L 26-01

4.2. FDM and FVM

4.2.1. Finite Difference Methods (FDM)

- **Classical Idea:**

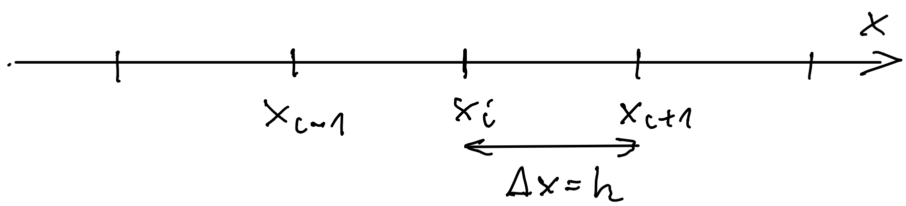
  Just replace partial derivatives by finite differences on a uniform grid:

$$u_{\bar{x}}(x_i,y) := \frac{u(x_i,y)-u(x_{i-1},y)}{\Delta x} \overset{O(h)}{\approx} \frac{\partial u}{\partial x}(x_i,y) \overset{O(h)}{\approx} \frac{u(x_{i+1},y)-u(x_i,y)}{\Delta x} =: u_x(x_i,y)$$

backward difference $\qquad h = \Delta x \qquad \approx \leftarrow O(h^2) \qquad$ forward difference

$$\frac{u(x_{i+1},y)-u(x_{i-1},y)}{2\Delta x} =: u_{\mathring{x}}(x_i,y)$$

central difference



Indeed, using Taylor's formula, we can easily find the approximation order:

$$u_{\bar{x}}(x_i,y) = \frac{u(x_i,y)-u(x_i-h,y)}{h} \qquad \exists\, \theta \in (0,1)$$

$$= \frac{1}{h}\Big[u(x_i,y) - \Big(u(x_i,y) - \frac{\partial u}{\partial x}(x_i,y)\,h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i-\theta h,y)\,h^2\Big)\Big]$$

$$= \frac{\partial u}{\partial x}(x_i,y) - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i+\theta h,y)\,h = \frac{\partial u}{\partial x}(x_i,y) + O(h)$$

$$u_x(x_i,y) = u_{\bar{x}}(x_{i+1},y) = \qquad \text{(mms)}$$

$$u_{\mathring{x}}(x_i,y) = \qquad \text{(mms)}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,y) \underset{O(h^2)}{\overset{\text{Taylor}}{\approx}} \frac{u(x_{i+1},y) - 2u(x_i,y) + u(x_{i-1},y)}{(\Delta x)^2} =: u_{\bar{x}x}(x_i,y) = u_{x\bar{x}}(x_i,y)$$

second difference quotient

Indeed, we have

$$u_{\bar{x}x}(x_i,y) = \frac{1}{h^2}\left[u(x_i+h,y) - 2u(x_i,y) + u(x_i-h,y)\right]$$

Taylor, $\exists \theta_{\pm i} \in (0,1)$

$$\overset{\text{Taylor}}{=} h^{-2}\Big[u(x_i,y) + \frac{\partial u}{\partial x}(x_i,y)\,h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i,y)\,h^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x_i,y)\,h^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(\underbrace{x_i^+}_{=x_i+\theta_{+i}h},y)\,h^4$$
$$- 2u(x_i,y) +$$
$$u(x_i,y) - \frac{\partial u}{\partial x}(x_i,y)\,h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(x_i,y)\,h^2 - \frac{1}{6}\frac{\partial^3 u}{\partial x^3}(x_i,y)\,h^3 + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}(\underbrace{x_i^-}_{=x_i-\theta_{-i}h},y)\,h^4\Big]$$

$$= \frac{\partial^2 u}{\partial x^2}(x_i,y) + \frac{1}{24}\left[\frac{\partial^4 u}{\partial x^4}(x_i+\theta_{+i}h,y) + \frac{\partial^4 u}{\partial x^4}(x_i-\theta_{-i}h,y)\right] h^2$$

$$= \frac{\partial^2 u}{\partial x^2}(x_i,y) + O(h^2)\text{, i.e.}$$

$$\left|\frac{\partial^2 u}{\partial x^2}(x_i,y) - u_{\bar{x}x}(x_i,y)\right| \leq \frac{1}{12} \max_{x\in[x_{i-1},x_{i+1}]} \left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| h^2$$

provided that $u \in C^4$.

FD-Scheme for the Poisson Equation:

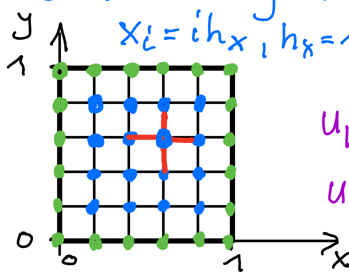
Let us consider the Dirichlet BVP for the Poisson Equation

$$(20)_{CF} \quad \begin{cases} -\Delta u(x,y) = f(x,y), & (x,y) \in \Omega = (0,1)^2, \\ \quad u(x,y) = g(x,y) := 0, & (x,y) \in \Gamma = \partial\Omega. \end{cases}$$

We now discretize $\overline{\Omega}$ by a uniform grid $\overline{\omega}_h = \omega_h \cup \partial\omega_h$

$x_i = ih_x$, $h_x = 1/n_x$

$y_j = jh_y$, $h_y = \frac{1}{n_y}$



$u_{ij} \approx u(x_i,y_j)$

$u_{ij} = u_h(x_i,x_j)$

$h_x = h_y = h$

$n_x = n_y = n$

and replace the 2nd-order derivatives in $(20)_{CF}$ by the corresponding 2nd-order finite differences:

$(20)_h$ Find grid function $u_h : \overline{\omega}_h \to \mathbb{R}$ such that

$$-\frac{u_{i-1,j} - 2u_{ij} + u_{i+1,j}}{h_x^2} - \frac{u_{i,j-1} - 2u_{ij} + u_{i,j+1}}{h_y^2} = f_{ij} := f(x_i, y_j) \quad (x_i,y_j)\in\omega_h$$

$$u_{ij} = g(x_i, y_j) \text{ for } (x_i,y_j)\in\partial\omega_h, \ (i,j)\in\partial\omega_h \qquad (i,j)\in\omega_h, \ i,j = \overline{1,n-1}$$

$$L_h u_h = f_h \text{ in } \omega_h$$
$$u_h = g_h := 0 \text{ on } \gamma_h = \partial\omega_h$$

$\Updownarrow$

$(\underline{20})_h$ Find $\underline{u}_h = [u_{ij}]_{i,j=1,n-1} \in \mathbb{R}^{N_h = (n-1)^2}$:

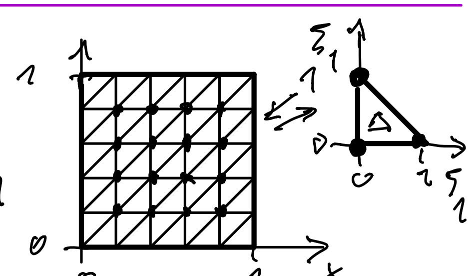
$$K_h \underline{u}_h = \underline{f}_h \text{ in } \mathbb{R}^{N_h},$$

with $\underline{f}_h = [f_{ij}]_{i,j=\overline{1,n-1}} \in \mathbb{R}^{N_h}$, $f_{ij} = f(x_i,y_j)$, $(i,j)\in\omega_h$.

Exercise 4.2:

Show that

$$K_h^{FDM} = \frac{1}{h^2} K_h^{FEM}, \quad \kappa = 1$$



but the right-hand sides are different:

$$\underline{f}_h^{FDM} \neq \frac{1}{h^2} \underline{f}_h^{FEM} \ !$$

How can you use this observation to get discretization error estimates in the H^1-seminorm:

$$|u - u_h^{FDM}|_{1,\Omega} := \|\nabla u - \nabla u_h^{FDM}\|_{L_2(\Omega)} \leq \ ?$$

on the basis of 1st Strang's Lemma, where $u_h^{FDM}(x) = \sum_{i,j} u_{ij} p_{ij}(x)$, FE-basis function.

Classical Convergence Analysis:

$(20)_{CF}$ $u : \overline{\Omega} \to \mathbb{R}$: $Lu = f$ in Ω, $u = g$ on $\Gamma = \partial\Omega$

$(20)_h$ $u_h : \overline{\omega} \to \mathbb{R}$: $L_h u_h = f_h$ in ω_h, $u_h = g_h$ on $\gamma_h = \partial\omega_h$

e.g. with $f_h(x_i,y_j) = f(x_i,y_j) \quad \forall (x_i,y_j) \in \omega_h \leftrightarrow (i,j) \in \omega_h$

$g_h(x_i,y_j) = g(x_i,y_j) \quad \forall (x_i,y_j) \in \gamma_h \leftrightarrow (i,j) \in \gamma_h$

The error $z_h = u - u_h : \overline{\omega}_h \to \mathbb{R}$ satisfies the error scheme:

$$L_h z_h = L_h u - L_h u_h = L_h u - f_h = L_h u - f = L_h u - Lu = \psi_h \text{ in } \omega_h$$

$z_h = 0$ on γ_h approximation error

$$(21)_h \quad \boxed{\begin{aligned} L_h z_h &= \psi_h := L_h u - Lu \text{ in } \omega_h \\ z_h &= 0 \text{ on } \gamma_h \end{aligned}}$$

Choose appropriate normed space X_h and Y_h of grid functions. Then $(21)_h$ leads to the estimate

$$(22) \quad \|z_h\|_{X_h} = \|L_h^{-1}\psi_h\|_{X_h} \le \underbrace{\|L_h^{-1}\|_{[Y_h,X_h]}}_{\le c_s \ne c_s(h)} \; \underbrace{\|\psi_h\|_{Y_h}}_{\le c_A(u)\,h^p}$$

Stability Approximation

$$\le c_S \, c_A(u)\, h^p \xrightarrow[h \to 0]{} 0$$

$$\boxed{\text{STABILITY + APPROXIMATION = DISCRETE CONV.}}$$

$[Y_h, X_h] = L(X_h, \omega_h)$ $\quad Y_h \quad X_h$

Let us return to our Example, the famous 5-star FD-scheme $(20)_h$ for the Poisson equation (20):

We choose: $X_h = \overset{\circ}{C}(\bar{\omega}_h)$, $\|z_h\|_{X_h} := \max_{(x,y)\in\bar{\omega}_h} |z_h(x,y)|$

$Y_h = C(\omega_h)$, $\|\psi_h\|_{Y_h} := \max_{(x,y)\in\omega_h} |\psi_h(x,y)|$

1. Approximation in $Y_h := C(\omega_h)$:

$$\|\psi_h\|_{C(\omega_h)} := \max_{(x,y)\in\omega_h} |L_h u(x,y) - Lu(x,y)|$$

$$= \max_{(x,y)\in\omega_h} \left| -u_{\bar{x}x}(x,y) - u_{\bar{y}y}(x,y) - \left(\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)\right)\right|$$

$$\le \frac{1}{12} \max_{(x,y)\in\bar{\Omega}} \left\{ \left|\frac{\partial^4 u}{\partial x^4}(x,y)\right| + \left|\frac{\partial^4 u}{\partial y^4}(x,y)\right| \right\} h^2$$

$$= c_A(u)\, h^2$$

2. Stability: $\|L_h^{-1}\|_{[C(\omega_h), \overset{\circ}{C}(\bar{\omega}_h)]} \le c_s$, i.e.

we have to show

$$\|z_h\|_{\overset{\circ}{C}(\bar{\omega}_h)} \le c_s \|\psi_h\|_{C_h(\omega_h)}, \quad L_h z_h = \psi_h$$

$$\max_{(x,y)\in\omega_h} |z_h(x,y)| \le c_s \max_{(x,y)\in\omega_h} |\psi_h(x,y)|$$

This estimate holds with the stability constant $c_s = 1/2$.

The proof is not trivial, see NUMERIK II, pp. 138-139.

STABILITY + APPROXIMATION = DISC. CONV.:

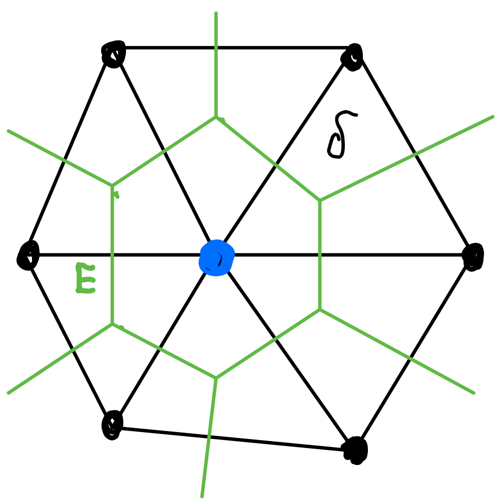
$$\max_{(x,y)\in\omega_h} |u(x,y) - u_h(x,y)| \le \frac{1}{2} c_A(u)\, h^2.$$

4.2.2. Finite Volume Method

= modern FDM on non-uniform grids
= Finite Integration Technik (FIT) in Electromagnetics!

- <u>Idea:</u> Discretize directly the <u>Balance Equation</u> that first arises in Modelling a physical process like heat conduction, see Lectures on Mathematical Modelling:

$x = (x_1, x_2) = (x, y)$



primary grid $\delta \in \mathcal{T}_h$

$x^{(i)}$

$B(x^{(i)})$

secondary grid

- Let us again consider the Dirichlet BVP for the Poisson equation $(20)_{CF}$ as model problem:

$$(20)_{CF} \qquad -\Delta u = f \text{ in } \Omega, \text{ and } u = g := 0 \text{ on } \Gamma = \partial\Omega$$

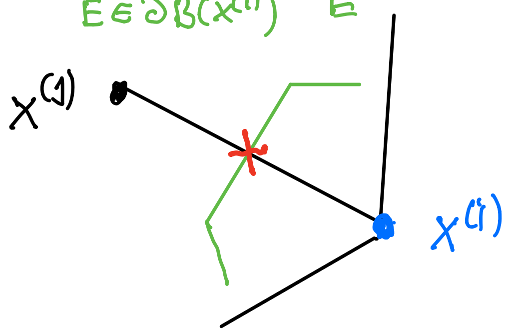
modelling

$$- \int_{B(x^{(i)})} \Delta u \, dx = \int_{B(x^{(i)})} f \, dx \quad \forall x^{(i)} \in \omega_h \; (\forall i \in \omega_h)$$

$\Downarrow$ Gauß

$$- \int_{\partial B(x^{(i)})} \frac{\partial u}{\partial n}(x) \, ds_x = \int_{B(x^{(i)})} f(x) \, dx \quad \text{Balance Equation}$$

$$Lu(x) = -\sum_{E \in \partial B(x^{(i)})} \int_E \frac{\partial u}{\partial n}(x)\, ds_x = \int_{B(x^{(i)})} f(x)\, dx$$



$\forall x^{(i)} \in \omega_h$

$\forall i \in \omega_h$

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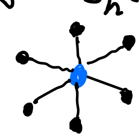
$$L_h u_h(x^{(i)}) = -\sum_{\substack{E \in \partial B(x^{(i)}) \\ j \in S(x^{(i)})}} \frac{u_h(x^{(1)}) - u_h(x^{(i)})}{|x^{(1)} - x^{(i)}|} |E| = \int_{B(x^{(i)})} f(x)\, dx$$

$$u_h(x^{(i)}) = g(x^{(i)}) := 0 \quad \forall i \in \gamma_h := \partial\omega_h$$

- **Result: FV-Scheme:** $u^{(i)} \approx u(x^{(i)})$

$$-\sum_{\substack{E \in \partial B(x^{(i)}) \\ j \in S_h(x^{(i)}) = \{\bullet\}}} \frac{u^{(1)} - u^{(i)}}{|x^{(0)} - x^{(i)}|} |E| = f^{(i)} := \int_{B(x^{(i)})} f(x)\, dx \quad \forall i \in \omega_h$$

numerical integration



difference star

$$u^{(i)} = g(x^{(i)}) \quad \forall i \in \gamma_h = \partial\omega_h = \gamma_{0h}$$

- **Convergence Analysis:** Stability + Appr. = discr. Conv.

$$\|u - u_h\|_{X_h(\bar\omega_h)} \le \|L_h^{-1}\|_{[Y_h, X_h]} \|L_h u - L u\|_{Y_h(\omega_h)}$$
$$\le C_S \cdot C_A(u)\, h^p$$