4.2.2.3. Discrete Convergence

- For the **DS (28)$_{PB}$** (similarly, for $(28)_{MD}$, ...)

(28)
$$\boxed{\begin{aligned} v = u_h : \overline{\omega}_h \to \mathbb{R} :\quad & L_h v(x) = f_h(x),\ x \in \omega && (28)_L \\ & \ell_h v(x) = g_h(x),\ x \in \gamma_n = \gamma_2 \cup \gamma_3 && (28)_\ell \\ & v(x) = g_1(x),\ x \in \gamma_e = \gamma_1 & & \end{aligned}}$$

error estimates in discrete norms follow from the STABILITY and the APPROXIMATION with respect to the corresponding norms:

① Discrete Convergence in the $\overset{\circ}{W}{}_2^1(\omega_h)$-norm:

(29)
$$\|\overbrace{u - v}^{\text{error } z}\|_{W_2^1(\omega_h)} \le c(u) \begin{cases} h, & \text{for (i) } u \in W_2^2(\Omega), \text{ regular } \omega \\ h^{3/2}, & \text{for (ii) } u \in W_2^3(\Omega), \text{ loc. non-uni} \\ h^2, & \text{for (iii) } u \in W_2^3(\Omega), \text{ unif.}, \ \Gamma = \Gamma_1, \text{ data smooth} \end{cases}$$

(25) (28)
(23)

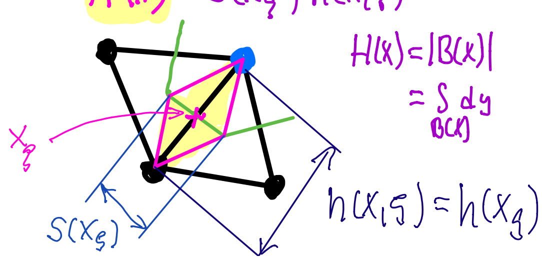
where $\overset{\circ}{W}{}_2^1(\omega) = \overset{\circ}{H}{}^1(\omega) := \{ z : \overline{\omega} \to \mathbb{R}^1 : z|_{\gamma_1} = 0 \}$ with the norm

$$\|z\|^2_{W_2^1(\omega)} := \underbrace{\sum_{x_\xi}{}' z_{\bar{n}}^2(x_\xi) H'(x_\xi)}_{=: |z|^2_{W_2^1(\omega)}} + \sum_{x \in \omega} z^2(x) H(x) + \sum_{x \in \gamma_n}{}' z^2(x) h(x)$$

$h(x,\xi) = |x - \xi|$

$H'(x) = S(x_\xi)\, h(x,\xi)$

$z_{\bar{n}}(x_\xi) := \dfrac{z(\xi) - z(x)}{h(x,\xi)}$



(iii) uniform grid: ⬡ ⊠ □

(ii) locally non-uniform grids:
uniformity is only perturbed for $O(h^{-1})$ triangles (along interfaces)

Proof technique:

Without loss of generality, we consider the pure Dirichlet problem ($\gamma_1 = \gamma$). The error scheme has the form

approximation error

(30) $$z = u - v : \bar{\omega} \to \mathbb{R}^1 : \quad L_h z(x) = \psi(x) \quad \forall x \in \omega = \overset{\circ}{\omega}$$
$$z(x) = 0 \quad \forall x \in \gamma = \gamma_1$$

Due to the general theory, we have to show

$\boxed{\overset{\circ}{W}{}_2^1(\omega) - W_2^{-1}(\omega) - \text{Stability}}$ + $\boxed{\text{Approx. } \|\psi\|_{W_2^{-1}(\omega)}}$ ⇒ $\boxed{\text{discr. conv. in } \overset{\circ}{W}{}_2^1(\omega)}$

a) (31) b) (32) (29)

a) $\overset{\circ}{W}{}_2^1(\omega) - W_2^{-1}(\omega)$ - STABILITY:

- Define the discrete $L_2(\omega)$ scalar product

$$(v, z) = (v, z)_{L_2(\omega)} := \sum_{x \in \omega} v(x)\, z(x)\, H(x)$$

- Multplying (30) by z, and estimating from below and above, we get the estimates

$$\tilde{\mu}_1 \|z\|^2_{\overset{\circ}{W}{}_2^1(\omega)} \leq (L_h z, z) = (\psi, z) \leq \underbrace{\left[\sup_{w} \frac{|(\psi, w)|}{\|w\|_{W_2^1(\omega)}} \right]}_{=: \|\psi\|_{W_2^{-1}(\omega)}} \|z\|_{W_2^1(\omega)}$$

(33) (↓) see L29-03

- Resultat:

(31) $$\boxed{\|z\|_{W_2^1(\omega)} \leq c_S \|\psi\|_{W_2^{-1}(\omega)} \quad \text{with } c_S = \tilde{\mu}_1^{-1}}$$

i.e. $\|L_h^{-1}\|_{[W_2^{-1}, \overset{\circ}{W}{}_2^1]} \leq c_S := 1/\tilde{\mu}_1$

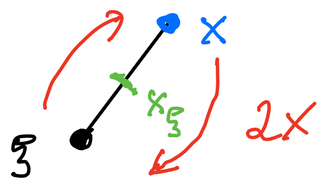
L 29-03

- Let us show the L_h-ellipticity estimate (33):

(33) $$(L_h z, z) = \sum_{x\in\omega}\Big[-\sum_{\xi\in S'(x)} \bar a(x_\xi)\frac{z(\xi)-z(x)}{h(x,\xi)} s(x_\xi) z(x)\Big] + \sum_{x\in\omega} \bar c(x) z^2(x) H(x)$$

$$L_h z(x) = -\frac{1}{H(x)}\sum_{\xi\in S'(x)} \bar a(x_\xi)\frac{z(\xi)-z(x)}{h(x,\xi)} s(x_\xi) + \bar c(x) z(x)$$

$$= \sum_{x\in\omega}\sum_{\xi\in S'(x)} \bar a(x_\xi)\Big(-\frac{z(\xi)-z(x)}{h(x,\xi)}\frac{z(x)}{h(x,\xi)}\Big) h(x,\xi)\, s(x_\xi) + \sum \bar c(x) z^2(x) H(x)$$



$$h(x,\xi) = h(\xi,x) = |x-\xi|$$

$$-(z(\xi)-z(x))z(x) - (z(x)-z(\xi))z(\xi)$$
$$= z^2(x) - 2z(\xi)z(x) + z^2(\xi) = (z(\xi)-z(x))^2$$

$$= \sum_{x_\xi} \bar a(x_\xi)\Big(\underbrace{\frac{z(\xi)-z(x)}{h(x,\xi)}}_{=:\, z_h(x_\xi)}\Big)^2 H'(x_\xi) + \sum_{x\in\omega} \bar c(x) z^2(x) H(x)$$

$$\geq \tilde\mu_1 \|z\|^2_{W_2^1(\omega)}, \text{ with } \tilde\mu_1 = \begin{cases} \min\{\bar a_1, c_1\}, & \text{if } \bar c(x) \geq c_1 = \text{const} > 0 \\ \bar a_1 (1+\tilde c_F^2)^{-1}, & \text{if } \bar c(x) \geq 0 \end{cases}$$
$$\forall x\in\omega_h\ \forall h \in Ⓗ$$

where $\bar a_1 = \text{const} > 0$: $\bar a(x_\xi) \geq \bar a_1 = \text{const} > 0\ \forall x,\xi \in \bar\omega_h\ \forall h \in Ⓗ$

$\tilde c_F = \text{const} > 0$: constant from the discrete Friedrichs ineq.:

(34) $$\boxed{\|z\|_{L_2(\omega)} \leq \tilde c_F\, |z|_{W_2^1(\omega)}\ \forall z \in \mathring{W}_2^1(\omega_h)\ \forall h \in Ⓗ}$$

Exercise 4.4:

Show the discrete Friedrichs inequality (34)!

Hint: Use the known Friedrichs inequality (Ch. 2) for $\tilde{z}_h \in V_{0h} \subset \overset{\circ}{W}{}_2^1(\Omega)$, $\tilde{z}_h \leftrightarrow z$:

$$\|\tilde{z}_h\|_{L_2(\Omega)} \le c_F \, |\tilde{z}_h|_{W_2^1(\Omega)} \quad \forall \tilde{z}_h \in V_{0h},$$

where V_{0h} is Courant's finite element space.

Exercise 4.5:

Show the relation

$$(L_h z, v) = \sum_{x_\xi} \tilde{a}(x_\xi) \frac{z(\xi) - z(x)}{h(\xi,x)} \frac{v(\xi) - v(x)}{h(\xi,x)} H'(x_\xi) +$$

$$+ \sum_{x \in \omega} \bar{c}(x)\, z(x)\, v(x)\, H(x),$$

from which and from (33) we immediately see that L_h, and, therefore, the corresponding matrix A_h are symmetric and positive definite.

b) APPROXIMATION Estimate: $\|\psi\|_{W_2^{-1}(\omega)} \leq ?$

By mapping to a reference domain, application of Bramble-Hilbert's Lemma, and return mapping,
and, under the assumptions
- (i) $u \in W_2^2(\Omega) = H^2(\Omega)$,
- $\bar{\omega} = \bar{\omega}_h$ – regular grid, i.e. $\mathcal{T}_\Delta$ – reg. triangul.,
- and additional smoothness requirements imposed on the data $\{a, c, f\}$,

we can prove the estimate

$$(32) \qquad |(\psi, z)| \leq c_A(u)\, h\, \|z\|_{W_2^1(\omega)} \quad \forall z$$

Indeed, let us consider the approximation error ψ in detail: $x \in \omega$

$$\psi(x) = L_h u(x) - L_h v(x) = L_h u(x) - \overset{,,f_h(x)}{\bar{f}(x)}$$

$$= L_h u(x) \underbrace{- \Big[-\frac{1}{H(x)} \int_{\partial B(x)} a(y) \frac{\partial u}{\partial n}(y)\, ds_y + \frac{1}{H(x)} \int_{B(x)} c\, u\, dy \Big] + \frac{1}{H(x)} \int_{B(x)} f(y)\, dy}_{\overset{(25)}{=} 0 = f_{IB} - L_{IB} u} - \bar{f}(x)$$

$$= \Big\{ -\frac{1}{H(x)} \sum_{\xi \in S'(x)} \bar{a}(x_\xi) \frac{u(\xi) - u(x)}{h(x,\xi)} s(x_\xi) - \Big[-\frac{1}{H(x)} \int_{\partial B(x)} a(y) \frac{\partial u}{\partial n}(y)\, ds_y \Big] \Big\}$$

$$+ \Big\{ \tilde{c}(x) u(x) - \frac{1}{H(x)} \int_{B(x)} c\, u\, dy \Big\} + \Big\{ \frac{1}{H(x)} \int_{B(x)} f(y)\, dy - \bar{f}(y) \Big\}$$

$$= \psi_a(x) + \psi_{cu}(x) + \psi_f(x)$$

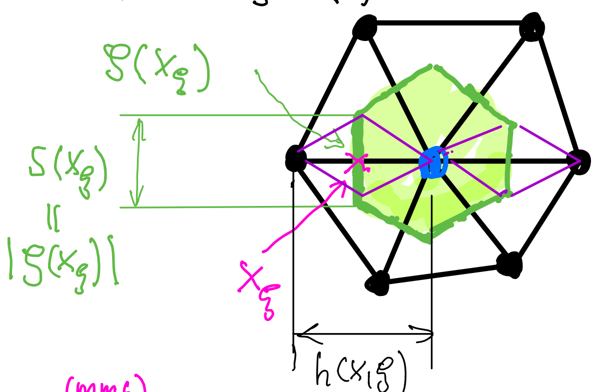
$$\Longrightarrow \quad |(\psi, z)| \le \underbrace{|(\psi_a, z)|}_{1)} + \underbrace{|(\psi_{cu}, z)|}_{\substack{= O(h) \\ \text{(mms)}}} + \underbrace{|(\psi_f, z)|}_{\substack{= 0 \\ \text{for } \bar{f}(x) = \frac{1}{H(x)} \int_{B(x)} f(y)\,dy}}$$

It remains to estimate 1) $|(\psi_a, z)| \le \, ?$

For simplicity, we assume that $a = \bar{a} = 1$. Then

$$(\psi_a, z) = \sum_{x \in \omega}' \psi_a(x)\, z(x)\, H(x)$$

$$= \sum_{x \in \omega}' \left\{ \sum_{\xi \in S'(x)} \left[-\frac{u(\xi) - u(x)}{h(x,\xi)} S(x_\xi) + \int_{\varsigma(x_\xi)} \frac{\partial u}{\partial n}(y)\, ds_y \right] \right\} z(x)$$



$H'(x) = S(x_\xi)\, h(x, \xi)$

$$\partial B(x) = \bigcup_{\xi \in S'(x)} \varsigma(x_\xi)$$

$$\overset{\text{(mms)}}{=} \sum_{x_\xi}' \underbrace{\left[\frac{1}{S(x_\xi)} \int_{\varsigma(x_\xi)} \frac{\partial u}{\partial n}\, ds_y - \frac{u(\xi) - u(x)}{h(x,\xi)} \right]}_{=: \, \ell_{x_\xi}(u)} \underbrace{\frac{z(\xi) - z(x)}{h(x,\xi)}}_{=: \, z_h(x_\xi)} \overbrace{h(x,\xi)\, S(x_\xi)}^{= H'(x)}$$

$$= \sum_{x_\xi}' \ell_{x_\xi}(u)\, z_h(x_\xi)\, H'(x_\xi)$$

$$\Longrightarrow \ |(\psi_{q1}, z)| = \sum_{x_q} \ell_{x_q}(u)\, z_h(x_q)\, H^1(x_q)$$

$$\leq \sqrt{\sum_{x_q} \ell_{x_q}^2(u)\, H^1(x_q)}\ \underbrace{\sqrt{\sum_{x_q} z_h^2(x_q)\, H^1(x_q)}}_{=:|z|_{W_2^1(\omega)}}$$

$$\leq \sqrt{\sum_{x_q} \ell_{x_q}^2(u)\, H^1(x_q)}\ \ \|z\|_{W_2^1(\omega)}$$

mapping + Bramble-Hilbert Lemma + return mapping

(mms)

$$\leq c\, h\, |u|_{W_2^2(\Omega)}\ \ \|z\|_{W_2^1(\omega)}$$

$$\leq C_A(u)\, h\, \|z\|_{W_2^1(\omega)}$$

with $C_A(u) = c\, |u|_{W_2^2(\Omega)}$.

2) Discrete Convergence in the $C(\omega)$ norm (2d):

$$(35) \quad \| \overbrace{\underset{(25)\,(23)}{u} - \underset{(28)}{v}}^{\text{error } z} \|_{C(\omega)} \le c(u)\, |\ln h|^{1/2} \begin{cases} h, & \text{for (i)} \\ h^{3/2}, & \text{for (ii)} \\ h^2, & \text{for (iii)} \end{cases} \quad \text{see (29)}$$

These estimates immediately follow from the $W_2^1(\omega)$-estimates (29) and the so-called "weak" $W_2^1(\omega)$-embedding in $C(\omega)$:

$$(36) \quad \| z \|_{C(\omega)} := \max_{x \in \omega} |z(x)| \le c\, |\ln h|^{1/2} \| z \|_{W_2^1(\omega)},$$

$\forall$ grid functions $z : \bar{\omega} \to \mathbb{R}$ with $z(x) = 0 \; \forall x \in \gamma = \gamma_1$

Inequality (36) follows from the corresponding inequality for finite element functions. Indeed, it holds

$$(36)_{FE} \quad \| \tilde{z}_h \|_{C(\bar{\Omega})} \le c\, |\ln h|^{1/2} \| \tilde{z}_h \|_{W_2^1(\Omega)}$$

$\forall\, \tilde{z}_h \in V_{0h}$ – Courant's FE space of cont. piecewise linear functions on the primary grid with $\tilde{z}_h(x) = 0 \;\; \forall x \in \Gamma_1$

$$\tilde{z}_h \overset{FE}{\updownarrow} \text{iso}$$

$$\underline{z}_h \in \mathbb{R}^n$$

$$\text{FV} \updownarrow \text{iso}$$

$$z_h(\cdot) : \bar{\omega}_h \to \mathbb{R}^1$$

Furthermore, we have: $\| \tilde{z}_h \|_{C(\bar{\Omega})} = \max_{x \in \omega} |z(x)|$ and

$$\underline{c}\, \| z \|_{W_2^1(\omega)} \le \| \tilde{z}_h \|_{W_2^1(\Omega)} \le \bar{c}\, \| z \|_{W_2^1(\omega)} \quad \text{(mms)}.$$

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SUMMARY: FVM and Discrete Convergence: $-\mathrm{div}(a\nabla u) + cu = f$ in Ω, $u = 0$ on $\Gamma = \partial\Omega$

$u : \overline{\Omega} \to \mathbb{R}$

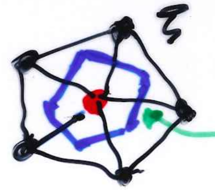
$$A \rightsquigarrow Lu(x) = -\frac{1}{|B(x)|} \sum_{E \in \partial B(x)}' \int_E a(y) \frac{\partial u}{\partial n}(y)\, ds_y + \frac{1}{|B(x)|} \int_{B(x)} c(y) u(y)\, dy$$

$v = u_h : \overline{\omega}_h \to \mathbb{R}$

$$A_h \rightsquigarrow L_h v(x) = -\frac{1}{|B(x)|} \sum_{\xi \in S'(x)}' \overline{a}(x_\xi) \frac{v(\xi) - v(x)}{|\xi - x|} s(x_\xi) + \overline{c}(x) v(x), \quad x \in \omega = \overset{\circ}{\omega}, \; v(x) = 0, \forall x \in \gamma, \; \omega_h = \overset{\circ}{\omega}_h$$

$$\begin{array}{ccccl}
D(L) := \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega) \subset X = \overset{\circ}{H}{}^1(\Omega) & \xrightarrow{L} & Y = H^{-1}(\Omega) & : & Lu = f := \frac{1}{|B(x)|} \int_{B(x)} f(y)\, dy : \; u \leftarrow E_h u_h \\
R_h \downarrow \; \uparrow E_h & & Q_h \downarrow \; \uparrow F_h & & \\
X_h = \overset{\circ}{H}{}^1(\omega) & \xrightarrow{L_h} & Y_h = H^{-1}(\omega) & : & L_h u_h = f_h := [f(x)]_{x \in \omega} : \; u_h \rightarrow R_h u
\end{array}$$

Restriction to the grid $\overline{\omega}$



<u>Discrete Error:</u> $z_h = R_h u - u_h \in X_h = \overset{\circ}{H}{}^1(\omega)$: $L_h z_h = \psi_h \Rightarrow z_h = L_h^{-1} \psi_h$

$$\Longrightarrow \|z_h\|_{X_h} \le \underbrace{\|L_h^{-1}\|_{L(Y_h, X_h)}}_{\le c_S} \cdot \underbrace{\|\psi_h(u)\|_{Y_h}}_{\le c_A(u)\, h^p} \le c_S\, c_A(u)\, h^p$$

$(p = 1$ if $u \in H^2)$

Stability + Approximation $\Rightarrow$ discrete Convergence

with the appr. error $\psi_h(u) := L_h R_h u - Q_h L u = -\frac{1}{|B(x)|} \left[\sum_{\xi \in S'(x)}' a(x_\xi) \frac{u(\xi) - u(x)}{|\xi - x|} s(x_\xi) - \sum_{E \in \partial B(x)}' \frac{1}{|E|} \int_E a(y) \frac{\partial u}{\partial n}(y)\, ds_y \right] + \ldots$