4.1.2. V_h - Elliplicity

- **Lemma 4.4:**

Let $\mathcal{T}_h = \{\delta_r : r \in \mathbb{R}_h\}$ be a (shape) regular triangulation as defined in Def. 3.3 : (3.8), (3.9), (3.10).

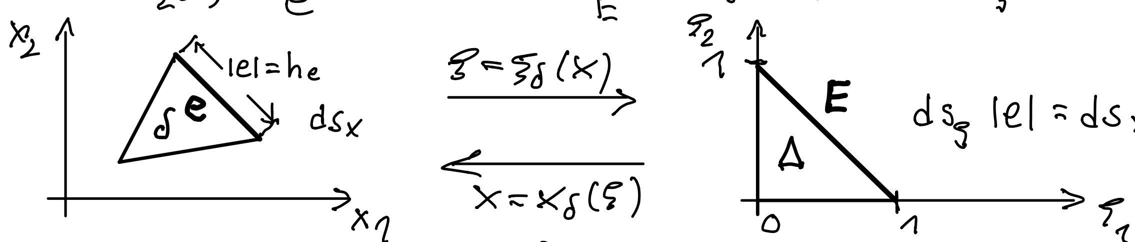
Then there is some positive constant $c_4 = c_4(K) > 0$:

(7) $$\|v\|_{L_2(e)} \leq c_4 h_\delta^{-1/2} \|v\|_{L_2(\delta)}$$

$$\forall v \in P_K(\delta) \ \forall e \in \partial\delta \ \forall \delta \in \mathcal{T}_h \ \forall h \in \Theta \quad (h_\delta = h_r \text{ for } \delta = \delta_r, r \in \mathbb{R}_h)$$

Proof: $\forall v \in P_K(\delta)$, we have

$$\|v\|^2_{L_2(e)} = \int_e (v(x))^2 ds_x = \int_E (v(x_\delta(\xi)))^2 |e| \, ds_\xi$$



$$= |e| \int_E \big(\underbrace{v(x_\delta(\xi))}_{=:\, \tilde{v}(\xi)}\big)^2 ds_\xi = |e| \, \|\tilde{v}(\xi)\|^2_{L_2(E)}$$

$$\leq |e| \, c_T^2(\Delta) \, \|\tilde{v}(\xi)\|^2_{H^1(\Delta)} \qquad \forall \tilde{v} \in H^1(\Delta)$$

↑ trace theorem (Ch. 2): $\|\tilde{v}\|_{L_2(E)} \leq \|\tilde{v}\|_{H^{1/2}(E)} \leq c_T \|\tilde{v}\|_{H^1(\Delta)}$

$$\leq |e| \, c_T^2(\Delta) \, c_{1,0}^2(\Delta) \, \|\tilde{v}(\xi)\|^2_{L_2(\Delta)}$$

↑ equivalence of all norms on finite dim. spaces

$\Delta \to \delta$

$$= |e| \, c_T^2(\Delta) \, c_{1,0}^2(\Delta) \int_\delta (v(x))^2 \, |J_\delta^{-1}| \, dx$$

$|J_\delta| = 2|\delta|$

$$= \frac{|e|}{|J_\delta|} \, c_T^2 \, c_{1,0}^2 \, \|v\|^2_{L_2(\delta)} \leq c_4^2 \, h_\delta^{-1} \, \|v\|^2_{L_2(\delta)} .$$

Def. 3.3 $\quad \dfrac{|e|}{|J_\delta|} \leq \dfrac{h_\delta^{d-1}}{\underline{c}_1 h_\delta^d} = \dfrac{1}{\underline{c}_1} h_\delta^{-1}$ $\qquad c_4^2 := \dfrac{c_T^2 \, c_{1,0}^2}{\underline{c}_1}$

q.e.d.

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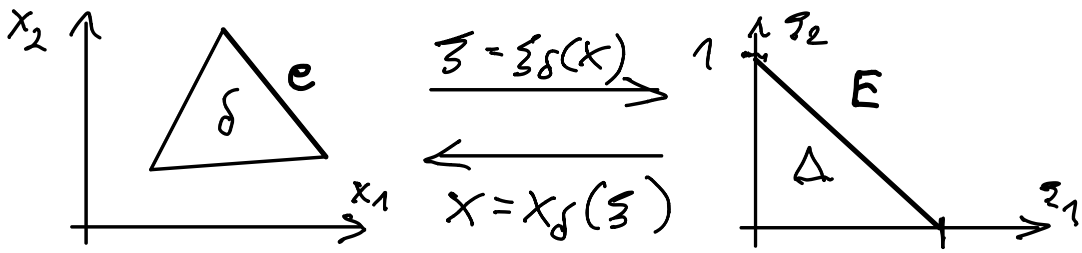
Alternative proof of the estimate

$$\|\tilde{v}\|^2_{L_2(E)} \leq c \, \|\tilde{v}\|^2_{L_2(\Delta)} \quad \forall \tilde{v} \in \mathbb{P}_k(\Delta)$$

$c = c_r^2(\Delta)\, c_{1,0}^2(\Delta) \quad (\uparrow)$

from the proof of Lemma 4.4, where $\tilde{v} = \tilde{v}(\xi) = v(x_\delta(\xi)) \in \mathbb{P}_k(\Delta)$:



$$\|\tilde{v}\|^2_{L_2(E)} = (\tilde{v}, \tilde{v})_{L_2(E)} = (M_E \underline{v}, \underline{v})$$

$\forall \tilde{v} \leftrightarrow \underline{v}$ $\quad M_E = M_E^T \geq 0$ SNN, but not regular!

$$\leq \lambda_{max} \, (\overset{SPD}{M_\Delta} \underline{v}, \underline{v})$$

$$= \lambda_{max} \, (\tilde{v}, \tilde{v})_{L_2(\Delta)} = \lambda_{max} \|\tilde{v}\|^2_{L_2(\Delta)},$$

i.e.

$$\lambda_{max} = \max_{\underline{v} \in \mathbb{R}^{|A_r|}} \frac{(M_E \underline{v}, \underline{v})}{(M_\Delta \underline{v}, \underline{v})} \iff \begin{array}{l} \text{EVP} \\ M_E \underline{v} = \lambda M_\Delta \underline{v}, \\ M_\Delta^{-1} M_E \underline{v} = \lambda \underline{v} \end{array}$$

i.e. $\lambda_{max} = \lambda_{max}(M_\Delta^{-1} M_E) = \lambda_{max}(k)$

polynomial degree

$$\Rightarrow \quad c = c(k) = \max_{E \subset \partial\Delta} \lambda_{max}(M_\Delta^{-1} M_E).$$

- Let us now define the so-called **dG norm**

(8) $$\|v\|_h^2 := \sum_{\delta\in\mathcal{T}_h} \|\nabla v\|^2_{L_2(\delta)} + \sum_{e\in\bar{E}} \frac{\alpha_e}{h_e} \|[v]_e\|^2_{L_2(e)}$$

that is indeed a norm on $\bar{V}_h = V_K(\mathcal{T}_h)$! (mms)
Now we are in the position to prove $\bar{V}_h$-ellipticity of the dG bilinear form $a_h(\cdot,\cdot)$:

- **Lemma 4.5:** ($\bar{V}_h$-ellipticity / coercivity)
Let $\beta = -1$ (SIPG). Then there exists a positive constant $\mu_3 \neq \mu_3(h)$ such that

(9) $$a_h(v,v) \geq \mu_3 \|v\|_h^2 \quad \forall v \in \bar{V}_h$$

holds provided that all α_e are sufficienty large, i.e. $\exists\, \alpha_0 = \mathrm{const} > 0$ (see proof): $\forall \alpha_e \geq \alpha_0$, (9) holds.

Proof: $\|\cdot\|_\delta = \|\cdot\|_{L_2(\delta)}$, $\|\cdot\|_e = \|\cdot\|_{L_2(e)}$. $\forall v = v_h \in \bar{V}_h = V_K(\mathcal{T}_h)$

$$a_h(v,v) \underset{SIPG}{=} \underbrace{\sum_\delta \|\nabla v\|^2_\delta}_{OK} \underbrace{- 2\sum_e (\{\nabla v\},[v])_e}_{(*)} + \underbrace{\sum_e \frac{\alpha_e}{h_e}\|[v]\|_e}_{OK}$$

$$(*) \leq \left(\tilde{c}\sum_\delta \|\nabla v\|^2_\delta\right)^{1/2} \cdot \left(\sum_e \frac{1}{\alpha_e}\frac{\alpha_e}{h_e}\|[v]\|^2_e\right)^{1/2}$$

$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$

$$\geq \Big[1 - \underbrace{\tilde{c}\,\varepsilon}\Big] \sum_\delta \|\nabla v\|^2_\delta + \sum_e \underbrace{\Big[1 - \frac{1}{\varepsilon\alpha_e}\Big]} \frac{\alpha_e}{h_e}\|[v]\|^2_e$$

$\tilde{c}\varepsilon = \frac{1}{2}$, i.e. $\varepsilon = \frac{1}{2\tilde{c}}$ $\longrightarrow$ $1 - \frac{2\tilde{c}}{\alpha_e} \geq \frac{1}{2}$, i.e. $\boxed{\alpha_e \geq 4\tilde{c}}$ $\forall e \in \bar{E}_h$

$$\geq \frac{1}{2}\sum_\delta \|\nabla v\|^2_\delta + \frac{1}{2}\sum_e \frac{\alpha_e}{h_e}\|[v]\|^2_e$$

$= \frac{1}{2}\|v\|_h^2$, i.e. $\mu_3 = \frac{1}{2}$ provided that $\alpha_e \geq 4\tilde{c}$ $\forall e \in \bar{E}_h$.

(q.e.d.)

It remains to show the "red" estimate

$$(*) = \sum_e (\{\nabla v\}, [v])_e$$

L_2-Cauchy $\leq \sum_e \|\{\nabla v\}\|_e \, h_e^{1/2} \, h_e^{-1/2} \, \|[v]\|_e$

Σ-Cauchy $\leq \Big(\sum_e h_e \underbrace{\|\{\nabla v\}\|_e^2}\Big)^{1/2} \Big(\sum_e h_e^{-1} \|[v]\|_e^2\Big)^{1/2}$

$e \in \partial E_h$ ✓

$e \in E_h$

$$= \Big\| \tfrac{1}{2}\big(\nabla v|_{\delta_1} + \nabla v|_{\delta_2}\big)\Big\|_e^2$$

$$\leq \tfrac{1}{4}\Big(\|\nabla v|_{\delta_1}\|_e + \|\nabla v|_{\delta_2}\|_e\Big)^2$$

Lemma 4.4. $\longrightarrow \leq \tfrac{1}{4}\Big(c_4 h_{\delta_1}^{-1/2} \|\nabla v\|_{\delta_1} + c_4 h_{\delta_2}^{-1/2} \|\nabla v\|_{\delta_2}\Big)^2$

$(a+b^2) \leq 2(a^2+b^2)$ $\leq \tfrac{1}{2} c_4^2 h_{\delta_1}^{-1} \|\nabla v\|_{\delta_1}^2 + \tfrac{1}{2} c_4^2 h_{\delta_2}^{-1/2} \|\nabla v\|_{\delta_2}^2$

$h_e \leq h_{\delta_i}$

$\frac{h_e}{h_{\delta_i}} \leq 1$ $\leq \Big(\tilde{c} \sum_\delta \|\nabla v\|_\delta^2\Big)^{1/2} \Big(\sum_e h_e^{-1} \|[v]\|_e^2\Big)^{1/2}$

$$\boxed{\tilde{c} = c_4^2 \, \frac{1}{2} \underbrace{\max_{\delta \in T_h} |E(\delta)|}_{= 3 \text{ for } \delta = \triangle \text{ in 2d}} = c_4^2 \, \frac{3}{2}}$$

i.e. $\alpha_e \geq 4\tilde{c} = 6 c_4^2 = \frac{6}{\varepsilon_1} c(K) = \alpha_0(K) =$ computable!

q.e.d.

Remark 4.6:

Lemma 4.5 is obviously also valid for $\beta = 0, +1$.

Proof = (mms).

Lemma 4.7: (V_h - boundedness)
Let $\beta = -1, 0, 1$; and $\mathcal{T}_h$ be a (shape) regular triangulation. Then the dG bilinear form $a_h(\cdot,\cdot): V_h \times V_h \longrightarrow \mathbb{R}$ is V_h - ¥ (continuous), i.e. $\exists \tilde{\mu}_4 \neq \tilde{\mu}_4(h)$:

$$(10) \quad |a_h(v,w)| \le \tilde{\mu}_4 \|v\|_h \|w\|_h \quad \forall v, w \in V_h := V_K(\mathcal{T}_h).$$

Proof: e.g., for $\beta = -1$ (SIPG):

$$a_h(v,w) = \sum_{\delta} (\nabla v, \nabla w)_\delta - \sum_{e} (\{\nabla v\}, [w])_e - \sum_{e} ([v], \{\nabla w\})_e + \sum_{e} \frac{\alpha_e}{h_e} ([v],[w])_e$$

$|a_h(v,w)| \le$ (mms)

Corollary 4.8: $\exists!\, u_h \longleftrightarrow \underline{u}_h$

1. V_h-ellipticity of the dG bilinear form already yields uniqueness of the dG solution
   $u_h \in V_h$: $(4)_h$ resp. $\underline{u}_h \in \mathbb{R}^{N_h}$: $K_h \underline{u}_h = \underline{f}_h$ $(\underline{4})_h$,
   and uniqueness implies existence (finite dimensional)!
2. K_h is always **positive definite** since
   $(K_h \underline{v}_h, \underline{v}_h)_{\cancel{\mathbb{R}^{N_h}}} = a_h(v_h, v_h) \ge \mu_3 \|v_h\|_h^2 > 0 \quad \forall \underline{v}_h \in \mathbb{R}^{N_h}\setminus\{0\} \leftrightarrow v_h \in V_h \setminus \{0\}$
3. In the SIPG ($\beta = -1$) case: K_h is SPD !
4. The following spectral equivalence inequalities are valid:

$$\mu_3 \|v_h\|_h^2 \le (K_h \underline{v}_h, \underline{v}_h) \le \tilde{\mu}_4 \|v_h\|_h^2$$

$$\mu_3 (\underset{\text{SPD}}{C_h} \underline{v}_h, \underline{v}_h) \le (K_h \underline{v}_h, \underline{v}_h) \le \tilde{\mu}_4 (\underset{\text{SPD}}{C_h} \underline{v}_h, \underline{v}_h)$$

$\forall \underline{v}_h \in \mathbb{R}_h$, $\underline{v}_h \longleftrightarrow v_h \in V_h$.

Remark:

The V_h-ellipticity and the V_h-boundedness is not sufficient to derive discretization error estimates, cf. 2nd STRANG's Lemma (= Theorem 3.16) !

Indeed:

$$\|u - u_h\|_h \le \underbrace{\|u - \overset{I_h u}{\overset{\|}{v_h}}\|_h}_{\text{interpolation error est. } \checkmark} + \underbrace{\|\overbrace{v_h - u_h}^{\in V_h}\|_h}$$

$$\mu_3 \|v_h - u_h\|_h^2 \le a_h(v_h - u_h, v_h - u_h)$$

$$\overset{\text{GO (6)}}{=} a_h(\underbrace{v_h - u}_{\substack{\notin V_h \,! \\ \in H^s(T_h) + V_h,\; s > 3/2,\ \text{e.g. } s=2}}, v_h - u_h)$$

(11)

$$\le \mu_4 \|v_h - u\|_{*h} \|v_h - u_h\|_h$$

Combining these estimates, we get

(12) $$\|u - u_h\|_h \le \left(1 + \frac{\mu_4}{\mu_3}\right) \inf_{v_h \in V_h} \|u - v_h\|_{*h} \le \left(1 + \frac{\mu_3}{\mu_4}\right) \|u - I_h u\|_{*h}$$

$v_h = I_h u$ (interp.)

Question:

$$\|\cdot\|_{*h}^2 = \|\cdot\|_h^2 + \sum_\delta h_\delta^2 |\cdot|^2_{\substack{2,\delta \\ 3/2+\varepsilon,\delta}}$$

?