}^d \text{ (bounded)}, \\ u(x) = g_1(x), \ x \in \Gamma_1, \\ \frac{\partial u}{\partial n}(x) = g_2(x), \ x \in \Gamma_2, \\ \frac{\partial u}{\partial n}(x) = \alpha(x)(g_3(x) - u(x)), \ x \in \Gamma_3 \end{cases}$$

$?\Downarrow\Uparrow?$

$$(2) \quad \begin{cases} \text{Find } u \in V_g = \{v \in V = H^1(\Omega) : v = g_1 \text{ on } \Gamma_1\} \text{ such that } \forall v \in V_0 : \\ \underbrace{\int\limits_\Omega (\nabla^T u \nabla v + cuv)\, dx + \int\limits_{\Gamma_3} \alpha uv\, ds}_{=a(u,v)} = \underbrace{\int\limits_\Omega fv\, dx + \int\limits_{\Gamma_2} g_2 v\, ds + \int\limits_{\Gamma_3} \alpha g_3 v\, ds}_{=\langle F,v\rangle}, \end{cases}$$

where $V_0 = \{v \in V = H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}$.

03 Show that the assumptions of the Lax-Milgram-Theorem are satisfied for the variational problem (2) under the assumptions (3) and the additional assumptions $b_i = 0$, $c(x) \geq 0$ for almost all $x \in \Omega$, $\alpha(x) \geq \underline{\alpha} = \text{const} > 0$ for almost all $x \in \Gamma_3$, and $\text{meas}_{d-1}(\Gamma_i) > 0, i = 1,2,3$! What happens in the case $\Gamma_1 = \emptyset$, and what happens in the case $\alpha = 0$?

04 In addition to assumption (3), let us assume that $c(x) \geq \underline{c} = \text{const} > 0$ for almost all $x \in \Omega, \Gamma_1 = \Gamma_3 = \emptyset$, and $b_i \not\equiv 0$. Provide conditions for the coefficients $b_i(\cdot)$ such that the assumptions of the Lax-Milgram-Theorem are satisfied !

○ <u>Hint:</u> For the estimate of the convection term $\sum\limits_{i=1}^{d} \int\limits_\Omega b_i \frac{\partial u}{\partial x_i} v\, dx$, make use of the ε-inequality (Young's inequality)

$$|ab| \leq \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2, \quad \forall a,b \in \mathbf{R}^1 \quad \forall \varepsilon > 0 \ !$$

05 Derive the variational formulation of the pure Neumann problem for the Poisson equation

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma := \partial\Omega, \tag{4}$$

and discuss the question of the existence and uniqueness of a generalized solution of (4) !

○ Hint:
Obviously, $u(x) + c$ with an arbitrary constant $c \in \mathbb{R}^1$ solves (4) provided that u is the solution of the BVP (4). There are the following ways to analyze the existence of a generalized solution:

1) Set up the variational formulation in $V = H^1(\Omega)$ and apply the FREDHOLM-Theory !
2) Set up the variational formulation in the factor-space $V = H^1(\Omega)|_{\text{ker}}$ with ker$= \{c : c \in \mathbb{R}^1\} = \mathbb{R}^1$ and apply the LAX-MILGRAM-Theorem !

06* Derive the variational formulation of the Dirichlet problem for the Helmholtz equation

$$-\Delta u - \omega^2 u = f \text{ in } \Omega = (0,1)^2 \subset \mathbb{R}^2 \quad \text{and} \quad u = 0 \text{ on } \Gamma := \partial\Omega, \tag{5}$$

where ω^2 is a given positive constant. Then discuss the problem of the existence and uniqueness of a generalized solution of (5) !

○ Hint: Apply the Fredholm theory to the operator equation

$$\text{Find } u \in V_0 : \ (I - K)u = \tilde{f} \text{ in } V_0$$

that arises from the setting

$$\underbrace{\int_\Omega [\nabla^T u \nabla v + uv]\, dx}_{:=[u,v]} - \underbrace{(1+\omega^2)\int_\Omega uv\, dx}_{:=[Ku,v]} = \underbrace{\int_\Omega fv\, dx}_{:=[\tilde{f},v]}$$

which is equivalent to the variational formulation of the Helmholtz equation.