4.2.2.2. Direct Finite Difference Approximation of the Balance Equations

- Let us consider a regular triangulation $\mathcal{T}_\Delta := \mathcal{T}_h$ (FEM, see Ch. 3)
  of the polygonally bounded Lip domain $\Omega \subset \mathbb{R}^2$:

$$\overline{\Omega} = \bigcup_{r \in R_h} \overline{\delta}_r \,, \quad \mathcal{T}_\Delta := \{ \overline{\delta}_r : r \in R_h \} \,, \; h \in \Theta :$$

$$\measuredangle \delta_r < \frac{\pi}{2} \; (PB: P_r \in \delta_r) \text{ resp. } \leq \frac{\pi}{2} \; (PB: P_r \in \overline{\delta}_r) \; \forall r \in R_h \; \forall h \in \Theta.$$

For simplicity, we construct the secondary grid $\mathcal{T}_B$ with the PB-method (Voroni mesh):

$$\overline{\Omega} = \bigcup_{x \in \overline{\omega}} \overline{B(x)} \,, \quad \mathcal{T}_B := \{ B(x) : x \in \overline{\omega} \},$$

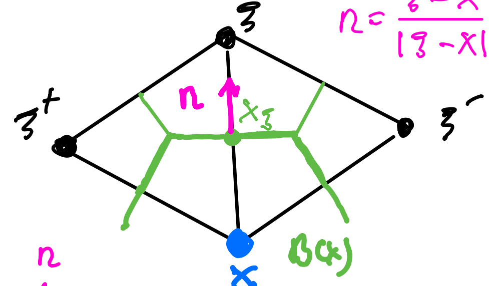
- We now consider the balance equation **(25)** $\quad B = B(x)$

$$(25)_{B(x)} \quad \forall x \in \omega = \overset{\circ}{\omega} \cup \gamma_n$$

$$-\int_{\partial B \setminus \partial B_n} (a \nabla u, n)\, ds + \int_B (b, \nabla u)\, dy + \int_B c u\, dy + \int_{\partial B_3} \varkappa u\, ds = \int_B f\, dy + \int_{\partial B_n} g\, ds$$

at all boxes $B(x)$, $x \in \omega$, of the Voroni-mesh and approximate

$$n = \frac{\xi - x}{|\xi - x|}$$

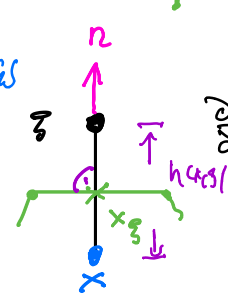


$\int_{\partial B} \ldots ds$ by quadrature formulas

$\int_B \ldots dx$ by quadrature formulas

$\frac{\partial u}{\partial n}$ by finite differences

$h(x,\xi) := |x - \xi|$



$$\frac{\partial u}{\partial n}(x_\xi) \approx \frac{u(\xi) - u(x)}{h(x,\xi)} = \frac{\partial u}{\partial n}(x_\xi) + O(h^2)$$

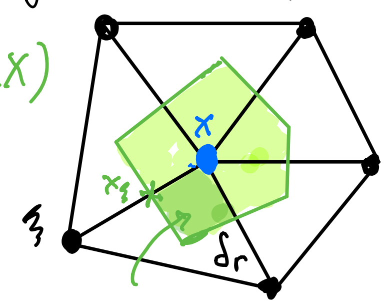
(CMMS)

for the case smooth (continuous) data, i.e.

$$(27)\quad \begin{cases} a(x) = a(x)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} : a(\cdot) \in C(\bar\Omega) \\ b(x) = \mathbb{0} \quad \text{(no convection, but see Remark 4.16)} \\ c(\cdot), f(\cdot) \in C(\bar\Omega) \\ \text{æ}(\cdot) \in C(\bar\Gamma_3),\ g_i(\cdot) \in C(\bar\Gamma_i),\ i = 1,2,3 : \end{cases}$$

a) $x \in \mathring{\omega}$:

$B(x)$



PB method

$|B(x)| = \text{meas } B(x) = O(h^2)$

$B_r = B_r(x) = B(x) \cap \delta_r \quad \forall r \in B_h(x) = \{r \in R_h : x \in \bar\delta_r\}$

$S_h(x) = \{x, \xi, \bullet, \bullet, \bullet, \bullet\} = \{\bullet, \bullet\bullet\bullet\bullet\bullet\} = S_h'(x) \cup \{x\}$

↳ difference star — neighbourhood of the difference star

Consider now balance equation (25) at the point $x \in \mathring{\omega}$ for the box $B(x)$:

$$-\oint_{\partial B(x)} \underbrace{(a\nabla u, \vec n)}_{= a\frac{\partial u}{\partial n}}\, ds + \cancel{\int_{B(x)} (b, \nabla u)\, dy} + \int_{B(x)} c u\, dy = \int_{B(x)} f(y)\, dy$$

① see Remark 4.16 ② ③

and approximate the terms ① - ③ directly:

(1) $$\int_{\partial B(x)} a \frac{\partial u}{\partial n}\, ds = \sum_{\xi \in S'(x)} \int_{\wp(x,\xi)} a \frac{\partial u}{\partial n}\, ds$$

$$\approx \sum_{\xi \in S'(x)} \bar{a}(x_\xi) \frac{u(\xi) - u(x)}{h(x,\xi)} s(x_\xi)$$

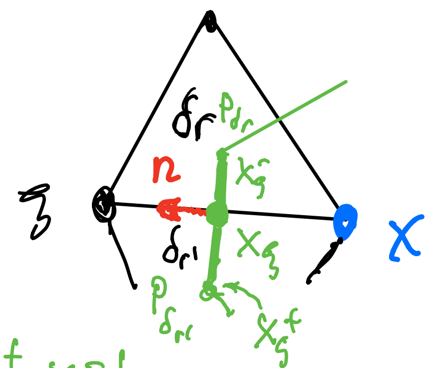
with $\bar{a}(x_\xi) = a(x_\xi)$ for $a \in C(\bar{\Omega})$

$h(x,\xi) = |x - \xi|$

x_ξ^-, x_ξ, x_ξ^+

$\wp(x,\xi) = \overline{x_\xi^-, x_\xi^+}$

$s(x_\xi) = |\wp(x,\xi)| = |x_\xi^+ - x_\xi^-|$



(2) $$\int_{B(x)} c u\, dy \approx \bar{c}(x)\, u(x)\, |B(x)|$$

with $\bar{c}(x) := c(x)$ for $c \in C(\bar{\Omega})$

(3) $$\int_{B(x)} f(y)\, dy \approx \bar{f}(x)\, |B(x)|$$ with $\bar{f}(x) = f(x)$ for $f \in C(\bar{\Omega})$.

<u>Result:</u> $u \mapsto V : \bar{\omega} \to \mathbb{R}$ (grid function)

$(28)_L$ $\quad L \mapsto L_h$ (difference operator)

$$\boxed{L_h V := -\frac{1}{|B(x)|} \sum_{\xi \in S'(x)} \bar{a}(x_\xi) \underbrace{\frac{V(\xi) - V(x)}{h(x,\xi)}}_{=: V_n(x_\xi)} s(x_\xi) + \bar{c}(x) V(x) = \bar{f}(x) =: f_h(x) \quad \forall x \in \omega}$$

$L_h V(x) = f_h(x)$

For <u>piecewise continuous data</u>, i.e. $a, c, f \in PC(\bar{\Omega})$, the interfaces must be covered by the primary grid as in the FEM, i.e. $a, c, f \in C^k(\bar{\delta}_r)$ $\forall r \in R_h$ $\forall h \in \textcircled{H}$:

① $\tilde{a}(x_q) := (a(\bar{x}_q^-)s(\bar{x}_q^-) + a(\bar{x}_q^+)s(\bar{x}_q^+)) / s(x_q)$

② $\breve{c}(x) := \sum\limits_{r \in B_h(x)}' \int\limits_{B_r(x)} c(y)\,dy \approx \dots$

③ $\bar{f}(x) := \sum\limits_{r \in B_h(x)}' \int\limits_{B_r(x)} f(y)\,dy \approx \dots$

Different approximation techniques and assembling technologies are possible, e.g., the elementwise procedure known from the FEM.

Exercise 4.3:

Rewrite $(28)_L$ in the form

$$(29) \quad L_h v(x) = A(x)v(x) - \sum_{\xi \in S'(x)}' B(x,\xi)v(\xi), \quad x \in \overset{\circ}{\omega},$$

and show that L_h is monotone! If $c(x) \geq \underline{c} = const > 0$ $\forall x \in \bar{\Omega}$, then L_h is even strongly monotone!

We recall that L_h is called (strongly) monotone if

$$A(x) > 0, \; B(x,\xi) > 0 \;\forall \xi \in S'(x) \;\forall x \in \overset{\circ}{\omega},$$

$$D(x) := L_h 1 = A(x) - \sum_{\xi \in S'(x)}' B(x,\xi) \geq 0 \quad \forall x \in \overset{\circ}{\omega}$$

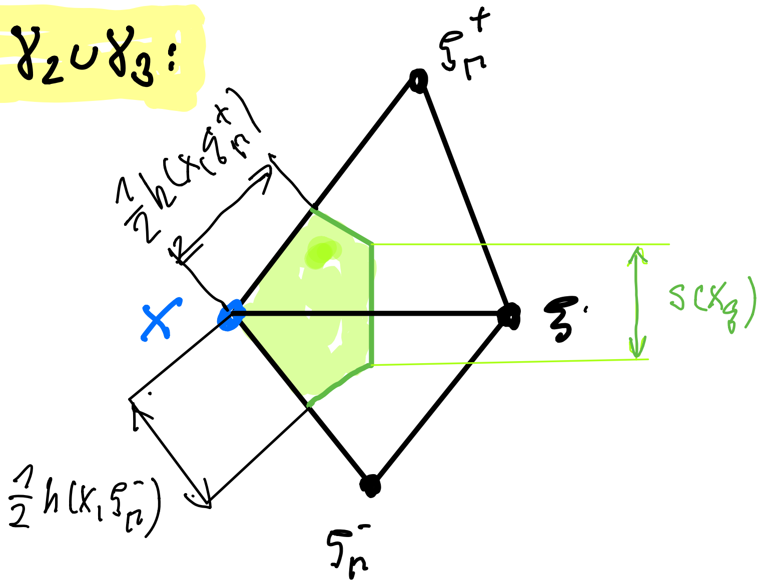
> 0 — " —

We mention the Difference Schemes with monotone diff. op. L_h possess nice properties like "discrete max. principle", see Nu II, Sec. 5.1.4.2.

ω – connected grid

b) $x \in \gamma_n := \gamma_2 \cup \gamma_3$:

e.g.
$\partial B_n = \partial B_3$
$= \partial B_3(x)$



(4) $$\int_{\partial B_3(x)} \varkappa u \, ds \approx \tilde{\varkappa}(x)\, u(x) \underbrace{\left[\frac{1}{2}\left(h(x,\xi_\Gamma^-) + h(x,\xi_\Gamma^+)\right)\right]}_{=:h(x)}$$

with $\tilde{\varkappa}(x) = \varkappa(x)$ if $\varkappa \in C(\Gamma_3)$.

(5) $$\int_{\partial B_n(x)} g \, ds \approx \tilde{g}(x)\, h(x) \quad \text{with } \tilde{g}(x) = g(x) \text{ if } g \in C(\Gamma_n)$$

Resultat: $u \mapsto v = u_h$, $\ell \mapsto \ell_h$, $x \in \gamma_n$ (γ_3)

$$-\frac{1}{h(x)} \sum_{\xi \in S'(x)} \bar{a}(x_\xi) \frac{v(\xi)-v(x)}{h(x,\xi)} s(x_\xi) + \underbrace{\underbrace{\frac{|B(x)|}{h(x)} \bar{c}(x) v(x)}_{=O(h)} + \bar{\varkappa}(x) v(x)}_{=:\bar{\varkappa}(x)v(x)} = \underbrace{\underbrace{\frac{|B(x)|}{h(x)} \bar{f}(x)}_{=O(h)} + \bar{g}(x)}_{=:\bar{g}(x)}$$

can be omitted !

(28)$_\ell$

$$\underbrace{-\frac{1}{h(x)} \sum_{\xi \in S'(x)} \bar{a}(x_\xi) \frac{v(\xi)-v(x)}{h(x,\xi)} + \bar{\varkappa}(x)\, v(x)}_{=:\ell_h v(x)} = \underbrace{\bar{g}(x)}_{=:q_h(x)}$$

For piecewise continuous $\varkappa$ and g, one has to modify the scheme accordingly, see Nu II, p. 155.

$(28)_L$ & $(28)_R$ yield DS: $A_h(x)\,u_h(x) = b_h(x),\ x \in \overline{\omega}_h$

System of FD Eqn. = ES: $A_h\, \underline{u}_h = \underline{b}_h$ in $\mathbb{R}^{N_h}$

(28)

$$\boxed{\begin{array}{l} \text{Find } u_h(\cdot) \equiv v(\cdot): \overline{\omega} \to \mathbb{R}^1: \\ L_h u_h(x) = f_h(x),\ x \in \overset{\circ}{\omega} \\ \ell_h u_h(x) = g_h(x),\ x \in \gamma_n := \gamma_2 \cup \gamma_3 \\ \quad u_h(x) = g_1(x),\ x \in \gamma_e := \gamma_1 \end{array}}$$

$\Big\}\ \hat{L}_h u_h(x) = \hat{f}_h(x),\ x \in \omega_h$

$\Big\}\ u_h(x) = g_1(x),\ x \in \gamma_h$

Exercise 4.3:

If $\bar{a}(x) > 0\ \forall x \in \omega$,

$\bar{c}(x) \geq 0\ \forall x \in \omega$ (resp. > 0 in ω), and

$\varkappa(x) \geq 0\ \forall x \in \gamma_n$ (resp. > 0 on $\gamma_n = \gamma_3$),

then the DS (28) is monotone (resp. strongly monotone)!

Consequences: discrete maximum principle,

Nu II

comparison theorem,

$C(\omega_h)$-$C(\omega_h)$ - stability,

A_h is an M-matrix etc.

Remark 4.16: Treatment of the Convection Term

The convection term

$$\int_{B(x)} (b, \nabla u)\, dy$$

can be approximated in such a way (upwind-approximation) that the monotonicity of L_h (and, therefore, the discrete maximum principle!) is preserved.

Starting point for deriving such approximations are the following relation: $\vec{n} = n$

$$\int_{B(x)} (b, \nabla u)\, dy \overset{PI}{=} \int_{\partial B(x)} (b, \vec{n}) \cdot u\, ds - \int_{B(x)} \operatorname{div} b \cdot u\, dy$$

$$= \int_{\partial B(x)} (b(y), \vec{n}(y)) [u(y) - u(x)]\, ds_y + \int_{B(x)} \operatorname{div} b(y) [u(x) - u(y)]\, dy$$

$$\underbrace{\left[\int_{B(x)} \operatorname{div} b(y)\, dy - \int_{\partial B(x)} (b, \vec{n})\, ds \right]}_{=0} u(x) = 0$$

$$= \qquad I(x) \qquad + \qquad g(x)$$

$$= I(x) = \int_{\partial B(x)} (b(y), \vec{n}(y)) [u(y) - u(x)]\, ds_y \approx \ldots$$

$\operatorname{div} b = 0$
(incompressible velocity field)

[B. Heinrich, 1987]