4.1.3. $V_{*h}\times V_h$ - Boundedness of the dG Bilinear Form

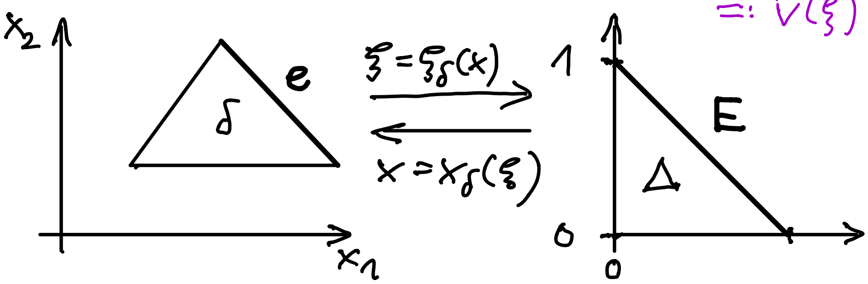
■ **Lemma 4.9:** Scaled trace inequality

Assume shape regular triangulation $\mathcal{T}_h$, cf. Def. 3.3.
Then there is some generic constant $c = \text{const} > 0$:

$$(13)\quad \|v\|_e^2 := \|v\|_{L_2(e)}^2 \le c\, h_\delta^{-1}\Big(\|v\|_\delta^2 + h_\delta^2 \underbrace{\|\nabla v\|_\delta^2}_{=|v|^2_{H^1(\delta)}}\Big)$$
$$\forall v \in H^1(\delta)\ \forall e \subset \partial\delta\ \forall \delta \in \mathcal{T}_h\ \forall h \in \Theta$$

Proof: cf. Lemma 4.4.: $\mathbb{P}_k(\delta) \longmapsto H^1(\delta)$! ~~inverse inequality~~

$$\|v\|^2_{L_2(e)} = \int_e (v(x))^2 ds_x = \int_E \Big(\underbrace{v(x_\delta(\xi))}_{=:\tilde v(\xi)}\Big)^2 \underbrace{|e|}_{=h_e \le h_\delta \approx O(h_\delta^{d-1})} ds_\xi$$



$$= |e| \int_E (\tilde v(\xi))^2 d\xi$$

$[\ldots] = \|\tilde v\|^2_{H^1(\Delta)}$

Ch. 2 trace th.

$$\le |e|\, c_T^2 \Big[\int_\Delta |v(x_\delta(\xi))|^2 d\xi + \int_\Delta |\nabla_\xi v(x_\delta(\xi))|^2 d\xi\Big]$$

$$= c_T^2 |e| \Big[\int_\delta |v(x)|^2 \frac{1}{|J_\delta|} dx + \int_\delta |J_\delta^T \nabla_x v(x)|^2 \frac{1}{|J_\delta|} dx\Big]$$

$\|J_\delta^T\| \le c_2 h_\delta$
$|e| = h_e \le h_\delta$
$|J_\delta| \ge c_1 h_\delta^d$
$d = 2$

$$\le \frac{c_T^2}{c_1} h_\delta^{-1} \Big[\int_\delta |v|^2 dx + c_2^2 h_\delta^2 \int_\delta |\nabla_x v|^2 dx\Big]$$

$$\le \frac{c_T^2}{c_1} \max\{1, c_2^2\}\, h_\delta^{-1} \Big\{\int_\delta |v|^2 dx + h_\delta^2 \int_\delta |\nabla_x v|^2 dx\Big\}.$$

q.e.d.

Remark:

1. (13) is obviously also valid for $d=3$!

$|e| \le c_e h_\delta^{d-1}$, $|\partial\delta| \ge c_1 h_\delta^d \Rightarrow \frac{|e|}{|\partial\delta|} \le \frac{c_e}{c_1} h_\delta^{-1}$ ✓

2. Using the sharp trace theorem (Ch. 2)

$$\|\tilde{v}\|_E := \|\tilde{v}\|_{L_2(E)} \le \underbrace{c_T(\varepsilon)}_{\varepsilon\to 0 \;\to\; \infty} \|\tilde{v}\|_{H^{1/2+\varepsilon}(\delta)} \quad \forall \tilde{v} \in H^{1/2+\varepsilon}(\delta),$$

we can easily show (mms)

$$(13)_\varepsilon \qquad \|v\|_e^2 \le c\, h_\delta^{-1} \left\{ \|v\|_\delta^2 + h^{2(\frac{1}{2}+\varepsilon)} |v|^2_{H^{1/2+\varepsilon}(\delta)} \right\}$$

$$\forall v \in H^{1/2+\varepsilon}(\delta)\ \forall e \subset \partial\delta\ \forall \delta \in \mathcal{T}_h\ \forall h \in \Theta,\ \varepsilon > 0 \text{ fix}.$$

Remark:

Inequality (13) can be used to prove the second statement in Lemma 3.18 (Clément):

$$\|v - I_h v\|_e := \|v - I_h v\|_{L_2(e)} \le c\, h_e^{1/2} |v|_{H^1(U(\delta_r))},$$

where $I_h v$ denotes the CLÉMENT interpolator, see Def. 3.17.

Indeed, using (13), we get

$$\|v - I_h v\|_e^2 \le c\, h_\delta^{-1} \Big(\|v - I_h v\|_\delta^2 + h_\delta^2 \underbrace{\|\nabla(v - I_h v)\|_\delta^2}_{|v - I_h v|^2_{H^1(\delta)}} \Big)$$

$$\le c\, h_\delta^{-1} \left(c_0^2 h_\delta^2 |v|^2_{H^1(U(\delta))} + h_\delta^2 c_1^2 |v|^2_{H^1(U(\delta))} \right)$$

$$\le c\, h_\delta^1 |v|^2_{H^1(U(\delta))} \le c\, h_e^1 |v|^2_{H^1(U(\delta))}.$$

generic constants. q.e.d.

- Let us consider the extended space

$$V_{*h} := \underset{\downarrow\atop H^{3/2+\varepsilon}(\mathcal{T}_h)}{H^2(\mathcal{T}_h)} + V_h \ni \underset{\downarrow\atop \text{solution of } (1)_{VF}}{u} - v_h \quad \forall v_h \in V_h,$$

and let us define the norm

$$(14) \quad \|v\|_{*h}^2 := \|v\|_h^2 + \sum_{\delta\in\mathcal{T}_h} h_\delta^2 |v|_{H^2(\delta)}^2$$

$$= \sum_{\delta\in\mathcal{T}_h} \|\nabla v\|_{L_2(\delta)}^2 + \sum_{e\in\mathcal{E}_h} \frac{\alpha_e}{h_e}\|[v]\|_{L_2(e)}^2 + \sum_{\delta\in\mathcal{T}_h} h_\delta^2 |v|_{H^2(\delta)}^2 .$$

- For all $v \in V_{*h}$ and $w_h \in V_h$, we can now estimate the SIPG ($\beta=-1$) dG bilinear form as follows:

$$(15) \quad |a_h(v,w_h)| = \Big| \sum_{\delta\in\mathcal{T}_h} (\nabla v, \nabla w_h)_\delta - \sum_{e\in\mathcal{E}_h} (\{\nabla v\},[w_h])_e$$

$$- \sum_{e\in\mathcal{E}_h} (\{\nabla w_h\},[v])_e + \sum_{e\in\mathcal{E}_h} \frac{\alpha_e}{h_e}([v]\,[w_h])_e \Big|$$

$$= |T_1 + T_2 + T_3 + T_4| \le \sum_{i=1}^4 |T_i|$$

T_1 and T_4 can easily be estimated by Cauchy:

$$|T_1| \le \Big(\sum_\delta \|\nabla v\|_\delta^2\Big)^{1/2} \Big(\sum_\delta \|\nabla w_h\|_\delta^2\Big)^{1/2},$$

$$|T_4| \le \Big(\sum_e \frac{\alpha_e}{h_e}\|[v]\|_e^2\Big)^{1/2} \Big(\sum_e \frac{\alpha_e}{h_e}\|[w_h]\|_e^2\Big)^{1/2}.$$

Using Lemma 4.4, we can estimate T_3 like (*) in the proof of Lemma 4.5; see Lecture 24:

$$|T_3| = \Big|\sum_e \big(\{\nabla \overset{\nabla_h}{w_h}\}, [v]\big)_e\Big|$$

$$\le \tilde{C}_3 \Big(\sum_\delta \|\nabla w_h\|_\delta^2\Big)^{1/2} \Big(\sum_e \frac{\alpha_e}{h_e} \|[v]\|_e^2\Big)^{1/2}$$

with $\tilde{C}_3^2 = \tilde{C} \max_e \alpha_e^{-1} = c_4^2 \, \frac{3}{2} \max_e \alpha_e^{-1}$.

In order to estimate T_2, we need Lemma 4.9:

$$|T_2| = \Big|\sum_e \big(\{\nabla \overset{\nabla_{*h}}{v}\}, [w_h]\big)_e\Big|$$

L_2-Cauchy
$$\le \sum_e \|\{\nabla v\}\|_e \Big(\frac{h_e}{\alpha_e}\Big)^{1/2} \Big(\frac{\alpha_e}{h_e}\Big)^{1/2} \|[w_h]\|_e$$

Σ-Cauchy
$$\le \Big(\sum_e \frac{h_e}{\alpha_e} \underbrace{\|\{\nabla v\}\|_e^2}\Big)^{1/2} \Big(\sum_e \frac{\alpha_e}{h_e} \|[w_h]\|_e^2\Big)^{1/2}$$

$e \in \partial E_h$ ✓

$e \in E_h$

$$= \Big\|\tfrac{1}{2}\big(\nabla v|_{\delta_1} + \nabla v|_{\delta_2}\big)\Big\|_e^2$$

$$\le \tfrac{1}{4}\big(\|\nabla v|_{\delta_1}\|_e + \|\nabla v|_{\delta_2}\|_e\big)^2$$

$(a+b)^2 \le 2(a^2+b^2)$
$$\le \tfrac{1}{2}\big[\|\nabla v|_{\delta_1}\|_e^2 + \|\nabla v|_{\delta_2}\|_e^2\big]$$

Lemma 4.9
$$\underset{(13)}{\le} \frac{c}{2}\Big[h_{\delta_1}^{-1}\|\nabla v\|_{\delta_1}^2 + h_{\delta_1}\|\nabla^2 v\|_{\delta_1}^2 + h_{\delta_2}^{-1}\|\nabla v\|_{\delta_2}^2 + h_{\delta_2}\|\nabla^2 v\|_{\delta_2}^2\Big]$$

$$\le \tilde{C}_4 \Big(\sum_\delta \|\nabla v\|_\delta^2 + \sum_\delta h_\delta^2 |v|_{2,\delta}^2\Big)^{1/2} \Big(\sum_e \frac{\alpha_e}{h_e} \|[w_h]\|_e^2\Big)^{1/2}$$

with a positive generic constant $\tilde{C}_4$.

Inserting these estimates, i.e. $|T_i| \leq \dots, i = 1,2,3,4$, into (15), and using Cauchy, we immediately get

$$(16) \quad |a_h(v, w_h)| \leq \mu_4 \|v\|_{*h} \|w_h\|_h$$
$$\forall v \in V_{*h} = H^2(\mathcal{T}_h) + V_h \quad \forall w_h \in V_h \quad \forall h \in \Theta$$

with a positiv generic constant $\mu_4 = \mu_4(\max_e \alpha_e^{-1}, k)$, i.e. we have proved the following Lemma:

Lemma 4.10: ($V_{*h} \times V_h$-boundedness)
Let $\beta = -1, 0, 1$, and $\mathcal{T}_h$ be a shape regular triangulation. Then the dG bilinear form $a_h(\cdot, \cdot)$ is bounded on $V_{*h} \times V_h$, i.e. $\exists\, \mu_4 \neq \mu_4(h)$ such that inequality (16) holds.

Theorem 4.11:
Ass.: Assume that the exact solution u of $(1)_{VF}$ belongs to $H^s(\mathcal{T}_h)$ for some $s \geq 2$ ($s > 3/2$), $\mathcal{T}_h$ - (shape) regular triangulation, and we also assume that the penalty parameters α_e are large enough, i.e. $\alpha_e \geq \alpha_0 \; \forall e \in \mathcal{E}_h$; cf. Lemma 4.5.

St.: Then there exists a positive constant $c \neq c(h)$:

$$(17) \quad \|u - u_h\|_h \leq c \Big[\sum_{\delta \in \mathcal{T}_h} h_\delta^{2(\min\{k+1, s\} - 1)} |u|^2_{H^s(\delta)} \Big]^{1/2}$$
$$\leq c\, h^{\min\{k+1, s\} - 1} \|u\|_{H^s(\mathcal{T}_h)}.$$

Proof: In the case of consistent schemes, second Strang's Lemma (= Theorem 3.16) implies estimate (40), see also Remark at the end of Lecture 20:

$$\|u - \underset{(4)_h}{\overset{dG}{u_h}}\|_h^2 \leq \left(1 + \frac{\mu_4}{\mu_3}\right)^2 \inf_{v_h \in \bar{V}_{gh} = \bar{V}_h} \|u - v_h\|_{*h}^2$$

$(1)_{VF}$

$$\leq \left(1 + \frac{\mu_4}{\mu_3}\right)^2 \|u - \mathrm{Int}_{\bar{V}_h}(u)\|_{*h}^2$$

nodal interpolator

$u \in H^s(T_h) \hookrightarrow C(\bar{T}_h)$, i.e.

$\tilde{u}_h = \mathrm{Int}_{\bar{V}_h = V_k(T_h)}(u)$ $\quad u \in H^s(\delta) \hookrightarrow C(\bar{\delta})$

since $s > 3/2$!

$$= (\cdot)^2 \sum_{\delta \in T_h} \|\nabla(u - \tilde{u}_h)\|_{L_2(\delta)}^2 + \sum_{e \in E_h} \frac{\alpha_e}{h_e} \|[u - \tilde{u}_h]\|_{L_2(e)}^2 + \sum_{\delta \in T_h} h_\delta^2 |u - \tilde{u}_h|_{H^2(\delta)}^2$$

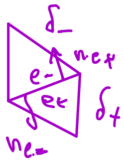
$$= \left(1 + \frac{\mu_4}{\mu_3}\right)^2 (T_1 + T_2 + T_3)$$

$$T_1 = \sum_{\delta \in T_h} \|\nabla(u - \tilde{u}_h)\|_{L_2(\delta)}^2 \leq c \sum_{\delta \in T_h} h_\delta^{2(\min\{k+1, s\} - 1)} |u|_{H^s(\delta)} \checkmark$$

Subsect. 3.4.2. Approximation Theorem

$$T_3 = \sum_{\delta \in T_h} h_\delta^2 |u - \tilde{u}_h|_{H^2(\delta)}^2 \leq c \sum_{\delta} h_\delta^2 h_\delta^{2(\min\{k+1, s\} - 2)} |u|_{H^s(\delta)} \checkmark$$

$$T_2 = \sum_{e \in E_h} \frac{\alpha_e}{h_e} \|[u - \tilde{u}_h]\|_{L_2(e)}^2$$



$e \in \partial E_h$: $[u - \tilde{u}_h]_e = u - \tilde{u}_h|_e$

$e \in E_h$: $[u - \tilde{u}_h]_e = (u - \tilde{u}_h)|_{e_+} \cdot n_{e_+} + (u - \tilde{u}_h)|_{e_-} n_{e_-}$

L25-07

$e \in \partial E:\ e_+$

$$\le \sum_e \frac{\alpha_e}{h_e} 2\left(\| u-\tilde u_h \|^2_{e_+} + \| u - \tilde u \|^2_{e_-} \right)$$

Lemma 4.9: (13)

$$\le 2c \sum_e \frac{\alpha_e}{h_e} \Big(h_{\delta_+}^{-1} \| u-\tilde u_h \|^2_{L_2(\delta_+)} + h_{\delta_+} | u-\tilde u_h |^2_{H^1(\delta_+)} +$$

$$+ \underbrace{h_{\delta_-}^{-1} \| u-\tilde u_h \|^2_{L_2(\delta_-)}}_{\le\, c\, h_{\delta_\pm}^{2\min\{k+1,s\}} |u|^2_{H^s(\delta_\pm)}} + h_{\delta_-} \underbrace{| u-\tilde u_h |^2_{H^1(\delta_-)}}_{\le\, c\, h_{\delta_\pm}^{2(\min\{k+1,s\}-1)} |u|^2_{H^s(\delta_\pm)}} \Big)$$

$$\le c' \sum_{\delta \in \mathcal{T}_h} h_\delta^{2(\min\{k+1,s\}-1)} |u|^2_{H^s(\delta)} \quad \checkmark$$

$$= \left(1 + \frac{\mu_4}{\mu_5}\right)^2 \left(T_1 + T_2 + T_3 \right)$$

$$\le c \sum_{\delta \in \mathcal{T}_h} h_\delta^{2(\min\{k+1,s\}-1)} |u|^2_{H^s(\delta)} .$$ q.e.d.

generic constant: $\left(1 + \frac{\mu_4}{\mu_3}\right)^2 c$

Remark 4.12:

1. Basically, we proved the refined estimate

$$(18)\qquad \|u-u_h\|_h \le \left[\sum_{\delta\in\mathcal{T}_h} c_\delta^2\, h_\delta^{2(\min\{k_\delta+1,\,s_\delta\}-1)}\, |u|^2_{H^{s_\delta}(\delta)}\right]^{1/2},$$

with $s_\delta > 3/2$, $u|_\delta \in H^{s_\delta}(\delta)$, $0 < c_\delta \le \bar{c} = \text{const}$
$\forall \delta \in \mathcal{T}_h$ $\forall h \in \Theta$

shape reg. triangulation that can be non-conform with hanging nodes!

2. One can also prove L_2-estimates like

$$(19)\qquad \|u-u_h\|_{L_2(\Omega)} \le c\, h^{\min\{k+1,\,s\}}\, |u|_{H^s(\mathcal{T}_h)} \qquad s \ge 2$$

provided that $u \in H^s(\mathcal{T}_h)$, $s \ge 2$, and that the adjoint problem is H^2-coercive; see Reference [7], Subsection 2.8.2.

References: [7] and [8] on the NuEPDE website!

[7] B. Rivière: Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation. SIAM, Phil., 2008.

[8] D.A. Di Pietro, A. Ern: Mathematical Aspects of Discontinuous Galerkin Methods, Springer, 2012.

$\Rightarrow$ low regularity cas $u \in H^s(\mathcal{T}_h)$ with $1 < s \le 3/2$ in 2d!
$\Rightarrow$ there is a "gap" in the theory in 3d.
This gap was closed by Z. Cai, X. Ye, and S. Zhang in their SINUM paper 2011, v. 49, No 5, p. 1761–1787.